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Normality of Monomial Ideals

Ibrahim Al-Ayyoub

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Abstract

Given the monomial ideal $I = (x_1^{\alpha_1}, \dots, x_n^{\alpha_n}) \subset K[x_1, \dots, x_n]$ where α_i are positive integers and K a field and let J be the integral closure of I . It is a challenging problem to translate the question of the normality of J into a question about the exponent set $\Gamma(J)$ and the Newton polyhedron $NP(J)$. A relaxed version of this problem is to give necessary or sufficient conditions on $\alpha_1, \dots, \alpha_n$ for the normality of J. We show that if $\alpha_i \in \{s, l\}$ with s and l arbitrary positive integers, then J is normal.

Introduction

Let I be an ideal in a Noetherian ring R. The integral closure of I is the ideal $\overline{I}$ that consists of all elements of R that satisfy an equation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0, \qquad a_i \in I^i$$

The ideal I is said to be integrally closed if $I = \overline{I}$. Clearly one has that $I \subseteq \overline{I} \subseteq \sqrt{I}$. An ideal is called normal if all of its positive powers are integrally closed. It is known that if R is a normal integral domain, then the Rees algebra $R[It] = \oplus_{n \in N} I^n t^n$ is normal if and only if I is a normal ideal of R . This brings up the importance of normality of ideals as the Rees algebra is the algebraic counterpart of blowing up a scheme along a closed subscheme.

It is well known that the integral closure of monomial ideal in a polynomial ring is again a monomial ideal, see [SH] or [Vit] for a proof. The problem of finding the integral closure for a monomial ideal I reduces to finding monomials r, integer i and monomials $m_1, m_2, \dots, m_i$ in I such that $r^i + m_1 m_2 \cdots m_i = 0$, see [SH]. Geometrically, finding the integral closure of monomial ideals I in $R = K[x_0, \dots, x_n]$ is the same as finding all the integer lattice points in the convex hull $NP(I)$ (the Newton polyhedron of I) in $\mathbb{R}^n$ of $\Gamma(I)$ (the Newton polytope of I) where $\Gamma(I)$ is the set of all exponent vectors of all the monomials in I. This makes computing the integral closure of monomial ideals simpler.

A power of an integrally closed monomial ideal need not be integrally closed. For example, let J be the integral closure of $I = (x^4, y^5, z^7) \subset K[x, y, z]$. Then J^2 is not integrally closed (observe that $y^3 z^3 \in J$ as $\left(y^3 z^3\right)^5 = y^5 y^5 y^5 z^7 z^8 \in I^5$. Now $x^2 y^4 z^5 \in \overline{J^2}$ since $\left(x^2 y^4 z^5\right)^2 = \left(x^4 \cdot y^5\right)\left(y^3 z^3 \cdot z^7\right) \in \left(J^2\right)^2$. On the other hand we used the algebra software Singular [GPS05] to show that $x^2 y^4 z^5 \notin J^2$). However, a nice result of Reid et al. [RRV, Proposition 3.1] states that if the first $n-1$ powers of a monomial ideal, in a polynomial ring of n variables over a field, are integrally closed, then the ideal is normal. For the case

$n=2$ this follows from the celebrated theorem of Zariski [ZS] that asserts that the product of integrally closed ideals in a 2-dimensional regular ring is again integrally closed.

In general, there is no good characterization for normal monomial ideals. It is a challenging problem to translate the question of normality of a monomial ideal I into a question about the exponent set $\Gamma(I)$ and the Newton polyhedron $NP(I)$. Under certain hypotheses, some necessary conditions are given. Faridi [Far] gives necessary conditions on the degree of the generators of a normal ideal in a graded domain. Vitulli [Vit] investigated the normality for special monomial ideals in a polynomial ring over a field.

For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ let $I(\boldsymbol{\alpha})$ be the integral closure of $(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) \subset K[x_1, \ldots, x_n]$. Reid et. al. [RRV] showed that if $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ with pairwise relatively prime entries, then the ideal $I(\boldsymbol{\alpha})$ is normal if and only if the additive submonoid $\Lambda = \langle 1/\alpha_1, \ldots, 1/\alpha_n \rangle$ of $\mathbb{Q}_{\geq}$ is quasinormal, that is, whenever $x \in \Lambda$ and $x \geq p$ for some $p \in \mathbb{N}$, there exist rational numbers $y_1, \ldots, y_p$ in Λ with $y_i \geq 1$ for all i such that $x = y_1 + \cdots + y_p$. Thus for the case where $\alpha_1, \ldots, \alpha_n$ are pairwise relatively prime, the normality condition on the n-dimensional monoid is reduced to the quasinormality condition on the 1-dimensional monoid. Another nice result of Reid et. al. [RRV] is that the monomial ideal $I(\boldsymbol{\alpha})$ is normal if $\gcd(\alpha_1, \ldots, \alpha_n) > n-2$. In particular, if $n=3$ and $\gcd(\alpha_1, \alpha_2, \alpha_3) \neq 1$, then $I(\boldsymbol{\alpha})$ is normal. Therefore, in $k[x_1, x_2, x_3]$ it remains to investigate the normality of $I(\boldsymbol{\alpha})$ whenever $\gcd(\alpha_1, \alpha_2, \alpha_3) = 1$ and the integers are not pairwise relatively prime.

A important result of Reid et. al. [RRV], which we use to improve our result in this paper, is as following. Choose i and set $c = \operatorname{lcm}(\alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_n)$. Put $\boldsymbol{\alpha}' = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + c, \alpha_{i+1}, \ldots, \alpha_n)$. If $I(\boldsymbol{\alpha}')$ is normal then $I(\boldsymbol{\alpha})$ is normal. Conversely, If $I(\boldsymbol{\alpha})$ is normal and $\alpha_i \geq c$, then $I(\boldsymbol{\alpha}')$ is normal.

The goal of this paper is to show that the integral closure of the ideal $(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) \subset K[x_1, \ldots, x_n]$ is normal provided that $\alpha_i \in \{s, l\}$ with s and l arbitrary positive integers. The following theorem provide us with a technique that we mainly depend on to prove the integral closedness.

Theorem 1 *(Proposition 15.4.1, [SH]) Let I be a monomial ideal in the polynomial ring $R = K[x_1, \ldots, x_n]$ with K a field. If I is primary to $(x_1, \ldots, x_n)$ and $\overline{I} \cap (I : (x_1, \ldots, x_n)) \subseteq I$, then I is integrally closed.*

Proposition 2 *(Corollary 5.3.2, [SH]) If $I \subseteq J$ are ideals in a ring R, then $J \subseteq \overline{I}$ if and only if each element in some generating set of J is integral over I.*

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Let $(x_1^s, \ldots, x_m^s, y_1^l, \ldots, y_n^l) \subset K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ with K a field, x_i and y_i indeterminates over K, and s and l positive integers such that (without loss of generality) $l \geq s$.

Notation 3 *For the remaining of this paper fix positive integers s and l with $l \geq s$ and let $\lambda_a = \left\lceil a\frac{l}{s} \right\rceil$ where a is any integer. Also, let k be any positive integer.*

Let x and y be positive integers and write $x = ts + r$ with $1 \leq r \leq s$. Then $y\left\lceil \dfrac{x}{s} \right\rceil = y\dfrac{x+s-r}{s} = y\dfrac{s-r}{s} + y\dfrac{x}{s} \leq y\dfrac{s-r}{s} + \left\lceil y\dfrac{x}{s} \right\rceil$. Therefore, $\left\lceil y\dfrac{x}{s} \right\rceil \geq y\left(\left\lceil \dfrac{x}{s} \right\rceil - \dfrac{s-r}{s}\right)$. This inequality helps to prove the following lemma which is a key in this paper.

Lemma 4 *If $i \in \{0, 1, \ldots, ks\}$, then $kl(ks-i-1) + \lambda_i \geq (ks-i)(\lambda_{ks-1} - \frac{s-r}{s})$, where $(ks-1)l = ts + r$ with $1 \leq r \leq s$.*

Proof. By the note before the lemma we have $kl(ks-i-1) + \lambda_i = \left\lceil \frac{[ks(ks-i-1)+i]l}{s} \right\rceil = \left\lceil (ks-i)\frac{(ks-1)l}{s} \right\rceil \geq (ks-i)\left(\left\lceil \frac{(ks-1)l}{s} \right\rceil - \frac{s-r}{s}\right) = (ks-i)\left(\lambda_{ks-1} - \frac{s-r}{s}\right)$. ■

Definition 5 *Let $F_k = \{x_{i_1} \cdots x_{i_{ks-a}} y_{j_1} \cdots y_{j_{\lambda_a}} \mid a = 0, 1, 2, \ldots, ks,\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_{ks-a} \leq m,$ and $1 \leq j_1 \leq j_2 \leq \cdots \leq j_{\lambda_a} \leq n\}$, J_k the ideal generated by all the monomials in F_k, and $I_k = (x_1^{ks}, \ldots, x_m^{ks}, y_1^{kl}, \ldots, y_n^{kl}) \subset K[x_1, \ldots, x_m, y_1, \ldots, y_n]$. Also, let $J = J_1$, $F = F_1$, and $I = I_1$.*

Lemma 6 *J_k is integral over the ideal I_k, that is, $J_k \subseteq \overline{I_k}$.*

Proof. By Proposition 2 it suffices to show that every element of F_k is integral over I_k. Note $x_{i_1}^{ksl} \cdots x_{i_{ks-a}}^{ksl} \in I_k^{l(ks-a)}$ and $y_{j_1}^{ksl} \cdots y_{j_{\lambda_a}}^{ksl} \in I_k^{s\lambda_a}$. Also note $l(ks-a) + s\lambda_a = ksl - la + s\left\lceil a\dfrac{l}{s} \right\rceil \geq ksl$. Therefore, $\left(x_{i_1} \cdots x_{i_{ks-a}} y_{j_1} \cdots y_{j_{\lambda_a}}\right)^{ksl} \in I_k^{ksl}$. ■

The figure below is an illustration of $J_3 \subset K[x, y, z]$ with $s = 2$, $l = 7$ and $I = (x^s, y^s, z^l)$. In this case $I_3 = (x^{3s}, y^{3s}, z^{3l}) = (x^6, y^6, z^{21})$ and $F_3 = \{x^i y^j z^{\lambda_{6-(i+j)}} \mid i+j = 0, 1, 2, 3, 4, 5, 6$ *and* $\lambda_a = \left\lceil \frac{7a}{2} \right\rceil \}$. The elements of F_3 are represented by black circles. From the figure it is clear that the set F_3 minimally generates $\overline{I_3}$.

Later we will prove that J_k is the integral closure of I_k.

Lemma 7 $J^k = J_k$.

Proof. We show $J_kJ = J_{k+1}$. Let $x_{i_1}\cdots x_{i_{s-a}}y_{j_1}\cdots y_{j_{\lambda_a}} \in F$ and $x_{i_1}\cdots x_{i_{ks-b}}y_{j_1}\cdots y_{j_{\lambda_b}} \in F_k$. Multiplying these two monomials we get $x_{h_1}\cdots x_{h_{(k+1)s-(b+a)}}y_{t_1}\cdots y_{t_{\lambda_a+\lambda_b}}$ (with $1 \leq h_1 \leq h_2 \leq \ldots \leq m$ and $1 \leq t_1 \leq t_2 \leq \ldots \leq n$). This is a multiple of $x_{h_1}\cdots x_{h_{(k+1)s-(b+a)}}y_{t_1}\cdots y_{t_{\lambda_{a+b}}} \in J_{k+1}$ as $\lambda_{a+b} \leq \lambda_a + \lambda_b$. To show the other inclusion let $x_{i_1}\cdots x_{i_{(k+1)s-a}}y_{j_1}\cdots y_{j_{\lambda_a}} \in F_{k+1}$. If $a \geq ks$, write $a = ks + r$ with $0 \leq r \leq s$, then $\lambda_a = \lambda_{ks+r} = \left\lceil (ks+r)\frac{l}{s}\right\rceil = kl + \lambda_r$. Thus this monomial equals $x_{i_1}\cdots x_{i_{s-r}}y_{j_1}\cdots y_{j_{\lambda_r+kl}}$. But $y_{j_1}\cdots y_{j_{kl}} \in F_k$ and $x_{i_1}\cdots x_{i_{s-r}}y_{j_{kl+1}}\cdots y_{j_{kl+\lambda_r}} \in F$ as $0 \leq s-r \leq s$. If $a < ks$, then $x_{i_1}\cdots x_{i_{(k+1)s-a}}y_{j_1}\cdots y_{j_{\lambda_a}} = x_{t_1}\cdots x_{t_s}x_{h_1}\cdots x_{h_{ks-a}}y_{j_1}\cdots y_{j_{\lambda_a}} \in JJ_k$ as $x_{t_1}\cdots x_{t_s} \in J$ and $x_{h_1}\cdots x_{h_{ks-a}}y_{j_1}\cdots y_{j_{\lambda_a}} \in J_k$. ■

The main goal of this paper is to prove the following theorem

Theorem 8 *The integral closure of the ideal* $(x_1^{\alpha_1},\ldots,x_n^{\alpha_n}) \subset K[x_1,\ldots,x_n]$ *is normal, where* $\alpha_i \in \{s,l\}$ *with* s *and* l *arbitrary positive integers. Or equivalently, the ideal* J *is normal.*

By Lemma 6 and since $I_k \subseteq J_k$ we have

$$I_k \subseteq J_k \subseteq \overline{I_k} \subseteq \overline{J_k}$$

We will use Theorem 1 to show that J_k is integrally closed, hence J_k is the integral closure of I_k. Therefore we need the following.

Remark 9 *Let* $R = K[x_1,\ldots,x_m,y_1,\ldots,y_n]$. *For* $1 \leq i \leq m$, *it is easy to see that* $(J_k : (x_i))/J_k$ *is generated by* $\{z_{i_1}\cdots z_{i_{ks-a-1}}w_{j_1}\cdots w_{j_{\lambda_a}} \mid a = 0,\ldots,ks-1;\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_{ks-a-1} \leq m$ *and* $1 \leq j_1 \leq j_2 \leq \cdots \leq j_{\lambda_a} \leq n\}$ *where* z_i *and* w_i *are the images of* x_i *and* y_i, *respectively, in* R/J_k. *Also, for* $1 \leq j \leq n$ *note that* $(J_k : (y_j))/J_k$ *is generated by* $\{z_{i_1}\cdots z_{i_{ks-b}}w_{j_1}\cdots w_{j_{\lambda_b -1}} \mid b = 1,\ldots,ks;\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_{ks-b} \leq m$ *and* $1 \leq j_1 \leq j_2 \leq \cdots \leq j_{\lambda_b} \leq n\}$. *As the intersection of two monomial ideals is generated by the set of the least common multiples of the generators of the two ideals, it follows that* $(J_k : (x_1,\ldots,x_m,y_1,\ldots,y_n))/J_k$ *is generated by* $\{z_{i_1}\cdots z_{i_{ks-e}}w_{j_1}\cdots w_{j_{\lambda_e -1}} \mid e = 1,\ldots,ks;$ $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{ks-e} \leq m$ *and* $1 \leq j_1 \leq j_2 \leq \cdots \leq j_{\lambda_e} \leq n\}$.

Lemma 10 *The ideal* J_k *is integrally closed.*

Proof. By Theorem 1 we need to show that none of the preimages, in $K[x_1,\ldots,x_m,y_1,\ldots,y_n]$, of the monomial generators of $(J_k : (x_1,\ldots,x_m,y_1,\ldots,y_n))/J_k$ is in $\overline{J_k}$. Assume not, that is, assume $\sigma = x_{i_1}\cdots x_{i_{ks-e}}y_{j_1}\cdots y_{j_{\lambda_e -1}} \in \overline{J_k}$ for some $e \in \{1,\ldots,ks\}$. This implies $\sigma^d \in J_k^d$ for some positive integer d, thus $\sigma^d = x_{i_1}^d x_{i_2}^d\cdots x_{i_{ks-e}}^d y_{j_1}^d\cdots y_{j_{\lambda_e -1}}^d$ equals the following product of products of the generators of J_k

$$\beta \prod_{1\leq j_1\leq\cdots\leq j_{kl}\leq n} (y_{j_1}y_{j_2}\cdots y_{j_{kl}})^{c_{j_1,\ldots,j_{kl}}}$$

$$\prod_{\substack{1\le i_1\le m\\ 1\le j_1\le\cdots\le j_{\lambda_{ks-1}}\le n}} (x_{i_1}y_{j_1}y_{j_2}\cdots y_{j_{\lambda_{ks-1}}})^{l_{i_1,j_1,\ldots,j_{\lambda_{ks-1}}}}$$

$$\prod_{\substack{1\le i_1\le i_2\le m\\ 1\le j_1\le\cdots\le j_{\lambda_{ks-2}}\le n}} (x_{i_1}x_{i_2}y_{j_1}y_{j_2}\cdots y_{j_{\lambda_{ks-2}}})^{l_{i_1,i_2,j_1,\ldots,j_{\lambda_{ks-2}}}}$$

$$\vdots$$

$$\prod_{\substack{1\le 1i_1\le i_2\le\cdots\le i_{ks-2}\le m\\ 1\le j_1\le\cdots\le j_{\lambda_2}\le n}} (x_{i_1}\cdots x_{i_{ks-2}}y_{j_1}\cdots y_{j_{\lambda_2}})^{l_{i_1,i_2,\ldots,i_{ks-2},j_1,\ldots,j_{\lambda_2}}}$$

$$\prod_{\substack{1\le i_1\le i_2\le\cdots\le i_{ks-1}\le m\\ 1\le j_1\le\cdots\le j_{\lambda_1}\le n}} (x_{i_1}\cdots x_{i_{ks-1}}y_{j_1}\cdots y_{j_{\lambda_1}})^{l_{i_1,i_2,\ldots,i_{ks-1},j_1,\ldots,j_{\lambda_1}}}$$

$$\prod_{1\le i_1\le i_2\le\cdots\le i_{ks}\le m} (x_{i_1}x_{i_2}\cdots x_{i_{ks}})^{l_{i_1,i_2,\ldots,i_{ks}}}$$

where β is some monomial, $c_{j_1,\ldots,j_{kl}}$ and $l_{i_1,\ldots,i_t,j_1,\ldots,j_{\lambda_{ks-t}}}$ (with $1 \le t \le ks$) are non-negative integers. For $1 \le t \le ks$ let $L_t = \sum_{\substack{1\le i_1\le i_2\le\cdots\le i_t\le m\\ 1\le j_1\le\cdots\le j_{\lambda_{ks-t}}\le n}} l_{i_1,\ldots,i_t,j_1,\ldots,j_{\lambda_{ks-t}}}$ and let $C = \sum_{1\le j_1\le\cdots\le j_{kl}\le n} c_{j_1,\ldots,j_{kl}}$. By summing powers we have

$$L_{ks} + L_{ks-1} + \cdots + L_3 + L_2 + L_1 + C = d \tag{1}$$

Also, by the total-degree count of the monomial $x_{i_1}\cdots x_{i_{ks-e}}$ we have the following equality

$$(ks)L_{ks} + (ks-1)L_{ks-1} + \cdots + 3L_3 + 2L_2 + L_1 + \varepsilon = (ks-e)d \tag{2}$$

where ε is the total-degree of the monomial $x_{i_1}\cdots x_{i_{ks-e}}$ in β. By the total-degree count of the monomial $y_1\cdots y_{j_{\lambda_e}-1}$ we must have the following inequality

$$\lambda_1 L_{ks-1} + \lambda_2 L_{ks-2} + \cdots + \lambda_{ks-3}L_3 + \lambda_{ks-2}L_2 + \lambda_{ks-1}L_1 + Ckl \le (\lambda_e - 1)d \tag{3}$$

We finish the proof by showing that (1), (2), and (3) can not hold simultaneously.

From (1) and (2)

$$C = (ks-1)L_{ks} + (ks-2)L_{ks-1} + \cdots + 2L_3 + L_2 + \varepsilon - (ks-e-1)d \tag{4}$$

Recall, $(ks-1)l = ts + r$ with $1 \le r \le s$ and $\lambda_{ks-1} < \lambda_{ks} = kl$. Now consider the left-hand side of (3)

$$
\begin{aligned}
& \lambda_1 L_{ks-1} + \lambda_2 L_{ks-2} + \cdots + \lambda_{ks-3} L_3 + \lambda_{ks-2} L_2 + \lambda_{ks-1} L_1 + Ckl \\
= \quad & \left\lceil \sum_{i=0}^{ks-1} [kl(ks-1-i) + \lambda_i] L_{ks-i} \right\rceil + \varepsilon kl - kl(ks-e-1)d \quad \text{(By (4))} \\
\geq \quad & \left\lceil \sum_{i=0}^{ks-1} (ks-i)(\lambda_{ks-1} - \frac{s-r}{s}) L_{ks-i} \right\rceil + \varepsilon kl - kl(ks-e-1)d \quad \text{(by Lemma 4)} \\
\geq \quad & (\lambda_{ks-1} - \frac{s-r}{s})(ks-e)d - kl(ks-e-1)d \qquad\qquad \text{(By (2))} \\
= \quad & \frac{(ks-1)l}{s}(ks-e)d - kl(ks-e-1)d \\
= \quad & \left(\frac{e}{s} l\right) d \\
> \quad & (\lambda_e - 1)d.
\end{aligned}
$$

This is a contradiction to (3) as required. ■

Proof. (of Theorem 8) The proof follows by the above lemma and Lemma 7. ■

We have already proved that if $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with the entries of $\boldsymbol{\alpha}$ consisting of two positive integers, then $I(\boldsymbol{\alpha})$, the integral closure of $(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) \subset K[x_1, \ldots, x_n]$, is normal. Noting that the ideal $I(x^4, y^5, z^7) \subset K[x, y, z]$ is not normal, the following question arises: when is $I(\boldsymbol{\alpha})$ normal provided that $\boldsymbol{\alpha}$ consists of three distinct positive integers? In the proposition below we give a partial answer for this question.

Theorem 11 *(Theorem 5.1, [RRV]) Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $c = \operatorname{lcm}(\alpha_1, \ldots, \alpha_{n-1})$. Let $I(\boldsymbol{\alpha})$ be the integral closure of $(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) \subset K[x_1, \ldots, x_n]$ and $I(\boldsymbol{\alpha}')$ the integral closure of $(x_1^{\alpha_1}, \ldots, x_{n-1}^{\alpha_{n-1}}, x_n^{\alpha_n + c}) \subset K[x_1, \ldots, x_n]$. If $I(\boldsymbol{\alpha}')$ is normal, then $I(\boldsymbol{\alpha})$ is normal. Conversely, If $I(\boldsymbol{\alpha})$ is normal and $\alpha_n \geq c$, then $I(\boldsymbol{\alpha}')$ is normal.*

Proposition 12 *If $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with $\alpha_i \in \{s, l\}$ for $i = 1, \ldots, n-1$ such that s divides l and l divides α_n, then $I(\boldsymbol{\alpha})$ is normal.*

Proof. We proceed by induction on the integer $q = \alpha_n / l$. By Theorem **8** the ideal $I(\boldsymbol{\alpha})$ is normal whenever $q = 1$. Note $l = \operatorname{lcm}\{s, l\}$ as s divides l. Assume $I(\boldsymbol{\alpha})$ is normal for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n-1}, ql)$ with $\alpha_i \in \{s, l\}$ for $i = 1, \ldots, n-1$. Then by the above Theorem $I(\boldsymbol{\alpha}')$ is normal where $\boldsymbol{\alpha}' = (\alpha_1, \ldots, \alpha_{n-1}, ql + l)$. ■

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Ibrahim Al-Ayyoub, assistant professor.
Department of Mathematics and Statistics.
Jordan University of Science and Technology.
P O Box 3030, Irbid 22110, Jordan.
Email address: iayyoub@just.edu.jo