FUNDAMENTAL DIVISORS ON FANO VARIETIES OF INDEX $n-3$

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ABSTRACT. Let X be a Fano manifold of dimension n and index $n-3$. Kawamata proved the non vanishing of the global sections of the fundamental divisor in the case $n=4$. Moreover he proved that if Y is a general element of the fundamental system then Y has at most canonical singularities. We prove a generalization of this result in arbitrary dimension.

1. INTRODUCTION

A Fano variety is an n-dimensional $\mathbb{Q}$-Gorenstein projective variety X with ample anticanonical divisor $-K_X$. The *index* of a Fano variety X is

$$i(X) = \sup\{t \in \mathbb{Q} \mid -K_X \sim_{\mathbb{Q}} tH,\ H \text{ ample, Cartier}\}.$$

If X has at most log terminal singularities, the Picard group $Pic(X)$ is torsion free. Therefore the Cartier divisor H such that $-K_X \sim_{\mathbb{Q}} i(X)H$ is determined up to isomorphism. It is called the *fundamental divisor* of X.

It is well known that $i(X) \leq n+1$. By the Kobayashi-Ochiai criterion, if $i(X) \geq n$ then X is isomorphic to a hyperquadric or to a projective space. Smooth Fano varieties of index $n-1$ have been classified by Fujita and smooth Fano varieties of index $n-2$ by Mukai. An important tool in Mukai's classification is Mella's generalization [11] of Shokurov's theorem [12] on the general anticanonical divisor of smooth Fano threefolds: a general fundamental divisor $Y \in |H|$ is smooth.
In this work we consider the case of Fano varieties of index $n-3$. In [5] Kawamata proved, for a Fano variety X of dimension 4, the non vanishing of the global sections of the anticanonical divisor. Moreover he proved that if Y is a general element of the anticanonical system then Y has at most canonical singularities. In this work we propose the following generalization of Kawamata's result:

Theorem 1.1. *Let X be a Fano variety of dimension $n \geq 4$ with at most Gorenstein canonical singularities and index $n-3$, with H fundamental divisor.*

- *If the dimension of X is $n = 4, 5$, then $h^0(X,H) \geq n-2$.*
- *If the tangent bundle T_X is H-semistable, then $h^0(X,H) \geq n-2$.*
- *Suppose that $h^0(X,H) \neq 0$ and let $Y \in |H|$ be a general element. Then Y has at most canonical singularities.*

Note that if X is a Fano manifold of dimension at least six and index $n-3$ then we know by Wiśniewski's work [13], [14] that either X has a very special structure or the Picard number equals one. In the latter case it is conjectured that the tangent bundle is always semistable. Thus our statement covers all Fano manifolds of index $n-3$ and Picard number one at least modulo this very interesting conjecture.

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Note also that if $n \geq 5$ and $Y \in |H|$ is a general element, then by our theorem and the adjunction formula the variety Y is a $(n-1)$-dimensional Fano variety with at most Gorenstein canonical singularities and index $(n-1)-3$. Thus we can apply our theorem inductively to construct a "ladder" (cf. [1] for the terminology)

$$X \supsetneq Y_1 \supsetneq Y_2 \supsetneq \ldots \supsetneq Y_{n-3}$$

such that $Y_{i+1} \in |H|_{Y_i}|$ and Y_{n-3} is a Calabi-Yau threefold with at most canonical singularities. This technique reduces the study of Fano varieties of index $n-3$ to the fourfold case. In particular we obtain:

Corollary 1.2. *Suppose that we are in the situation of Theorem 1.1. Then the base locus of $|H|$ has dimension at most two.*

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2. Preliminaries

We will work over $\mathbb{C}$ and use the standard notation from [4]. In the following $\equiv$, $\sim$ and $\sim_{\mathbb{Q}}$ will respectively indicate numerical, linear and $\mathbb{Q}$-linear equivalence of divisors. The following definitions are taken from [9] and [10].

Definition 2.1. *Let (X, Δ) be a pair, $\Delta = \sum a_i \Delta_i$ with $a_i \in \mathbb{Q}^+$. Suppose that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f\colon Y \to X$ be a birational morphism, Y normal. We can write*

$$K_Y \equiv f^*(K_X + \Delta) + \sum a(E_i, X, \Delta) E_i.$$

where $E_i \subseteq Y$ are distinct prime divisors and $a(E_i, X, \Delta) \in \mathbb{R}$. Furthermore we adopt the convention that a nonexceptional divisor E appears in the sum if and only if $E = f_*^{-1} D_i$ for some i and then with coefficient $a(E, X, \Delta) = -a_i$.
The $a(E_i, X, \Delta)$ are called discrepancies.

Definition 2.2. *We set*

$$\mathrm{discrep(X, \Delta) = \inf\{a(E, X, \Delta) \mid E\, exceptional\, divisor\, over\, X\}}.$$

A pair (X, Δ) is defined to be

- *klt (kawamata log terminal) if* $\mathrm{discrep(X, \Delta)} > -1$ *and* $\lfloor \Delta \rfloor = 0$,
- *plt (purely log terminal) if* $\mathrm{discrep(X, \Delta)} > -1$,
- *lc (log canonical) if* $\mathrm{discrep(X, \Delta)} \geq -1$.

Definition 2.3. *Let (X, Δ) be a klt pair, D an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. The log canonical threshold of D for (X, Δ) is*

$$\mathrm{lct((X, \Delta), D) = \sup\{t \in \mathbb{R}^+ | (X, \Delta + tD)\, is\, lc\}}.$$

Definition 2.4. *Let (X, Δ) be a lc pair, $f\colon X' \to X$ a log resolution. Let $E \subseteq X'$ be a divisor on X' of discrepancy -1. Such a divisor is called a log canonical place. The image $f(E)$ is called center of log canonicity of the pair. If we write*

$$K_{X'} \equiv \mu^*(K_X + \Delta) + E,$$

we can equivalently define a place as an irreducible component of $\lfloor -E \rfloor$. We denote $CLC(X,\Delta)$ the set of all centers.

Definition 2.5. *Let (X,Δ) be a log canonical pair. A minimal center for (X,Δ) is an element of $CLC(X,\Delta)$ that is minimal with respect to inclusion.*

Definition 2.6. *Let (X,Δ) be a log canonical pair. A center W is said to be exceptional if there exists a log resolution $\mu\colon X' \to X$ for the pair (X,Δ) such that:*

- *there exists only one place $E_W \subseteq X'$ whose image in X is W;*
- *for every place $E' \neq E_W$, we have $\mu(E) \cap W = \emptyset$.*

Lemma 2.7. *Let X be a normal Gorenstein projective variety and Y an effective Cartier divisor. Suppose that (X,Y) is a non plt pair with and let c be its log canonical threshold. If all the minimal centers of (X,cY) have codimension one, then $c \leq 1/2$.*

Proof. If Y is not reduced, then $c \leq 1/2$.
If Y is reduced then we claim that (X,cY) has a minimal center of codimension at least two. Suppose first that $c < 1$. Since Y is reduced the discrepancy of every $E \subseteq Y$ of codimension one equals to c. Then E cannot be a center. Suppose now that $c = 1$. The pair (X,Y) is not plt and Y is reduced, so by definition there exists a center of codimension at least two. □

Lemma 2.8. [1, Lemma 5.1] *Let X be a normal variety and Δ a divisor on X such that (X,Δ) is klt. Let H be an ample Cartier divisor on X and $Y \in |H|$ a general element. Suppose that $(X,\Delta+Y)$ is not plt and let c be the log canonical threshold. Then the union of all the centers of log canonicity of $(X,\Delta+cY)$ is contained in the base locus of $|H|$.*

3. Non vanishing of the fundamental divisor

Let X be a Fano variety of dimension $n \geq 4$ and index $n-3$ with at most Gorenstein canonical singularities. Let H be the fundamental divisor. Then we have $-K_X \cong (n-3)H$. We want to study the non vanishing of $h^0(X,H)$. To do so, since by Kawamata-Viehweg vanishing theorem $\chi(X,H) = h^0(X,H)$, we look for an "explicit" expression for $\chi(X,H)$.
For $j \in \{-1 \ldots -(n-4)\}$ we have, by Kodaira vanishing and Serre duality,

$$\chi(X,jH) = (-1)^n h^n(X,jH) = (-1)^n h^0(X,-(n-3+j)H) = 0. \tag{1}$$

Hence we can write

$$\chi(X,tH) = \frac{H^n}{n!}\prod_{j=1}^{n-4}(t+j)(t^4+at^3+bt^2+ct+d). \tag{2}$$

By Serre duality, the polynomial $\chi(X,tH)$ has the following symmetry property for every integer t:

$$\chi(X,tH) = (-1)^n\chi(X,K_X-tH) = (-1)^n\chi(X,-(n-3+t)H). \tag{3}$$

By [2, Cor 1.4.4] there exists a birational map $\mu\colon X' \to X$ where X' has at most Gorenstein terminal singularities and $\mu^*K_X = K_{X'}$. Since canonical singularities are rational we have the equality

$$\chi(X,tH) = \chi(X',t\mu^*H).$$

Moreover, for a projective variety X' with at most Gorenstein terminal singularities and D a Cartier divisor on X', we have the following Riemann-Roch-formula

$$\chi(X',tD) = \frac{D^n}{n!}t^n + \frac{-K_{X'}D^{n-1}}{2(n-1)!}t^{n-1} + \frac{(K_{X'}^2+c_2(X'))D^{n-2}}{12(n-2)!}t^{n-2} + p(t) + \chi(X',\mathcal{O}_{X'}), \tag{4}$$

where $p(t)$ is a polynomial of degree $n-3$ and with no constant term.

By using the equalities (1) and (3) and applying (4) to $D = \mu^*H$ it is possible to compute the coefficients a, b, c, d in 2 and obtain

$$\begin{aligned} a &= 2(n-3) \\ c &= (n-3)(b-(n-3)^2) \\ b &= \tfrac{-n^4+8n^3+9n^2-160n+264}{24} + \tfrac{n(n-1)}{12}\tfrac{c_2(X)H^{n-2}}{H^n} \\ d &= \tfrac{n(n-1)(n-2)(n-3)\chi(X,\mathcal{O}_X)}{H^n}. \end{aligned}$$

Since $\chi(X,\mathcal{O}_X) = \chi(X',\mathcal{O}_{X'}) = 1$ for any Fano variety, we obtain the following lemma.

Lemma 3.1. *Let X be a Fano variety of dimension $n \geq 4$ and index $n-3$ with at most Gorenstein canonical singularities. Let H be a fundamental divisor. Let $\mu\colon X' \to X$ be a birational morphism with X' terminal and $K_{X'} = \mu^*K_X$. Then*

$$\chi(X,H) = \frac{H^n}{24}(-n^2+7n-8) + \frac{c_2(X')\mu^*H^{n-2}}{12} + n - 3.$$

Proposition 3.2. *Let X be a Fano variety with at most Gorenstein canonical singularities. Suppose that the dimension of X is $n = 4, 5$, and the index $n-3$. Then $h^0(X,H) \geq n-2$.*

Proof. We remark that if $n = 4, 5$ then $-n^2+7n-8 > 0$. By [7, Corollary 6.2] we have that $c_2(X')\mu^*H^{n-2} \geq 0$. Thus

$$\begin{aligned} h^0(X,H) &= \chi(X,H) = \tfrac{H^n}{24}(-n^2+7n-8) + \tfrac{c_2(X')\mu^*H^{n-2}}{12} + n - 3 \\ &\geq \tfrac{H^n}{24}(-n^2+7n-8) + n - 3 \geq n-2. \end{aligned}$$

□

Proposition 3.3. *Let X be a Fano variety of dimension $n \geq 4$ with at most Gorenstein canonical singularities and index $n-3$, with H fundamental divisor. Suppose that the tangent bundle T_X is H-semistable. Then $h^0(X,H) \geq n-2$.*

Proof. If T_X is H-semistable then $T_{X'}$ is μ^*H-semistable. Since X' is terminal, its tangent bundle $T_{X'}$ is a Q-sheaf in codimension 2 and satisfies the conditions of the Bogomolov inequality [7, Lemma 6.5]. In our situation the inequality becomes

$$c_2(X')\mu^*H^{n-2} \geq \frac{n-1}{2n}c_1(X')^2\mu^*H^{n-2} = \frac{(n-1)(n-3)^2}{2n}H^n.$$

We conclude by using it in the formula of Lemma 3.1. □

4. Structure of a general element in $|H|$.

Now we look for some result about the regularity of a general section $Y \in |H|$. By inversion of adjunction [9, Thm.7.5] the last part of Theorem 1.1 is equivalent to proving that the pair (X, Y) is plt. This is the object of the following

Proposition 4.1. *Let X be a Fano variety of dimension n and index $n-3$ with at most Gorenstein canonical singularities. Let H be a fundamental divisor and suppose that $h^0(X,H) \neq 0$. Let $Y \in |H|$ be a general element. Then (X,Y) is plt.*

Proof. We argue by contradiction and suppose that (X, Y) is not plt.

Let c be the log canonical threshold of (X, Y). By Lemma 2.8 the pair (X, cY) is plt in the complement of the base locus of $|H|$. Since (X, cY) is properly lc there exist a minimal center W.

By the perturbation technique [8, Thm 8.7.1] we can find some rational numbers $c_1, c_2 \ll 1$ and an effective $\mathbb{Q}$-divisor $A \sim_{\mathbb{Q}} c_1 H$ such that W is exceptional for the pair $(X, \Delta + (1-c_2)cY + A)$.
By Kawamata's subadjunction formula [6] for every $\varepsilon > 0$ there exists an effective $\mathbb{Q}$-divisor B_W on W such that (W, B_W) is a klt pair and

$$\begin{aligned} K_W + B_W &\sim_{\mathbb{Q}} (K_X + \Delta + (1-c_2)cY + A + \varepsilon H)|_W \\ &\sim_{\mathbb{Q}} (-(n-3) + (1-c_2)c + c_1 + \varepsilon)H|_W. \end{aligned}$$

Set $\eta = -c_2 c + c_1 + \varepsilon$, then η is arbitrary small and

$$K_W + B_W \sim_{\mathbb{Q}} -(n-3-c-\eta)H|_W. \tag{5}$$

Let Z be the union of all log canonical centers of the pair $(X, \Delta + (1-c_2)cY + A)$. Let $\mathcal{I}_Z$ be the ideal sheaf of Z. We consider the exact sequence

$$0 \to \mathcal{I}_Z(H) \to \mathcal{O}_X(H) \to \mathcal{O}_Z(H) \to 0.$$

By the Nadel vanishing theorem,

$$H^1(X, \mathcal{I}_Z(H)) = 0.$$

Thus we obtain the short exact sequence

$$0 \to H^0(X, \mathcal{I}_Z(H)) \to H^0(X, \mathcal{O}_X(H)) \to H^0(Z, \mathcal{O}_Z(H)) \to 0.$$

By Lemma 2.8 we know that Z is contained in the base locus of $|H|$, so

$$H^0(X, \mathcal{I}_Z(H)) \cong H^0(X, \mathcal{O}_X(H))$$

hence $h^0(Z, \mathcal{O}_Z(H)) = 0$.
Since W is a connected component of Z, we have

$$h^0(W, \mathcal{O}_W(H)) = 0$$

If W has dimension at most two, by [5, Prop 4.1] applied to $D = H|_W$ we obtain $h^0(W, \mathcal{O}_W(H)) \neq 0$, a contradiction.
If $\dim W$ is at least three, then (W, B_W) is log Fano of index $i(W) \geq n-3-c-\eta$. Suppose that

$$n - 3 - c - \eta > \dim W - 3,$$

this implies $h^0(W, \mathcal{O}_W(H)) \neq 0$ by [5, Theorem 5.1], a contradiction. Thus we are left with the case

$$\dim W \geq n - c - \eta.$$

Since $c \leq 1$ and η is arbitrary small, this implies $\dim W = n - 1$. This holds for all centers W, then $c < 1/2$ by Lemma 2.7, thus we have $\dim W \geq n - 1/2$, a contradiction. □

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