Duals of Ann-categories

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Abstract

Dual monoidal category $\mathcal{C}^*$ of a monoidal functor $F:\mathcal{C}\to\mathcal{V}$ has been constructed by S. Majid. In this paper, we extend the construction of dual structures for an Ann-functor $F:\mathcal{B}\to\mathcal{A}$. In particular, when $F=id_{\mathcal{A}}$, then the dual category $\mathcal{A}^*$ is indeed the center of $\mathcal{A}$ and this is a braided Ann-category.

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1 Introduction

Categories with quasi-symmetry appeared under the heading "braided monoidal categories" in a connection with low dimensional topology [5], as well as in the context of quantum groups [6].

The concept "*dual of monoidal category*" appeared in [9] in the following case. The Hopf algebra can be built via a monoidal category $\mathcal{C}$ and a functor $F:\mathcal{C}\to\mathbf{Vec}$. This event can be generalized as **Vec** is replaced by a monoidal category $\mathcal{V}$. Now, if F is a monoidal functor, then $\mathcal{C}$ is called *functored* on $\mathcal{V}$, or $(\mathcal{C},F)$ is called a $\mathcal{V}$*-category* in A. Grothendieck's terminology [4]. In this situation, S. Majid built the monoidal category $(\mathcal{C},F)^*=(\mathcal{C}^*,F^*)$, named "*full dual category*" of $(\mathcal{C},F)$. The objects of $(\mathcal{C},F)^*$ are pairs (V,u_V), consisting of $V\in\mathcal{C}$ and a natural transformation $u_V=(u_{V,X}:V\otimes FX\to FX\otimes V)$ satisfying the compatition with the monoidal functor $(F,\widetilde{F})$. The full subcategory $(\mathcal{C},F)^\circ$ consists of objects (V,u_V) where $u_{V,X}$ are isomorphisms. It is interesting when $\mathcal{V}=\mathcal{C}$ and $F=id$, then $(\mathcal{C},F)^\circ$ is a braided monoidal category, called the *center* $Z(\mathcal{C})$ of the monoidal category $\mathcal{C}$.

The notion of the center of a monoidal category appeared first in [5], [9]. It was a construction of a braided tensor category from an arbitrary tensor category. Then, the center of a category appears as a tool to study categorical groups [1] and graded categorical groups [3].

The detail proofs of the construction of $(\mathcal{C}, F)^*$ have showed in [10]. Concurrently, in [10], S. Majid enriched the results of the dual categories and established links between dual categories and braided groups.

Monoidal categories were considered in a more general situation due to M. Laplaza with the name *distributivity category* [7]. After, A. Fröhllich and C. T. C. Wall [2] presented the concept of *ring-like category*. These two concepts are categorifications of the concept of commutative rings, as well as a generalization of the category of modules over a commutative ring R. The overlap of these two concepts has been proved in [14].

In order to have descriptions of structures, and a cohomological classification, N. T. Quang [11] has introduced the concept of *Ann-categories*, as a categorification of the concept of rings, with requirements of invertibility of objects and morphisms of the under-lying category, similar to those of categorical groups (see [1], [3]). In [13], N. T. Quang proved that each congruence class of an Ann-category $\mathcal{A}$ is completely defined by three invariants: the ring $\Pi_0(\mathcal{A})$ of congruence classes of objects of $\mathcal{A}$, the $\Pi_0(\mathcal{A})$-bimodule $\Pi_1(\mathcal{A})$ of automorphisms of *zero* object, and an element in the cohomology group $H^3_{MacL}(R, M)$ due to Mac Lane [8]. The concept of *braided Ann-categories* is considered in [14], in which authors built the *center* of an Ann-category, an extension of the center construction of a monoidal category presented by A. Joyal and R. Street [5]. This motivation leads to the purpose of this paper is to construct a dual Ann-category of an arbitrary Ann-category (in Section 3). This gives us a new framework of the concept of Ann-categories, which is very close to the ring extension problem. We also note that the center of an Ann-category is a dual over $\mathcal{A}$. Thus, in the duals over $\mathcal{A}$ there always exist braided Ann-categories.

In this paper, we sometimes denote by XY the tensor product of two objects X, Y instead of $X \otimes Y$.

2 Some basic definitions

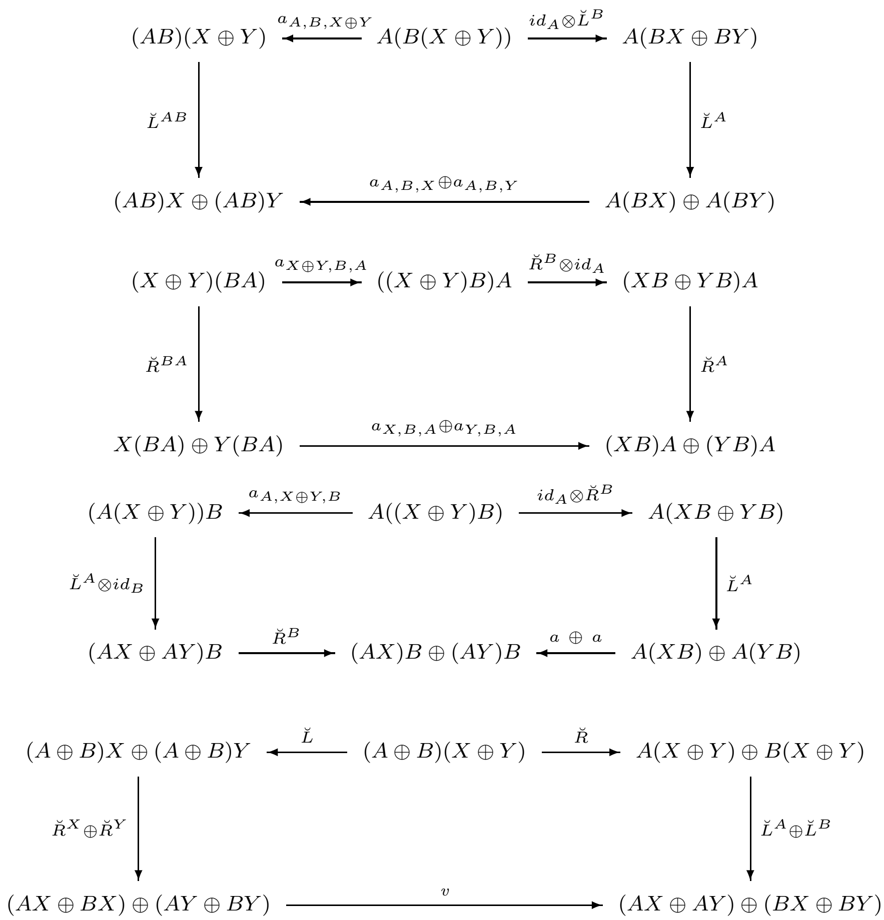
Definition 2.1 ([11]). *An* Ann-category *consists of:*
(i) Category $\mathcal{A}$ together with two bifunctors $\oplus, \otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$.
(ii) A fixed object $O \in \mathcal{A}$ together with naturality constraints a^+, c^+, g, d such that $(\mathcal{A}, \oplus, a^+, c^+, (O, g, d))$ is a symmetric categorical group.
(iii) A fixed object $I \in \mathcal{A}$ together with naturality constraints a, l, r such that $(\mathcal{A}, \otimes, a, (I, l, r))$ is a monoidal A-category.
(iv) Natural isomorphisms $\mathfrak{L}, \mathfrak{R}$

$$\begin{aligned} \mathfrak{L}_{A,X,Y} : \quad & A \otimes (X \oplus Y) \quad \to \quad (A \otimes X) \oplus (A \otimes Y), \\ \mathfrak{R}_{X,Y,A} : \quad & (X \oplus Y) \otimes A \quad \to \quad (X \otimes A) \oplus (Y \otimes A), \end{aligned}$$

such that the following conditions are satisfied:
(Ann-1) For each $A \in \mathcal{A}$*, the pairs* $(L^A, \breve{L}^A), (R^A, \breve{R}^A)$ *defined by relations:*

$$L^A = A \otimes -, \qquad R^A = - \otimes A,$$
$$\breve{L}^A_{X,Y} = \mathfrak{L}_{A,X,Y}, \qquad \breve{R}^A_{X,Y} = \mathfrak{R}_{X,Y,A}$$

are $\oplus$*-functors which are compatible with* a^+ *and* c^+*.*
(Ann-2) The following diagrams commute for all objects $A, B, X, Y \in \mathcal{A}$*:*



where $v = v_{U,V,Z,T} : (U \oplus V) \oplus (Z \oplus T) \to (U \oplus Z) \oplus (V \oplus T)$ *is the unique morphism built from* a^+, c^+, id *in the symmetric monoidal category* $(\mathcal{A}, \oplus)$*.*
(Ann-3) For the unit object $I \in \mathcal{A}$ *of the operation* $\otimes$*, we have the following relations for all objects* $X, Y \in \mathcal{A}$*:*

$$l_{X\oplus Y} = (l_X \oplus l_Y) \circ \breve{L}^I_{X,Y}, \quad r_{X\oplus Y} = (r_X \oplus r_Y) \circ \breve{R}^I_{X,Y}.$$

Definition 2.2. *Let* $\mathcal{A}$ *and* $\mathcal{A}'$ *be Ann-categories. An* Ann-functor *from* $\mathcal{A}$ *to* $\mathcal{A}'$ *is a triple* $(F, \breve{F}, \widetilde{F})$*, where* $(F, \breve{F})$ *is a symmetric monoidal functor respect to the operation* $\oplus$*,* $(F, \widetilde{F})$ *is an A-functor (i.e. an associativity functor)* *respect to the operation $\otimes$, satisfying the two following commutative diagrams for all $X, Y, Z \in Ob(\mathcal{A})$:*

$$\begin{array}{ccccc}
F(X(Y\oplus Z)) & \xleftarrow{\widetilde{F}} & FX.F(Y\oplus Z) & \xleftarrow{id\otimes\breve{F}} & FX(FY\oplus FZ)\\
\Big\downarrow{\scriptstyle F(\mathfrak{L})} & & & & \Big\downarrow{\scriptstyle \mathfrak{L}'}\\
F(XY\oplus XZ) & \xleftarrow{\breve{F}} & F(XY)\oplus F(XZ) & \xleftarrow{\widetilde{F}\oplus\widetilde{F}} & FX.FY\oplus FX.FZ
\end{array}$$

$$\begin{array}{ccccc}
F((X\oplus Y)Z) & \xleftarrow{\widetilde{F}} & F(X\oplus Y).FZ & \xleftarrow{\breve{F}\otimes id} & (FX\oplus FY).FZ\\
\Big\downarrow{\scriptstyle F(\mathfrak{R})} & & & & \Big\downarrow{\scriptstyle \mathfrak{R}'}\\
F(XZ\oplus YZ) & \xleftarrow{\breve{F}} & F(XZ)\oplus F(YZ) & \xleftarrow{\widetilde{F}\oplus\widetilde{F}} & FX.FZ\oplus FY.FZ
\end{array}$$

Definition 2.3. *A* braided Ann-category $\mathcal{A}$ *is an Ann-category* $\mathcal{A}$ *together with a braid c such that $(\mathcal{A}, \otimes, a, c, (I, l, r))$ is a braided tensor category, concurrently c satisfies the following relation:*

$$(c_{A,X} \oplus c_{A,Y}) \circ \breve{L}^A_{X,Y} = \breve{R}^A_{X,Y} \circ c_{A,X\oplus Y},$$

and the condition $c_{O,O} = id$.

Let us recall a result which has been known of an Ann-category.

Proposition 2.4 ([11, Proposition 3.1])**.** *In the Ann-category $\mathcal{A}$, there exist uniquely the isomorphisms:*

$$\hat{L}^A : A \otimes O \to A, \qquad \hat{R}^A : O \otimes A \to A$$

such that $(L^A, \breve{L}^A, \hat{L}^A), (R^A, \breve{R}^A, \hat{R}^A)$ are the functors which are compatible with the unit constraints of the operator $\oplus$ (also called U-functors).

3 Duals of Ann-categories

In this section, we shall build *duals of Ann-categories* based on the construction of duals of monoidal categories by S. Majid [9].

Let $\mathcal{A}$ be an Ann-category. An Ann-category $\mathcal{B}$ is *functored* over $\mathcal{A}$ if there is an Ann-functor $F : \mathcal{B} \to \mathcal{A}$.

First, let us recall that an Ann-category is called *almost strict* if all its natural constraints, except for the commutativity constraint and the left distributivity constraint, are identities. Each Ann-category is Ann-equivalent to an almost strict Ann-category of the type (R, M) (see [12]). In this category, for each $A \in Ob(\mathcal{A})$, there exists an object $A' \in Ob(\mathcal{A})$ such that

$$A \oplus A' = O. \tag{1}$$

So, hereafter, we always assume that $\mathcal{A}$ is an almost strict Ann-category and satisfies the condition (1) and the Ann-functor $F : \mathcal{B} \to \mathcal{A}$ satisfies the conditions $F(O) = O, F(I) = I$.

Definition 3.1. *Let $\mathcal{A}$ be an Ann-category. Let $(\mathcal{B}, F)$ be a functored Ann-category over $\mathcal{A}$. A right $(\mathcal{B}, F)$-module is a pair (A, u_A) consisting of an object A in $\mathcal{A}$ and a natural transformation $u_{A,X} : A \otimes F(X) \to F(X) \otimes A$ such that $u_{A,I} = id$ and the following diagrams commute:*

$$\begin{array}{ccccc}
A\otimes(FX\oplus FY) & \xrightarrow{\breve{L}^A_{FX,FY}} & (A\otimes FX)\oplus(A\otimes FY) & \xrightarrow{u_{A,X}\oplus u_{A,Y}} & (FX\otimes A)\oplus(FY\otimes A) \\
\downarrow{\scriptstyle id\otimes\breve{F}} & & & & \downarrow{\scriptstyle id} \\
A\otimes F(X\oplus Y) & \xrightarrow{u_{A,X\oplus Y}} & F(X\oplus Y)\otimes A & \xleftarrow{\breve{F}\otimes id} & (FX\oplus FY)\otimes A
\end{array} \tag{2}$$

$$\begin{array}{ccccc}
A\otimes(FX\otimes FY) & \xrightarrow{u_{A,X}\otimes id} & FX\otimes A\otimes FY & \xrightarrow{id\otimes u_{A,Y}} & FX\otimes FY\otimes A \\
\downarrow{\scriptstyle id\otimes\widetilde{F}} & & & & \downarrow{\scriptstyle \widetilde{F}\otimes id} \\
A\otimes F(X\otimes Y) & & \xrightarrow{\quad u_{A,X\otimes Y}\quad} & & F(X\otimes Y)\otimes A
\end{array} \tag{3}$$

A morphism $f : (A, u_A) \to (B, u_B)$ between right $(\mathcal{B}, F)$-modules is a morphism $f : A \to B$ in $\mathcal{A}$ such that the following diagram commutes for all $X \in \mathcal{B}$:

$$\begin{array}{ccc}
A\otimes FX & \xrightarrow{u_{A,X}} & FX\otimes A \\
\downarrow{\scriptstyle f\otimes id} & & \downarrow{\scriptstyle id\otimes f} \\
B\otimes FX & \xrightarrow{u_{B,X}} & FX\otimes B
\end{array} \tag{4}$$

Let $(\mathcal{B}, F)$ be a functored Ann-category over $\mathcal{A}$. We consider the category $\mathcal{B}^* = (\mathcal{B}, F)^*$ defined as follows. The objects of $\mathcal{B}^*$ are right $(\mathcal{B}, F)$-modules. The morphisms of $\mathcal{B}^*$ are morphisms between right $(\mathcal{B}, F)$-modules.

Now, we shall equip the operators and the structures for $\mathcal{B}^*$ so that $\mathcal{B}^*$ becomes an Ann-category.

Lemma 3.2. *For any two objects $(A, u_A), (B, u_B)$ in $\mathcal{B}^*$, $(A \oplus B, u_{A\oplus B})$ is an object of $\mathcal{B}^*$, where $u_{A\oplus B}$ is defined by:*

$$u_{A\oplus B,X} = \mathfrak{L}^{-1}_{FX,A,B} \circ (u_{A,X} \oplus u_{B,X}), \text{ for all } X \in \mathcal{A}.$$

Proof. Since $u_{A,I} = id,\ u_{B,I} = id,\ \mathfrak{L}_{FI,A,B} = \mathfrak{L}_{I,A,B} = id$, we have $u_{A\oplus B,I} = id$.

To prove that $u_{A\oplus B}$ satisfies the diagram (2), we consider the diagram (5) (see page 12). In the diagram (5), the regions (I), (VII) commute thanks to the determination of $u_{A\oplus B}$, the region (II) commutes thanks to the naturality of $\mathfrak{R} = id$, the regions (III), (VI) commute since $\mathcal{A}$ is an Ann-category, the region (V) commutes thanks to the naturality of $\mathfrak{L}$, the region (VIII) commutes thanks to the naturality of v, the perimeter commutes since $(A, u_A), (B, u_B)$ satisfy the diagram (2). Therefore, the region (IV) commutes, i.e., $(A \oplus B, u_{A\oplus B})$ satisfies the diagram (2).

To prove that $u_{A\oplus B}$ satisfies the diagram (3), we consider the diagram (6) (see page 13). In the diagram (6), the regions (I), (II) commute thanks to the naturality of $\mathfrak{R} = id$, the regions (III), (VI), (VIII) commute thanks to the determination of $u_{A\oplus B}$, the regions (IV), (X) commute since $\mathcal{A}$ is an Ann-category, the regions (VII), (IX) commute thanks to the naturality of $\mathfrak{L}$, the perimeter commutes thanks to u_A, u_B satisfy the diagram (3). Therefore, the region (V) commutes, i.e., $u_{A\oplus B}$ satisfies the diagram (3). So, $(A \oplus B, u_{A\oplus B})$ is an object of $\mathcal{B}^*$. □

By Lemma 3.2, we can determine the operator "+" of $\mathcal{B}^*$ where the sum of two objects is defined by

$$(A, u_A) + (B, u_B) = (A \oplus B, u_{A\oplus B}),$$

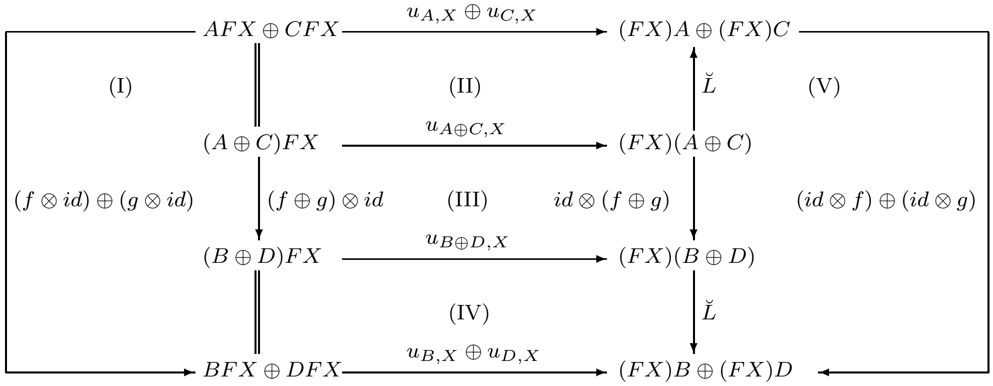
and the sum of two morphisms is the sum of morphisms in $\mathcal{A}$.

Proposition 3.3. *$\mathcal{B}^*$ is a symmetric categorical group where the associativity constraint is strict, the unit constraint is $((O, u_{O,X} = \hat{L}^{-1}_{FX}), id, id)$, and the commutativity constraint is $c^+_{(A,u_A),(B,u_B)} = c^+_{A,B}$.*

Proof. Assume that $f : (A, u_A) \to (B, u_B)$ and $g : (C, u_C) \to (D, u_D)$ are two morphisms in the category $\mathcal{B}^*$. We shall prove that

$$f + g = f \oplus g$$

satisfies the diagram (4), so it is a morphism of $\mathcal{B}^*$. We consider the diagram:

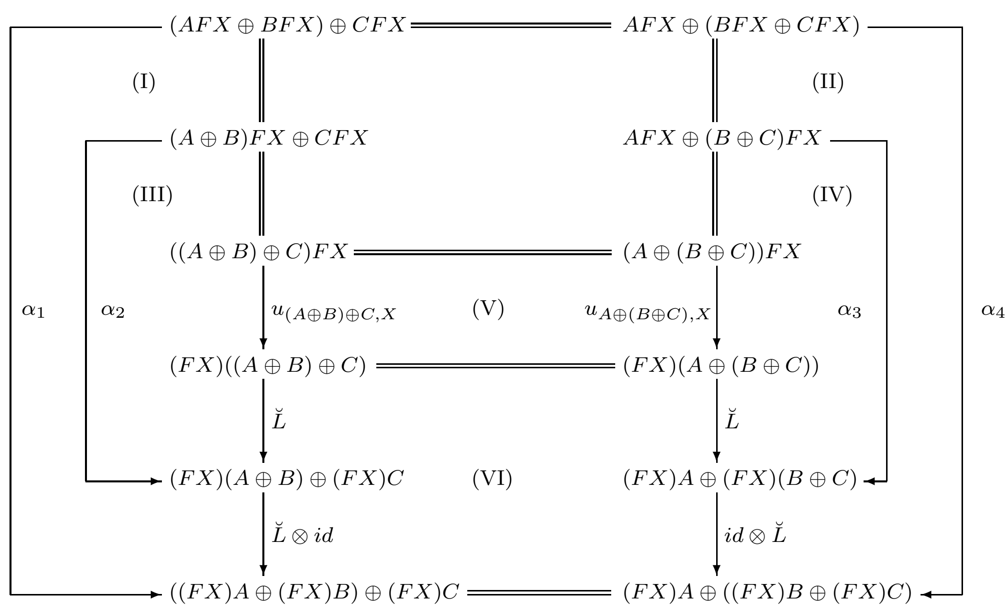


In this diagram, the region (I) commutes thanks to the naturality of $\mathfrak{R} = id$, the region (II) commutes thanks to the determination of $u_{A\oplus C}$, the region (IV) commutes thanks to the determination of $u_{B\oplus D}$, the region (V) commutes thanks to the naturality of $\mathfrak{L}$; each component of the perimeter commutes since f and g are morphisms of $\mathcal{B}^*$. So, the perimeter commutes. Therefore, the region (III) commutes, i.e., $f + g = f \oplus g$ is a morphism of $\mathcal{B}^*$.

Next, we prove that $a^+ = id$ is a morphism

$$((A, u_A) + (B, u_B)) + (C, u_C) \to (A, u_A) + ((B, u_B) + (C, u_C))$$

in $\mathcal{B}^*$. We consider the following diagram:



where $\alpha_1 = (u_{A,X} \oplus u_{B,X}) \oplus u_{C,X}$ $\qquad$ $\alpha_2 = u_{A\oplus B,X} \oplus u_{C,X}$

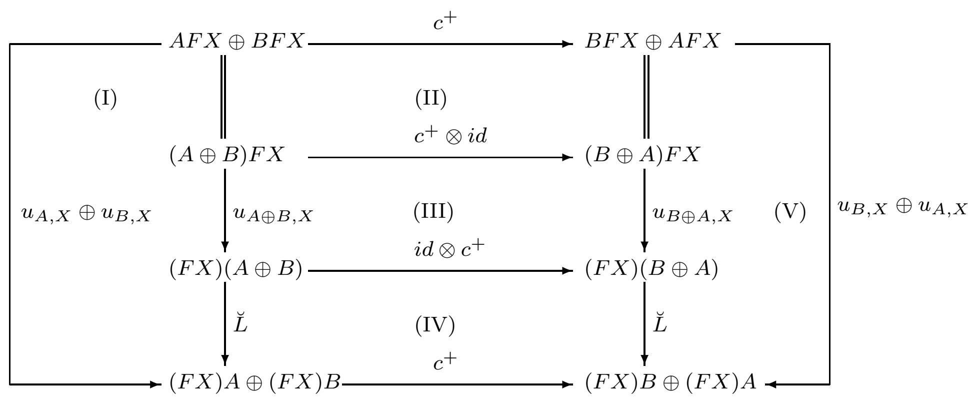
$\alpha_3 = u_{A,X} \oplus u_{B\oplus C,X}$ $\qquad$ $\alpha_4 = u_{A,X} \oplus (u_{B,X} \oplus u_{C,X})$

In the above diagram, the region (I) commutes thanks to the determination of $u_{A\oplus B}$, the region (II) commutes thanks to the determination of $u_{B\oplus C}$, the region (III) commutes thanks to the determination of $u_{(A\oplus B)\oplus C}$, the region (IV) commutes thanks to the determination of $u_{A\oplus(B\oplus C)}$, the region (VI) commutes since $\mathcal{A}$ is an Ann-category, the perimeter commutes thanks to the naturality of $a^+ = id$. Therefore, the region (V) commutes, i.e., $a^+ = id$ is a morphism of $\mathcal{B}^*$.

To prove that c^+ is the morphism

$$(A, u_A) + (B, u_B) \to (B, u_B) + (A, u_A)$$

in $\mathcal{B}^*$, we consider the following diagram. In this diagram, the region (I) commutes thanks to the determination of $u_{A\oplus B}$, the regions (II), (IV) commute since $\mathcal{A}$ is an Ann-category, the region (V) commutes thanks to the determination of $u_{B\oplus A}$, the perimeter commutes thanks to the naturality of c^+. Therefore, the region (III) commutes, i.e., c^+ is a morphism in $\mathcal{B}^*$.



One can verify that $((O, u_{O,X} = \hat{L}^{-1}_{FX}), id, id)$ is the unit constraint of $\mathcal{B}^*$. Finally, we shall prove that each object of $\mathcal{B}^*$ is invertible.

Let (A, u_A) be an object of $\mathcal{B}^*$. By the condition (1), there exsits an object $A' \in Ob(\mathcal{A})$ such that

$$A \oplus A' = O.$$

The family of natural transformations $u_{A',X} : A' \otimes FX \to FX \otimes A'$ is defined by:

$$u_{A,X} \oplus u_{A',X} = \mathfrak{L}_{FX,A,A'} \circ u_{O,X}.$$

One can prove that $(A', u_{A'})$ is the invertible object of the object (A, u_A) in the category $\mathcal{B}^*$. $\square$

Lemma 3.4. *For any two objects* $(A, u_A), (B, u_B)$ *of* $\mathcal{B}^*$, $(A \otimes B, u_{A\otimes B})$ *is an object of* $\mathcal{B}^*$, *where* $u_{A\otimes B}$ *is defined by:*

$$u_{A\otimes B,X} = (u_{A,X} \otimes id_B) \circ (id_A \otimes u_{B,X}), \text{ for all } X \in \mathcal{A}.$$

Proof. Let $(A, u_A), (B, u_B)$ be two objects of $\mathcal{B}^*$. Since $u_{A,I} = id$ and $u_{B,I} = id$, we have $u_{A\otimes B,I} = id$. Moreover, by Theorem 3.3 [9], $u_{A\otimes B}$ satisfies the diagram (3).

Finally, to prove that $u_{A\otimes B}$ satisfies the diagram (2), we consider the diagram (7) (see page 14). In the diagram (7), the region (I) commutes since (B, u_B) satisfies the diagram (2), the regions (II), (VII) and (IX) commute thanks to the naturality of $a^+ = id$, the region (III) commutes thanks to the naturality of $\mathfrak{L}$, the regions (IV), (XI) and the perimeter commutes since $\mathcal{A}$ is an Ann-category, the regions (VI), (VIII) commute thanks to the determination of u_{AB}, the region (X) commutes since (A, u_A) satisfies the diagram

(2), the region (XII) commutes thanks to the naturality of $\mathfrak{R} = id$. Therefore, the region (V) commutes, i.e., (AB, u_{AB}) satisfies the diagram (2). So $(A \otimes B, u_{A\otimes B})$ is an object of $\mathcal{B}^*$. $\square$

By Lemma 3.4, we can determine the operator "$\times$" of $\mathcal{B}^*$ where the product of two objects is defined by

$$(A, u_A) \times (B, u_B) = (A \otimes B, u_{A\otimes B}),$$

and the tensor product of two morphisms is the tensor product of two morphisms in $\mathcal{A}$.

Proposition 3.5. *$\mathcal{B}^*$ is a strict monoidal category.*

Proof. Assume that $f : (A, u_A) \to (B, u_B)$ and $g : (C, u_C) \to (D, u_D)$ are two morphisms in the category $\mathcal{B}^*$. By Theorem 3.3 [9], the morphism

$$f \times g = f \otimes g : (A, u_A) \times (C, u_C) \to (B, u_B) \times (D, u_D)$$

satisfies the diagram (4), i.e., $f \times g$ is a morphism in $\mathcal{B}^*$.

The composition of two morphisms in $\mathcal{B}^*$ is the normal composition. By Theorem 3.3 [9], $\mathcal{B}^*$ has the associativity constraint be strict. One can easily prove that (I, id) is an object in $\mathcal{B}^*$ and it together with the strict constraints $l = id, r = id$ is the unit constraint of the operator $\times$ in $\mathcal{B}^*$. $\square$

Theorem 3.6. *$\mathcal{B}^*$ is an Ann-category with the distributivity constraints are given by*

$$\mathfrak{L}_{(A,u_A),(B,u_B),(C,u_C)} = \mathfrak{L}_{A,B,C},\ \ \mathfrak{R}_{(A,u_A),(B,u_B),(C,u_C)} = id.$$

Proof. By Proposition 3.3, $(\mathcal{B}^*, +)$ is a symmetric categorical group. By Proposition 3.5, $(\mathcal{B}^*, \times)$ is a monoidal category. One can prove that

$$\mathfrak{L} : (A, u_A) \times ((B, u_B) + (C, u_C)) \to (A, u_A) \times (B, u_B) + (A, u_A) \times (C, u_C),$$
$$\mathfrak{R} = id : ((A, u_A + (B, u_B)) \times (C, u_C) \to (A, u_A) \times (C, u_C) + (B, u_B) \times (C, u_C)$$

are morphisms in $\mathcal{B}^*$.

Moreover, the constraints $a^+ = id, c^+, a = id, \mathfrak{L}, \mathfrak{R} = id$ of the Ann-category $\mathcal{A}$ satisfy the conditions (Ann-1), (Ann-2), (Ann-3), so, in the category $\mathcal{B}^*$, they also satisfy these conditions. Thus $\mathcal{B}^*$ is an Ann-category. $\square$

The following proposition is obvious.

Proposition 3.7. *$\mathcal{B}^*$ is functored over $\mathcal{A}$ with the forgetful Ann-functor*

$$F^* : \mathcal{B}^* \to \mathcal{A}.$$

Example 1. The center of an Ann-category $\mathcal{A}$

Let $\mathcal{A}$ be an Ann-category. Let $\mathcal{B} = \mathcal{A}$ and $F = id$. Then $\mathcal{B}^* = \mathcal{C}_{\mathcal{A}}$, where $\mathcal{C}_{\mathcal{A}}$ is the center of the Ann-category $\mathcal{A}$ which is built in [14]. This is a braided Ann-category with the quasi-symmetric

$$c_{(A,u_A),(B,u_B)} = u_{A,B} : A \otimes B \to B \otimes A.$$

Next, we shall apply above results to build the dual Ann-category of the pair $(\mathcal{B}, F)$, where $\mathcal{B} = (R', M', f')$, $\mathcal{A} = (R, M, f)$ are Ann-categories.

Example 2. Duals of an Ann-category of the type (R, M)

Let R be a ring and M be a R-bimodule. An Ann-category of the type (R, M) is a category $\mathcal{I}$ whose objects are elements of R, and whose morphisms are automorphisms, $(x, a) : x \to x,\ \forall a \in M$. The composition of morphisms is the addition in M. The two operators $\oplus$ and $\otimes$ of $\mathcal{I}$ are given by

$$\begin{aligned} x \oplus y = x + y, &\quad (x, a) \oplus (y, b) = (x + y, a + b), \\ x \otimes y = x.y, &\quad (x, a) \otimes (y, b) = (xy, xb + ay). \end{aligned}$$

All constraints of $\mathcal{I}$ are strict, except for the left distributivity constraint and the commutativity constraint given by

$$\begin{aligned} \mathfrak{L}_{x,y,z} &= (\bullet, \lambda(x, y, z)) : x(y + z) \to xy + xz, \\ c^+_{x,y} &= (\bullet, \eta(x, y)) : x + y \to y + x, \end{aligned}$$

where $\lambda : R^3 \to M, \eta : R^2 \to M$ are functions satisfying the some certain coherence conditions (for detail, see [12], [13]).

Let $\mathcal{A}$ be an almost strict Ann-category of the type (R, M) and $\mathcal{B}$ be an almost strict Ann-category of the type (R', M'). Let $(F, \breve{F}, \widetilde{F}) : \mathcal{B} \to \mathcal{A}$ be an Ann-functor. Then, by Theorem 4.3 [15], F is a functor of the type (p, q), i.e.,

$$F(x) = p(x), F(x, a) = (p(x), q(a)),$$

where $p : R' \to R$ is a ring homomorphism and $q : M' \to M$ is a group homomorphism and

$$q(xa) = p(x)q(a), \quad q(ax) = q(a)p(x), \text{ for all } x \in R, a \in M.$$

Moreover, $\breve{F}, \widetilde{F}$ are associated, respectively, to μ, ν which satisfy some certain coherence conditions (for detail, see Theorem 4.4 [15]).

According to the above steps, each object of $\mathcal{B}^*$ is a pair (r, u_r), where r is in the centerization of $Imp = p(R')$ in the ring R, (i.e., $rp(x) = p(x)r \ \forall x \in R'$) and $u_r : R' \to M$ is a function satisfying the condition $u_{r,1} = 0$ and the two following conditions for all $x, y \in R'$:

$$\begin{aligned} u(r,x) - u(r,x+y) + u(r,y) &= \mu(x,y)r + r\mu(x,y) - \lambda(r,px,py), \\ xu(r,y) - u(r,xy) + u(r,x)y &= r\nu(x,y) - \nu(x,y)r. \end{aligned}$$

We now describe a morphism $f : (r, u_r) \to (s, u_s)$ of $\mathcal{B}^*$. Since $f : r \to s$ is a morphism in the Ann-category $\mathcal{A}$, $s = r$, and $f = (r, a)$ with $a \in M$.

From the commutation of the diagram (4), we have

$$p(x)a = ap(x), \text{ for all } x \in R'.$$

Now, $\mathcal{B}^*$ is an Ann-category with the two operators given by

$$\begin{aligned} (r, u_r) + (s, u_s) &= (r + s, u_{r+s}), \\ (r, u_r) \times (s, u_s) &= (rs, u_{rs}), \end{aligned}$$

where

$$\begin{aligned} u_{r+s,x} &= u_{r,x} + u_{s,x} - \lambda(px, r, s), \\ u_{rs,x} &= u_{r,x}s + r.u_{s,x}, \end{aligned}$$

and $f + g = f \oplus g$, $f \times g = f \otimes g$ where $f : (r, u_r) \to (r, u_r)$, $g : (s, u_s) \to (s, u_s)$.

All constraints of $\mathcal{B}^*$ are strict, except for the commutativity constraint and the left distributivity constraint given by

$$\begin{aligned} c^+_{(r,u_r),(s,u_s)} &= c^+_{r,s} = (\bullet, \eta(r,s)), \\ \mathfrak{L}_{(r,u_r),(s,u_s),(t,u_t)} &= \mathfrak{L}_{r,s,t} = (\bullet, \lambda(r,s,t)). \end{aligned}$$

The invertible object of the object (r, u_r) respect to the operator $+$ is $(-r, u_{-r})$, where $-r$ is the opposite element of r in the group $(R, +)$ and $u_{-r} : R' \to M$ is given by:

$$u_{-r,x} = \lambda(px, r, -r) - u_{r,x}.$$

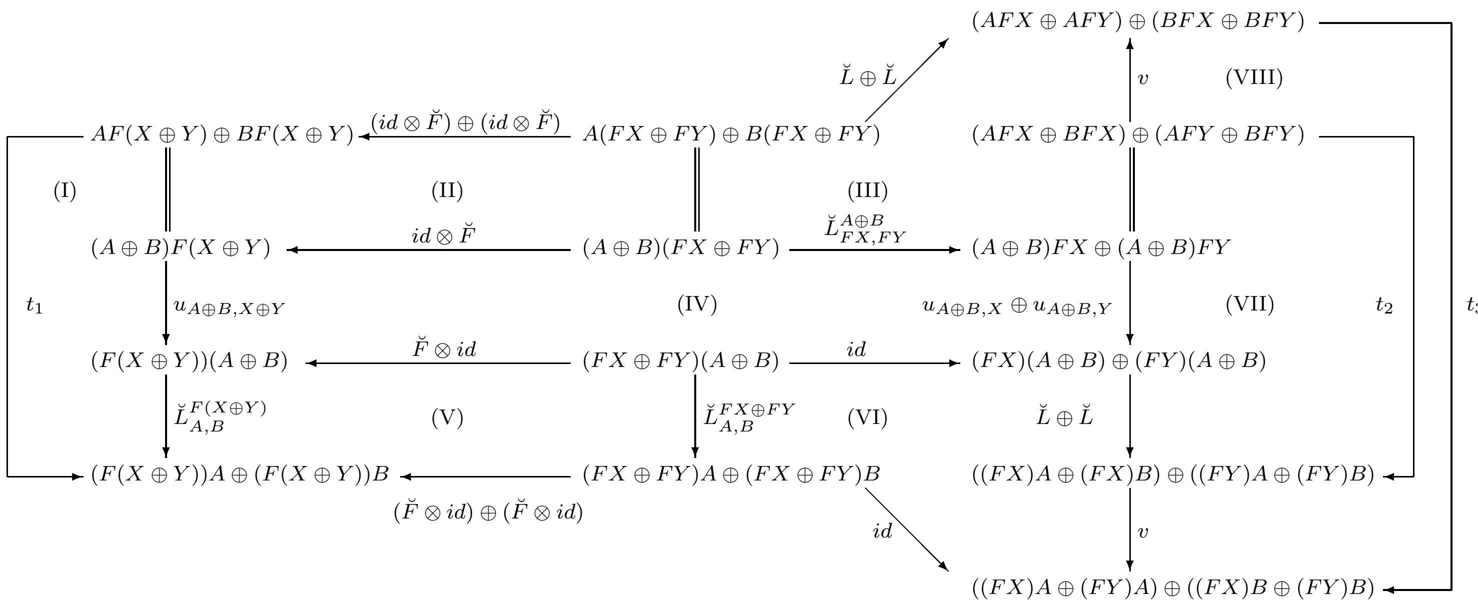


Diagram (5)

where $t_1 = u_{A,X \oplus Y} \oplus u_{B,X \oplus Y}$

$t_2 = (u_{A,X} \oplus u_{B,X}) \oplus (u_{A,Y} \oplus u_{B,Y})$

$t_3 = (u_{A,X} \oplus u_{A,Y}) \oplus (u_{B,X} \oplus u_{B,Y})$

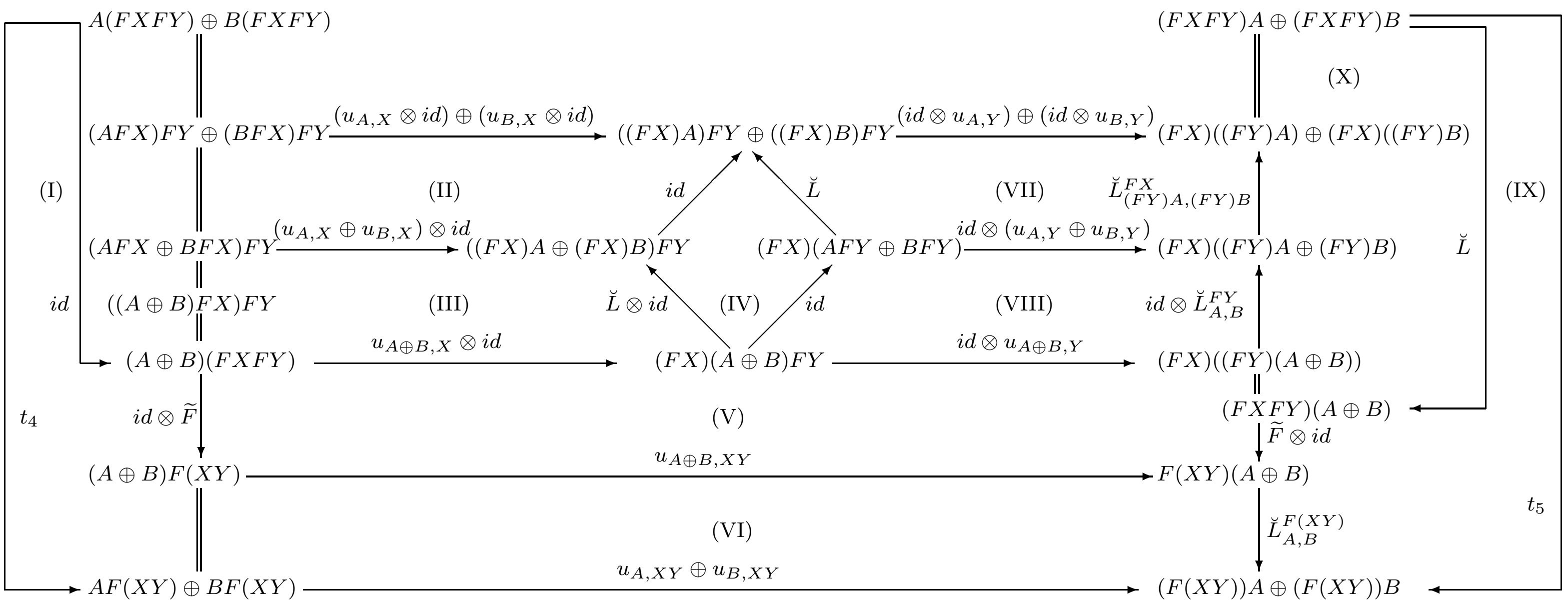


Diagram (6)

where $t_4 = (id_A \otimes \widetilde{F}_{X,Y}) \oplus (id_A \otimes \widetilde{F}_{X,Y})$

$t_5 = (\widetilde{F}_{X,Y} \otimes id_A) \oplus (\widetilde{F}_{X,Y} \otimes id_B)$

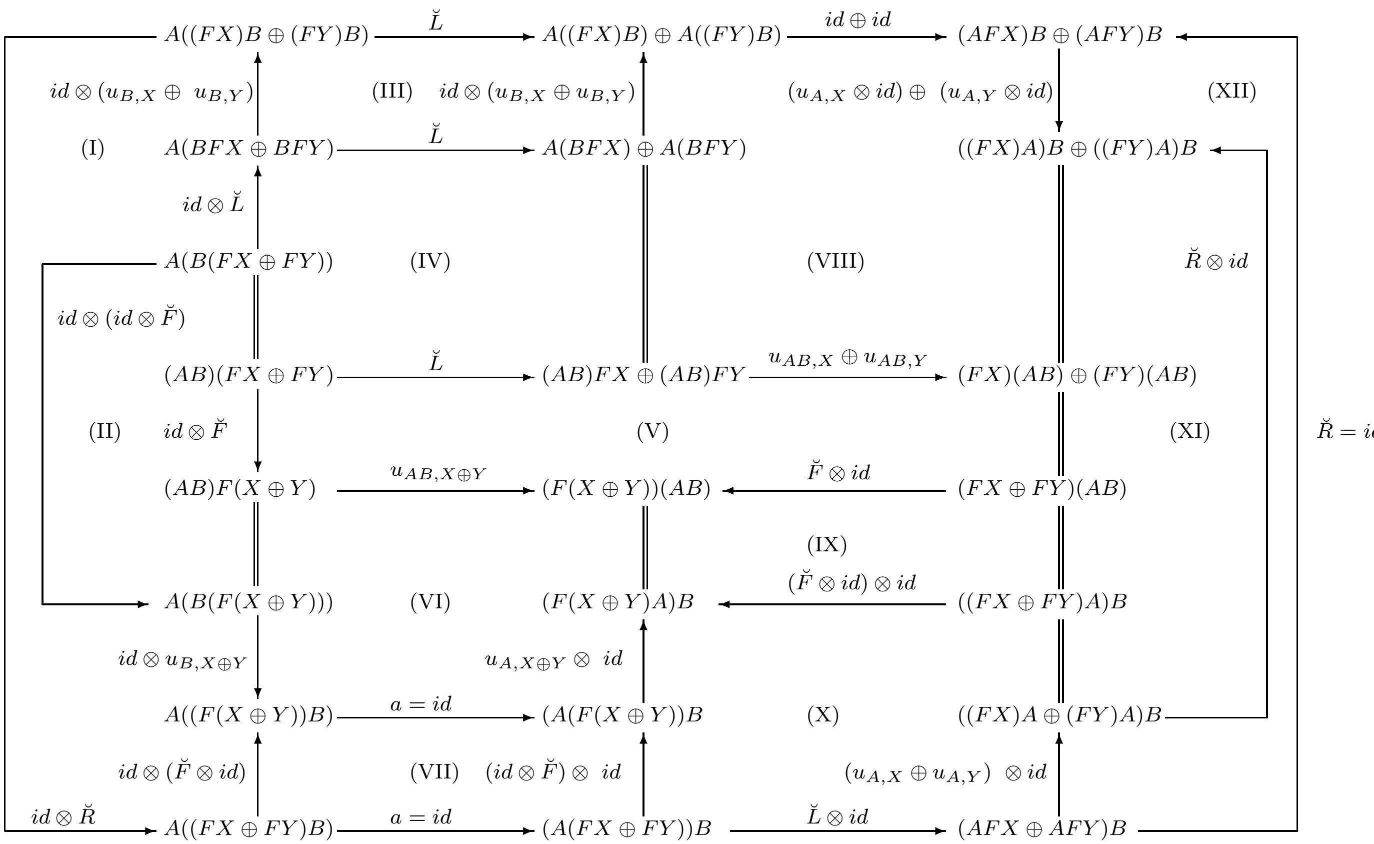

A((FX)B ⊕ (FY)B)
Ľ
A((FX)B) ⊕ A((FY)B)
id ⊕ id
(AFX)B ⊕ (AFY)B
id ⊗ (u_{B,X} ⊕ u_{B,Y})
(III)
id ⊗ (u_{B,X} ⊕ u_{B,Y})
(u_{A,X} ⊗ id) ⊕ (u_{A,Y} ⊗ id)
(XII)
(I)
A(BFX ⊕ BFY)
Ľ
A(BFX) ⊕ A(BFY)
((FX)A)B ⊕ ((FY)A)B
id ⊗ Ľ
A(B(FX ⊕ FY))
(IV)
(VIII)
Ř ⊗ id
id ⊗ (id ⊗ F̌)
(AB)(FX ⊕ FY)
Ľ
(AB)FX ⊕ (AB)FY
u_{AB,X} ⊕ u_{AB,Y}
(FX)(AB) ⊕ (FY)(AB)
(II)
id ⊗ F̌
(V)
(XI)
Ř = id
(AB)F(X ⊕ Y)
u_{AB,X⊕Y}
(F(X ⊕ Y))(AB)
F̌ ⊗ id
(FX ⊕ FY)(AB)
(IX)
A(B(F(X ⊕ Y)))
(VI)
(F(X ⊕ Y)A)B
(F̌ ⊗ id) ⊗ id
((FX ⊕ FY)A)B
id ⊗ u_{B,X⊕Y}
u_{A,X⊕Y} ⊗ id
A((F(X ⊕ Y))B)
a = id
(A(F(X ⊕ Y))B
(X)
((FX)A ⊕ (FY)A)B
id ⊗ (F̌ ⊗ id)
(VII)
(id ⊗ F̌) ⊗ id
(u_{A,X} ⊕ u_{A,Y}) ⊗ id
id ⊗ Ř
A((FX ⊕ FY)B)
a = id
(A(FX ⊕ FY))B
Ľ ⊗ id
(AFX ⊕ AFY)B

Diagram (7)

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