DETECTING ENTANGLEMENT OF STATES BY ENTRIES OF THEIR DENSITY MATRICES

XIAOFEI QI AND JINCHUAN HOU

Abstract. For any bipartite systems, a universal entanglement witness of rank-4 for pure states is obtained and a class of finite rank entanglement witnesses is constructed. In addition, a method of detecting entanglement of a state only by entries of its density matrix with respect to some product basis is obtained.

1. Introduction

Let H and K be separable complex Hilbert spaces. Recall that a quantum state is an operator $\rho \in \mathcal{B}(H \otimes K)$ which is positive and has trace 1. Denote by $\mathcal{S}(H)$ the set of all states on H. If H and K are finite dimensional, $\rho \in \mathcal{S}(H \otimes K)$ is said to be separable if ρ can be written as

$$\rho = \sum_{i=1}^{k} p_i \rho_i \otimes \sigma_i,$$

where ρ_i and σ_i are states on H and K respectively, and p_i are positive numbers with $\sum_{i=1}^{k} p_i = 1$. Otherwise, ρ is said to be inseparable or entangled (ref. [1, 16]). For the case that at least one of H and K is of infinite dimension, by Werner [21], a state ρ acting on $H \otimes K$ is called separable if it can be approximated in the trace norm by the states of the form

$$\sigma = \sum_{i=1}^{n} p_i \rho_i \otimes \sigma_i,$$

where ρ_i and σ_i are states on H and K respectively, and p_i are positive numbers with $\sum_{i=1}^{n} p_i = 1$. Otherwise, ρ is called an entangled state.

Entanglement is a basic physical resource to realize various quantum information and quantum communication tasks such as quantum cryptography, teleportation, dense coding and key distribution [16]. It is very important but also difficult to determine whether or not a state in a composite system is separable or entangled. It is obvious that every separable state has a positive partial transpose (the PPT criterion). For 2×2 and 2×3 systems, that is, for the

PACS. 03.67.Mn, 03.65.Ud, 03.65.Db.

Key words and phrases. Quantum states, separability, entanglement witnesses, positive linear maps.

This work is partially supported by National Natural Science Foundation of China (No. 10771157), Research Grant to Returned Scholars of Shanxi (2007-38) and the Foundation of Shanxi University.

case $\dim H = \dim K = 2$ or $\dim H = 2,\ \dim K = 3$, a state is separable if and only if it is a PPT state, that is, has positive partial transpose (see [7, 17]), but the PPT criterion has no efficiency for PPT entangled states appearing in the higher dimensional systems. In [3], the realignment criterion for separability in finite-dimensional systems was found, which says that if $\rho \in \mathcal{S}(H \otimes K)$ is separable, then the trace norm of its realignment matrix ρ^R is not greater than 1. The realignment criterion was generalized to infinite dimensional system by Guo and Hou in [6]. A most general approach to characterize quantum entanglement is based on the notion of entanglement witnesses (see [7]). A self-adjoint operator W acting on $H \otimes K$ is said to be an entanglement witness (briefly, EW), if W is not positive and $\mathrm{Tr}(W\rho) \geq 0$ holds for all separable states ρ. It was shown in [7] that, a state ρ is entangled if and only if it is detected by some entanglement witness W, that is, $\mathrm{Tr}(W\rho) < 0$. However, constructing entanglement witnesses is a hard task. There was a considerable effort in constructing and analyzing the structure of entanglement witnesses for finite and infinite dimensional systems [2, 4, 14, 15, 20] (see also [10] for a review). Recently, Hou and Qi in [14] showed that every entangled state can be recognized by an entanglement witness W of the form $W = cI + T$ with I the identity operator, c a nonnegative number and T a finite rank self-adjoint operator and provided a way how to construct them.

Another important criterion for separability of states is the positive map criterion [7, Theorem 2], which claims that a state $\rho \in \mathcal{S}(H \otimes K)$ with $\dim H \otimes K < \infty$ is separable if and only if $(\Phi \otimes I)\rho \geq 0$ holds for all positive linear maps $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$. Hou [13] generalized the positive map criterion to the infinite dimensional systems and obtained the following result.

Finite rank elementary operator criterion. ([13, Theorem 4.5]) *Let H, K be complex Hilbert spaces and ρ be a state acting on $H \otimes K$. Then the following statements are equivalent.*

(1) *ρ is separable;*

(2) *$(\Phi \otimes I)\rho \geq 0$ holds for every finite-rank positive elementary operator $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$.*

Recall that a linear map $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ is an elementary operator if there are operators $A_1, A_2, \cdots, A_r \in \mathcal{B}(H, K)$ and $B_1, B_2, \cdots, B_r \in \mathcal{B}(K, H)$ such that $\Phi(X) = \sum_{i=1}^r A_i X B_i$ for all $X \in \mathcal{B}(H)$. It is known that an elementary operator Φ is finite rank positive if and only if there exist finite rank operators $C_1, \ldots, C_k, D_1, \cdots, D_l \in \mathcal{B}(H, K)$ such that $(D_1, \cdots, D_l)$ is a contractive local combination of $(C_1, \cdots, C_k)$ and $\Phi(X) = \sum_{i=1}^k C_i X C_i^\dagger - \sum_{j=1}^l D_j X D_j^\dagger$ for all $X \in \mathcal{B}(H)$ (ref. [13] and the references therein).

Therefore, by the finite rank elementary operator criterion, a state ρ on $H \otimes K$ is entangled if and only if there exists a finite rank positive elementary operator $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ such that $(\Phi \otimes I)\rho$ is not positive. Here Φ must be not completely positive (briefly, NCP). Thus it is also important and interesting to find as many as possible finite rank positive elementary operators that are NCP, and then, to apply them to detect the entanglement of states. In [18], some new finite rank positive elementary operators were constructed and then applied to get some new entangled states that can not be detected by the PPT criterion and the realignment criterion.

Due to the Choi-Jamiołkowski isomorphism, any EW on finite dimensional system $H \otimes K$ corresponds to a linear positive map $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$. In fact, for system $H \otimes K$ of any dimension, if $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ is a normal positive completely bounded linear map, and if ρ_0 is an entangled state on $H \otimes K$, then $W = (\Phi \otimes I)\rho_0$ is an entanglement witness whenever W is not positive (see lemma 2.1). Recall that a linear map $\Delta : \mathcal{B}(H) \to \mathcal{B}(K)$ is said to be completely bounded if $\Delta \otimes I$ is bounded; is said to be normal if it is weakly continuous on bounded sets, or equivalently, if it is ultra-weakly continuous (i.e., if $\{A_\alpha\}$ is a bounded net and there is $A \in \mathcal{B}(H)$ such that $\langle x|A_\alpha|y\rangle$ converges to $\langle x|A|y\rangle$ for any $|x\rangle, |y\rangle \in H$, then $\langle \phi|\Delta(A_\alpha)|\psi\rangle$ converges to $\langle \phi|\Delta(A)|\psi\rangle$ for any $|\phi\rangle, |\psi\rangle \in K$. ref. [5, pp.59]).

The finite rank elementary operator criterion, together with lemma 2.1, gives a way of constructing finite rank entanglement witnesses from finite rank positive elementary operators for both finite and infinite dimensional bipartite systems. In the present paper, we construct a rank-4 entanglement witness W that has some what "universal" property for pure states in any bipartite systems $H \otimes K$. We show that, for such a rank-4 entanglement witness W, a pure state ρ is entangled if and only if there exist unitary operators U on H and V on K such that $\mathrm{Tr}((U \otimes V)W(U^\dagger \otimes V^\dagger)\rho) < 0$. In addition, if ρ is a mixed state such that $\mathrm{Tr}((U \otimes V)W(U^\dagger \otimes V^\dagger)\rho) < 0$, then ρ is 1-distillable (see theorem 2.2). We also construct a class of entanglement witnesses from the finite rank positive elementary operators obtained in [18] (see theorem 3.1).

So far, by our knowledge, there is no methods of recognizing the entanglement of a state by merely the entries of its density matrix. Another interesting result of this paper gives a way of detecting the entanglement of a state in a bipartite system by only a part of entries of its density matrix (see theorems 3.2, 3.3). This method is simple, computable and practicable because it provide a way to recognize the entanglement of a state by some suitably chosen entries of its matrix representation with respect to some given product basis. As an illustration, some new examples of entangled states that can be recognized by this way are proposed, which also provides some new entangled states that can not be detected by the PPT criterion and the realignment criterion (see examples 3.4, 3.5).

Recall that a bipartite state ρ is called n-distillable, if and only if maximally entangled bipartite pure states, e.g. $|\psi\rangle = \frac{1}{2}(|11'\rangle + |22'\rangle)$, can be created from n identical copies of the state ρ by means of local operations and classical communication; is called distillable if it is n-distillable for some n. It has been shown that all entangled pure states are distillable. However it is a challenge to give an operational criterion of distillability for general mixed states [8]. In [9], it was shown that a density matrix ρ is distillable if and only if there are some projectors P, Q that map high dimensional spaces to two-dimensional ones such that the state $(P\otimes Q)\rho^{\otimes n}(P\otimes Q)$ is entangled for some n copies.

2. Universal entanglement witnesses for pure states

In this section we will give a simple necessary and sufficient condition for separability of pure states in bipartite composite systems of any dimension.

Before stating the main result in this section, we give a basic lemma.

Lemma 2.1. *Let H, K be complex Hilbert spaces of any dimension and let $\Phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be a positive normal completely bounded linear map. Then, for any entangled state ρ_0 on $H\otimes K$, $W = (\Phi\otimes I)\rho_0$ is an entanglement witness whenever W on $H\otimes K$ is not positive.*

Proof. Because Φ is completely bounded, $W = (\Phi\otimes I)\rho_0$ is a bounded self-adjoint operator on $H\otimes K$. Note that $\mathcal{B}(H) = \mathcal{T}(H)^*$, where $\mathcal{T}(H)$ denotes the Banach space of all trace class operators on H endowed with the trace norm. Then the normality of Φ implies that there exists a bounded linear map $\Delta : \mathcal{T}(H) \to \mathcal{T}(H)$ such that $\Phi = \Delta^*$. We claim that Δ is also positive. In fact, for any unit vector $|\phi\rangle \in H$ and any positive operator $A \in \mathcal{B}(H)$, we have

$$\mathrm{Tr}(A\Delta(|\phi\rangle\langle\phi|)) = \mathrm{Tr}(\Phi(A)(|\phi\rangle\langle\phi|)) = \langle\phi|\Phi(A)|\phi\rangle \geq 0.$$

This implies that $\Delta(|\phi\rangle\langle\phi|)$ is positive for any $|\phi\rangle$. So, Δ is a positive linear map.

Now, for any separable state $\rho \in \mathcal{S}(H\otimes K)$, we have

$$\mathrm{Tr}(W\rho) = \mathrm{Tr}((\Phi\otimes I)\rho_0\cdot\rho) = \mathrm{Tr}(\rho_0\cdot(\Delta\otimes I)\rho) \geq 0$$

since $(\Delta\otimes I)\rho \geq 0$. So, if W is not positive, then it is an entanglement witness. □

Since every elementary operator is normal and completely bounded, by Lemma 2.1, if Φ is a positive elementary operator and if ρ_0 is an entangled state, then $W = (\Phi\otimes I)\rho_0$ is an entanglement witness whenever W is not positive. Also note that, if W is an entanglement witness, then for any positive number b, bW is an entanglement witness, too.

Let W be an entanglement witness on $H\otimes K$. We say that W is universal (for all states) if, for any entangled state ρ on $H\otimes K$, there exist unitary operators U on H and V on K such that $\mathrm{Tr}((U\otimes V)W(U^\dagger\otimes V^\dagger)\rho) < 0$; W is universal for pure states if, for any entangled pure state ρ on

$H\otimes K$, there exist unitary operators U on H and V on K such that $\mathrm{Tr}((U\otimes V)W(U^\dagger\otimes V^\dagger)\rho) < 0$.

The following is the main result of this section, which gives a universal entanglement witness of rank-4 for pure states. Particularly, we conclude that the separability of pure states can be determined by a special class of rank-4 witnesses, and every 1-distillable state can be detected by one of such rank-4 entanglement witnesses. However, we do not know whether or not there exists a universal entanglement witness for all states.

Let $\mathcal{U}(H)$ (resp. $\mathcal{U}(K)$) be the group of all unitary operators on H (resp. on K).

Theorem 2.2. *Let H and K be Hilbert spaces and let $\{|i\rangle\}_{i=1}^{\dim H\leq\infty}$ and $\{|j'\rangle\}_{j=1}^{\dim K\leq\infty}$ be any orthonormal bases of H and K, respectively. Let*

$$W=|1\rangle|2'\rangle\langle 1|\langle 2'|-|1\rangle|1'\rangle\langle 2|\langle 2'|-|2\rangle|2'\rangle\langle 1|\langle 1'|+|2\rangle|1'\rangle\langle 2|\langle 1'|. \tag{2.1}$$

Then W is an entanglement witness of rank-4. Moreover, the following statements are true.

(1) *If ρ is a pure state, then ρ is separable if and only if*

$$\mathrm{Tr}((U\otimes V)W(U^\dagger\otimes V^\dagger)\rho)\geq 0 \tag{2.2}$$

hold for all $U\in\mathcal{U}(H)$ and $V\in\mathcal{U}(K)$. So W is a universal entanglement witness for pure states.

(2) *Let ρ be a state. If there exist $U\in\mathcal{U}(H)$ and $V\in\mathcal{U}(K)$ such that $\mathrm{Tr}((U\otimes V)W(U^\dagger\otimes V^\dagger)\rho)<0$, then ρ is entangled and 1-distillable.*

Proof. We first prove that W is an entanglement witness. It is obvious that W is not positive. Define a map $\Phi:\mathcal{B}(H)\to\mathcal{B}(H)$ by

$$\begin{aligned}\Phi(A)= \quad & E_{11}AE_{11}^\dagger+E_{22}AE_{22}^\dagger+E_{12}AE_{12}^\dagger\\ & +E_{21}AE_{21}^\dagger-(E_{11}+E_{22})A(E_{11}+E_{22})^\dagger\end{aligned} \tag{2.3}$$

for every $A\in\mathcal{B}(H)$, where $E_{ij}=|i\rangle\langle j|\in\mathcal{B}(H)$. It is obvious that Φ is a positive map because the map

$$\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\mapsto\begin{pmatrix} a_{22} & -a_{12}\\ -a_{21} & a_{11}\end{pmatrix}$$

on $M_2(\mathbb{C})$ is positive. Note that $W=2(\Phi\otimes I)\rho_+$, where $\rho_+=|\psi_+\rangle\langle\psi_+|$ with $|\psi_+\rangle=\frac{1}{\sqrt{2}}(|11'\rangle+|22'\rangle)$. Thus, by Lemma 2.1, W is an entanglement witness.

If ρ is separable, then $\mathrm{Tr}((U\otimes V)W(U^\dagger\otimes V^\dagger)\rho)\geq 0$ as $(U^\dagger\otimes V^\dagger)\rho(U\otimes V)$ are separable. Conversely, assume that $\rho=|\psi\rangle\langle\psi|$ is inseparable. Consider its Schmidt decomposition $|\psi\rangle=\sum_{k=1}^{N_\psi}\delta_k|k,k'\rangle$, where $\delta_1\geq\delta_2\geq\cdots>0$ with $\sum_{k=1}^{N_\psi}\delta_k^2=1$, $\{|k\rangle\}_{k=1}^{N_\psi}$ and $\{|k'\rangle\}_{k=1}^{N_\psi}$ are orthonormal in H and K, respectively. As $|\psi\rangle$ is inseparable, we must have its Schmidt number $N_\psi\geq 2$. Thus $\rho=\sum_{k,l=1}^{N_\psi}\delta_k\delta_l|k,k'\rangle\langle l,l'|$. Up to unitary equivalence, we may assume that $\{|k\rangle\}_{k=1}^2 = \{|i\rangle\}_{i=1}^2$ and $\{|k'\rangle\}_{k'=1}^2 = \{|j'\rangle\}_{j=1}^2$. Then $\mathrm{Tr}(W\rho) = \mathrm{Tr}(-\delta_1\delta_2|11'\rangle\langle 11'| - \delta_1\delta_2|22'\rangle\langle 22'|) = -2\delta_1\delta_2 < 0$. Hence the statement (1) is true.

For the statement (2), assume that there exist $U \in \mathcal{U}(H)$ and $V \in \mathcal{U}(K)$ such that $\mathrm{Tr}((U\otimes V)W(U^\dagger \otimes V^\dagger)\rho) < 0$. Then ρ is entangled. Moreover, ρ has a matrix representation

$$\rho = \sum_{i,j,k,l} \alpha_{ijkl}|Ui\rangle|Vj'\rangle\langle Uk|\langle Vl'|.$$

Thus, one gets

$$\begin{aligned}
0 >\ & \mathrm{Tr}((U\otimes V)W(U^\dagger\otimes V^\dagger)\rho) = \mathrm{Tr}(W(U^\dagger\otimes V^\dagger)\rho(U\otimes V)) \\
=\ & \mathrm{Tr}(\textstyle\sum_{i,j,k,l}\alpha_{ijkl}(|1\rangle|2'\rangle\langle 1|\langle 2'| - |1\rangle|1'\rangle\langle 2|\langle 2'| - |2\rangle|2'\rangle\langle 1|\langle 1'| + |2\rangle|1'\rangle\langle 2|\langle 1'|) \\
& \cdot(U^\dagger\otimes V^\dagger)|Ui\rangle|Vj'\rangle\langle Uk|\langle Vl'|(U\otimes V)) \\
=\ & \mathrm{Tr}(\textstyle\sum_{i,j,k,l}\alpha_{ijkl}(|1\rangle|2'\rangle\langle 1|\langle 2'| - |1\rangle|1'\rangle\langle 2|\langle 2'| - |2\rangle|2'\rangle\langle 1|\langle 1'| + |2\rangle|1'\rangle\langle 2|\langle 1'|) \\
& \cdot|i\rangle|j'\rangle\langle k|\langle l'|) \\
=\ & -\alpha_{2211} - \alpha_{1122}.
\end{aligned}$$

Now let P and Q be the projectors from H and K onto the two dimensional subspaces spanned by $\{|1\rangle, |2\rangle\}$ and $\{|1'\rangle, |2'\rangle\}$, respectively. Then

$$\mathrm{Tr}(P\otimes Q)(U\otimes V)W(U^\dagger\otimes V^\dagger)(P\otimes Q)\rho(P\otimes Q)) = -\alpha_{2211} - \alpha_{1122} < 0,$$

which implies that $(P\otimes Q)\rho(P\otimes Q)$ is entangled. It follows from [9] that ρ is 1-distillable. The proof is complete. $\square$

3. Detecting entanglement of states by their entries

In this section, we give a method of detecting entanglement of a state in any bipartite system only by some entries of its matrix representation.

Let H and K be complex Hilbert spaces of any dimension with $\{|i\rangle\}_{i=1}^{\dim H}$ and $\{|j'\rangle\}_{j=1}^{\dim K}$ be orthonormal bases of them respectively. Denote $E_{ij} = E_{i,j} = |i\rangle\langle j|$, which is an operator from H into H. Let $n \le \min\{\dim H, \dim K\}$ be a positive integer. By [18, Remark 5.2], for any permutation κ of $(1, 2, \cdots, n)$, the linear map $\Phi_\kappa : \mathcal{B}(H) \to \mathcal{B}(H)$ defined by

$$\Phi_\kappa(A) = (n-1)\sum_{i=1}^n E_{ii}AE_{ii}^\dagger + \sum_{i=1}^n E_{i,\kappa(i)}AE_{i,\kappa(i)}^\dagger - (\sum_{i=1}^n E_{ii})A(\sum_{i=1}^n E_{ii})^\dagger \tag{3.1}$$

for every $A \in \mathcal{B}(H)$ is a positive elementary operator that is not completely positive if $\kappa \ne \mathrm{id}$. Then, for any unitary operators U and V on H, the map $\Phi_\kappa^{U,V}$ defined by

$$\begin{aligned}
\Phi_\kappa^{U,V}(A) = \ & (n-1)\textstyle\sum_{i=1}^n (VE_{ii}U)A(VE_{ii}U)^\dagger + \sum_{i=1}^n (VE_{i,\kappa(i)}U)A(VE_{i,\kappa(i)}U)^\dagger \\
& -(\textstyle\sum_{i=1}^n VE_{ii}U)A(\sum_{i=1}^n VE_{ii}U)^\dagger
\end{aligned} \tag{3.2}$$

for every $A \in \mathcal{B}(H)$ is positive, too. Let $\rho_+ = |\psi_+\rangle\langle\psi_+|$, where

$$|\psi_+\rangle = \frac{1}{\sqrt{n}}(|1\rangle|1'\rangle + |2\rangle|2'\rangle + \cdots + |n\rangle|n'\rangle).$$

Then, by Lemma 2.1, we get a class of entanglement witnesses of the form

$$W_\kappa^{U,V} = n(\Phi_\kappa^{U,V} \otimes I)\rho_+ = (\Phi_\kappa^{U,V}(E_{ij})). \tag{3.3}$$

Note that $W_\kappa^{U,V}$ is of finite rank because ρ_+ is.

Particularly, for permutations π, σ of $(1, 2, \cdots, n)$, if U and V are the unitary operators defined by $U^\dagger|i\rangle = |\pi(i)\rangle$, $V|i\rangle = |\sigma(i)\rangle$ for $i = 1, 2, \cdots n$ and $U^\dagger|i\rangle = |i\rangle$, $V|i\rangle = |i\rangle$ for $i > n$, then we have

$$\begin{array}{rl} \Phi_\kappa^{\pi,\sigma}(A) = & \Phi_\kappa^{U,V}(A) = (n-1)\sum_{i=1}^n E_{\sigma(i),\pi(i)} A E_{\sigma(i),\pi(i)}^\dagger \\ & +\sum_{i=1}^n E_{\sigma(i),\pi(\kappa(i))} A E_{\sigma(i),\pi(\kappa(i))}^\dagger - (\sum_{i=1}^n E_{\sigma(i),\pi(i)})A(\sum_{i=1}^n E_{\sigma(i),\pi(i)})^\dagger \end{array} \tag{3.4}$$

for every A. And correspondingly, we get entanglement witnesses of the concrete form

$$W_\kappa^{\pi,\sigma} = (\Phi_\kappa^{\pi,\sigma}(E_{ij})), \tag{3.5}$$

where

$$\Phi_\kappa^{\pi,\sigma}(E_{ij}) = -E_{\sigma(\pi^{-1}(i)),\sigma(\pi^{-1}(j))} \tag{3.6}$$

if $1 \le i \ne j \le n$,

$$\Phi_\kappa^{\pi,\sigma}(E_{ii}) = (n-2)E_{\sigma(\pi^{-1}(i)),\sigma(\pi^{-1}(i))} + E_{\sigma(\kappa^{-1}\pi^{-1}(i)),\sigma(\kappa^{-1}\pi^{-1}(i))} \tag{3.7}$$

if $1 \le i \le n$, and

$$\Phi_\kappa^{\pi,\sigma}(E_{ij}) = 0 \tag{3.8}$$

if $i > n$ or $j > n$.

Thus we have proved the following result.

Theorem 3.1. *Let H and K be complex Hilbert spaces of any dimension with $\{|i\rangle\}_{i=1}^{\dim H \le \infty}$ and $\{|j'\rangle\}_{j=1}^{\dim K \le \infty}$ be orthonormal bases of them respectively. For any positive integer $2 \le n \le \min\{\dim H, \dim K\}$ and any permutations κ, π, σ of $(1, 2, \cdots, n)$ with $\kappa \ne \mathrm{id}$, the finite rank operator $W_\kappa^{\pi,\sigma}$ defined by*

$$\begin{array}{rl} W_\kappa^{\pi,\sigma} = & (n-2)\sum_{i=1}^n |\sigma\pi^{-1}(i), i'\rangle\langle\sigma\pi^{-1}(i), i'| \\ & +\sum_{i=1}^n |\sigma\kappa^{-1}\pi^{-1}(i), i'\rangle\langle\sigma\kappa^{-1}\pi^{-1}(i), i'| \\ & -\sum_{1\le i\ne j\le n} |\sigma\pi^{-1}(i), i'\rangle\langle\sigma\pi^{-1}(j), j'| \end{array}$$

is an entanglement witness.

Assume that $\dim H = \dim K = n$. By applying the witnesses $W_\kappa^{\pi,\sigma}$ in Theorem 3.1, we get a method of detecting the entanglement of states by the entries of their density matrix. Write the product basis of $H \otimes K$ in the order

$$\begin{aligned}\{|e_1\rangle = |1\rangle|1'\rangle, |e_2\rangle = |2\rangle|1'\rangle, \cdots, |e_n\rangle = |n\rangle|1'\rangle, |e_{n+1}\rangle = |1\rangle|2'\rangle,\\ \cdots, |e_{n^2-1}\rangle = |(n-1)\rangle|n'\rangle, |e_{n^2}\rangle = |n\rangle|n'\rangle\}.\end{aligned} \tag{3.9}$$

Then every state $\rho \in \mathcal{S}(H \otimes K)$ has a matrix representation $\rho = (\alpha_{kl})_{n^2\times n^2}$.

Theorem 3.2. *Let $\rho \in \mathcal{B}(H \otimes K)$ with $\dim H = \dim K = n < \infty$ be a state with the matrix representation $\rho = (\alpha_{kl})_{n^2\times n^2}$ with respect to the product basis in Eq.(3.9). If there exist distinguished positive integers $(i-1)n < k_i, h_i \leq in$, $i = 1, 2, \cdots, n$ such that*

$$\sum_{i=1}^n k_i = \sum_{i=1}^n h_i = \frac{1}{2}n(n^2+1), \tag{3.10}$$

and

$$(n-2)\sum_{i=1}^n \alpha_{k_ik_i} + \sum_{i=1}^n \alpha_{h_ih_i} - \sum_{1\leq i\neq j\leq n} \alpha_{k_ik_j} < 0, \tag{3.11}$$

then ρ is entangled.

Proof. Eq.(3.10) implies that, there exist permutations π_1 and σ_1 such that $(k_1, k_2 - n, \cdots, k_n - (n-1)n) = \pi_1(1, 2, \cdots, n)$ and $(h_1, h_2 - n, \cdots, h_n - (n-1)n) = \sigma_1(1, 2, \cdots, n)$. It is clear that $\pi_1(i) \neq \sigma_1(i)$ as $k_i \neq h_i$ for every $i = 1, 2, \cdots, n$.

For any permutations κ, π and σ, by Theorem 3.1, we have

$$\begin{aligned}\mathrm{Tr}(W_\kappa^{\pi,\sigma}\rho) = \quad & (n-2)\textstyle\sum_{i=1}^n \alpha_{\sigma(\pi^{-1}(i))+(i-1)n,\sigma(\pi^{-1}(i))+(i-1)n}\\ & +\textstyle\sum_{i=1}^n \alpha_{\sigma(\kappa^{-1}\pi^{-1}(i))+(i-1)n,\sigma(\kappa^{-1}\pi^{-1}(i))+(i-1)n}\\ & -\textstyle\sum_{i\neq j}^n \alpha_{\sigma(\pi^{-1}(i))+(i-1)n,\sigma(\pi^{-1}(j))+(j-1)n}.\end{aligned} \tag{3.12}$$

Comparing Eq.(3.11) with Eq.(3.12), we have to find permutations κ, π and σ so that

$$\pi_1(i) = \sigma(\pi^{-1}(i)) \quad \text{and} \quad \sigma_1(i) = \sigma(\kappa^{-1}\pi^{-1}(i)) \tag{3.13}$$

for each i, that is, $\pi_1 = \sigma\pi^{-1}$ and $\sigma_1 = \sigma\kappa^{-1}\pi^{-1}$. Take $\pi = \mathrm{id}$. Then we get $\sigma = \pi_1$ and $\sigma_1 = \sigma\kappa^{-1} = \pi_1\kappa^{-1}$. Thus, $\kappa = \sigma_1^{-1}\pi_1$, $\pi = \mathrm{id}$ and $\sigma = \pi_1$ satisfy Eq.(3.13). With such κ, π and σ, by Eqs.(3.11) and (3.12), we have

$$\mathrm{Tr}(W_\kappa^{\pi,\sigma}\rho) = (n-2)\sum_{i=1}^n \alpha_{k_ik_i} + \sum_{i=1}^n \alpha_{h_ih_i} - \sum_{1\leq i\neq j\leq n} \alpha_{k_ik_j} < 0.$$

Hence, ρ is entangled with $W_\kappa^{\pi,\sigma}$ an entanglement witness for it. $\square$

The general version of Theorem 3.2 is the following result, which is applicable for bipartite systems of any dimension.

Theorem 3.3. *Let H and K be complex Hilbert spaces with $\{|i\rangle\}_{i=1}^{\dim H\leq\infty}$ and $\{|j'\rangle\}_{j=1}^{\dim K\leq\infty}$ be orthonormal bases of them respectively. Assume that ρ is a state on $H\otimes K$ and $n\leq\min\{\dim H,\dim K\}$ is a positive integer. If there exist permutations π and σ of $(1,2,\cdots,n)$ with $\pi(i)\neq\sigma(i)$ for any $i=1,2,\cdots,n$ such that*

$$(n-2)\sum_{i=1}^{n}\langle\pi(i),i'|\rho|\pi(i),i'\rangle+\sum_{i=1}^{n}\langle\sigma(i),i'|\rho|\sigma(i),i'\rangle-\sum_{1\leq i\neq j\leq n}\langle\pi(i),i'|\rho|\pi(j),j'\rangle<0, \quad (3.14)$$

then ρ is entangled.

The idea of the proof of Theorem 3.3 is the same as that of Theorem 3.2 and we omit it here.

Theorems 3.2 and 3.3 tell us, some times we can detect the entanglement of a state by suitably chosen n^2+n entries of its matrix representation with respect to some product basis, where $n\leq\min\{\dim H,\dim K\}$.

To illustrate how to use Theorem 3.2 and Theorem 3.3 to detect entanglement of a state, we give some examples.

Example 3.4. Let q_1,q_2,q_3 be nonnegative numbers with $q_1+q_2+q_3=1$ and let $a,b,c\in\mathbb{C}$ with $|a|^2\leq q_2q_3$, $|b|^2\leq q_2q_3$, $|c|^2\leq q_2q_3$. Let ρ be a state of 3×3 system with matrix representation

$$\rho=\frac{1}{3}\begin{pmatrix} q_1 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & q_1\\ 0 & q_3 & a & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \bar{a} & q_2 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & q_2 & 0 & b & 0 & 0 & 0\\ q_1 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & q_1\\ 0 & 0 & 0 & \bar{b} & 0 & q_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & q_3 & c & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{c} & q_2 & 0\\ q_1 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & q_1 \end{pmatrix}. \quad (3.15)$$

Note that, ρ in Eq.(3.15) is a new kind of states, and ρ degenerates to the state as that in [18, Example 3.3] when $a=b=c=0$.

We claim that, if $q_2<q_1$ or $q_3<q_1$, then ρ is entangled.

In fact, choosing $(k_1,k_2,k_3)=(1,5,9)$, $(h_1,h_2,h_3)=(3,4,8)$ or $(2,6,7)$, we have

$$\sum_{i=1}^{3}\alpha_{k_ik_i}+\sum_{i=1}^{3}\alpha_{h_ih_i}-\sum_{1\leq i\neq j\leq 3}\alpha_{k_ik_j}=\frac{1}{3}(3q_1+3q_2-6q_1)=q_2-q_1$$

or

$$\sum_{i=1}^{3}\alpha_{k_ik_i}+\sum_{i=1}^{3}\alpha_{h_ih_i}-\sum_{1\leq i\neq j\leq 3}\alpha_{k_ik_j}=\frac{1}{3}(3q_1+3q_3-6q_1)=q_3-q_1.$$

By Theorem 3.2, we see that ρ is entangled if $q_2 < q_1$ or $q_3 < q_1$.

It is clear that the partial transpose of ρ in Eq.(3.15) with respect to the first subsystem is

$$\rho^{T_1} = \frac{1}{3}\begin{pmatrix} q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_3 & \bar{a} & q_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & q_2 & 0 & 0 & 0 & q_1 & 0 & 0 \\ 0 & q_1 & 0 & q_2 & 0 & \bar{b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & q_3 & 0 & q_1 & 0 \\ 0 & 0 & q_1 & 0 & 0 & 0 & q_3 & \bar{c} & 0 \\ 0 & 0 & 0 & 0 & 0 & q_1 & c & q_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1 \end{pmatrix}.$$

Particularly, if we take $q_1 = \frac{1}{5}$, $q_2 = \frac{1}{10}$, $q_3 = \frac{7}{10}$ and $a = b = c = \frac{1}{20}$, then, by what proved above, we see that ρ is PPT entangled because its partial transpose has eigenvalues

$$\{\frac{1}{60}(8 \pm \sqrt{61}), \frac{1}{4}, \frac{1}{4}, \frac{1}{60}, \frac{1}{60}, \frac{1}{15}, \frac{1}{15}, \frac{1}{15}\}$$

that are all positive.

Example 3.5. Let ρ be a state in 4×4 systems with the matrix

$$\rho = \frac{1}{4}\begin{pmatrix} q_1 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & 0 & q_1 \\ 0 & q_4 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{a} & q_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_2 & q_2 & 0 & 0 & 0 & 0 & q_2 & 0 & 0 & 0 & 0 & q_2 & 0 \\ 0 & 0 & 0 & q_2 & q_2 & 0 & 0 & 0 & 0 & q_2 & 0 & 0 & 0 & 0 & q_2 & 0 \\ q_1 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & q_4 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{b} & q_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_3 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_2 & q_2 & 0 & 0 & 0 & 0 & q_2 & 0 & 0 & 0 & 0 & q_2 & 0 \\ q_1 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{c} & 0 & 0 & q_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_4 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{d} & q_3 & 0 & 0 \\ 0 & 0 & 0 & q_2 & q_2 & 0 & 0 & 0 & 0 & q_2 & 0 & 0 & 0 & 0 & q_2 & 0 \\ q_1 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & 0 & q_1 \end{pmatrix}, \qquad (3.16)$$

where $q_i \geq 0$ with $\sum_{i=1}^4 q_i = 1$, $|a|^2$, $|b|^2$, $|c|^2$ and $|d|^2$ are all $\leq q_3 q_4$. ρ defined by Eq.(3.16) is also a new example, and when $a = b = c = d = 0$ we get states in [18, Example 4.4].

We claim that, if $q_i < q_1$ for some $i \in \{2, 3, 4\}$; or if $q_i < q_2$ for some $i \in \{1, 3, 4\}$, then ρ is entangled.

In fact, we can take

$$(k_1, k_2, k_3, k_4) = (1, 6, 11, 16) \quad \text{and} \quad (h_1, h_2, h_3, h_4) = (2, 7, 12, 13),$$

or

$$(k_1, k_2, k_3, k_4) = (1, 6, 11, 16) \quad \text{and} \quad (h_1, h_2, h_3, h_4) = (3, 8, 9, 14),$$

or

$$(k_1, k_2, k_3, k_4) = (1, 6, 11, 16) \quad \text{and} \quad (h_1, h_2, h_3, h_4) = (4, 5, 10, 15),$$

or

$$(k_1, k_2, k_3, k_4) = (4, 5, 10, 15) \quad \text{and} \quad (h_1, h_2, h_3, h_4) = (1, 6, 11, 16),$$

or

$$(k_1, k_2, k_3, k_4) = (4, 5, 10, 15) \quad \text{and} \quad (h_1, h_2, h_3, h_4) = (2, 7, 12, 13),$$

or

$$(k_1, k_2, k_3, k_4) = (4, 5, 10, 15) \quad \text{and} \quad (h_1, h_2, h_3, h_4) = (3, 8, 9, 14).$$

Then, it follows from the first three choices that

$$2\sum_{i=1}^{4} \alpha_{k_i k_i} + \sum_{i=1}^{4} \alpha_{h_i h_i} - \sum_{1 \leq i \neq j \leq 3} \alpha_{k_i k_j} = q_i - q_1$$

with $i = 2, 3, 4$. Hence, by Theorem 3.2 we see that ρ is entangled if there exists some $i \in \{2, 3, 4\}$ such that $q_i < q_1$. Similarly, by the last three choices one sees that ρ is entangled if there exists some $i \in \{1, 3, 4\}$ such that $q_i < q_2$.

The kind of states in Eq.(3.16) allow us give some new examples of entangled states that can not be recognized by PPT criterion and the realignment criterion. It is obvious that the partial transpose of ρ in Eq.(3.16) with respect to the first subsystem is

$$\rho^{T_1}=\frac{1}{4}\begin{pmatrix}
q_1&0&0&0&0&0&0&q_2&0&0&0&0&0&0&0&0\\
0&q_4&\bar{a}&0&q_1&0&0&0&0&0&0&q_2&0&0&0&0\\
0&a&q_3&0&0&0&0&0&q_1&0&0&0&0&0&0&q_2\\
0&0&0&q_2&0&0&0&0&0&0&0&0&q_1&0&0&0\\
0&q_1&0&0&q_2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&q_1&0&0&q_2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&q_4&\bar{b}&0&q_1&0&0&q_2&0&0&0\\
q_2&0&0&0&0&0&b&q_3&0&0&0&0&0&q_1&0&0\\
0&0&q_1&0&0&q_2&0&0&q_3&0&0&\bar{c}&0&0&0&0\\
0&0&0&0&0&0&q_1&0&0&q_2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&q_1&0&0&q_2&0&0\\
0&q_2&0&0&0&0&0&0&c&0&0&q_4&0&0&q_1&0\\
0&0&0&q_1&0&0&q_2&0&0&0&0&0&q_4&\bar{d}&0&0\\
0&0&0&0&0&0&0&q_1&0&0&q_2&0&d&q_3&0&0\\
0&0&0&0&0&0&0&0&0&0&0&q_1&0&0&q_2&0\\
0&0&q_2&0&0&0&0&0&0&0&0&0&0&0&0&q_1
\end{pmatrix}$$

and that the realignment of ρ is

$$\rho^{R}=\frac{1}{4}\begin{pmatrix}
q_1&0&0&0&0&q_4&\bar{a}&0&0&a&q_3&0&0&0&0&q_2\\
0&q_1&0&0&0&0&0&0&0&0&0&0&q_2&0&0&0\\
0&0&q_1&0&0&0&0&0&0&0&0&0&0&q_2&0&0\\
0&0&0&q_1&0&0&0&0&0&0&0&0&0&0&q_2&0\\
0&0&0&q_2&q_1&0&0&0&0&0&0&0&0&0&0&0\\
q_2&0&0&0&0&q_1&0&0&0&0&q_4&\bar{b}&0&0&b&q_3\\
0&q_2&0&0&0&0&q_1&0&0&0&0&0&0&0&0&0\\
0&0&q_2&0&0&0&0&q_1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&q_2&q_1&0&0&0&0&0&0&0\\
0&0&0&0&q_2&0&0&0&0&q_1&0&0&0&0&0&0\\
q_3&0&0&\bar{c}&0&q_2&0&0&0&0&q_1&0&c&0&0&q_4\\
0&0&0&0&0&0&q_2&0&0&0&0&q_1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&q_2&q_1&0&0&0\\
0&0&0&0&0&0&0&0&q_2&0&0&0&0&q_1&0&0\\
0&0&0&0&0&0&0&0&0&q_2&0&0&0&0&q_1&0\\
q_4&\bar{d}&0&0&d&q_3&0&0&0&0&q_2&0&0&0&0&q_1
\end{pmatrix}.$$

If we take $q_1 = \frac{1}{20}$, $q_2 = \frac{1}{10}$, $q_3 = q_4 = \frac{17}{40}$ and $a = b = c = d = \frac{1}{40}$, ρ is PPT entangled because $q_1 < q_2$ and its partial transpose ρ^{T_1} has eigenvalues

$$\{0.0054, 0.0054, 0.0069, 0.0069, 0.0223, 0.0223, 0.0235, 0.0235,$$
$$0.0821, 0.0821, 0.1027, 0.1027, 0.1212, 0.1212, 0.1359, 0.1359\}$$

that are all positive. Moreover, the trace norm of the realignment ρ^R of ρ is $\|\rho^R\|_1 \doteq 0.8303 < 1$. Hence, we get another example of entangled states that is PPT and cannot be detected by the realignment criterion.

It is not difficult to give some examples of applying Theorem 3.3 to infinite dimensional systems based on examples 3.4 and 3.5.

4. Conclusions

Let H and K be Hilbert spaces and let $\{|i\rangle\}_{i=1}^{\dim H \leq \infty}$ and $\{|j'\rangle\}_{j=1}^{\dim K \leq \infty}$ be any orthonormal bases of H and K, respectively. By the finite rank elementary operator criterion [13], a state ρ on $H \otimes K$ is entangled if and only if there exists a finite rank positive elementary operator $\Phi : \mathcal{B}(H) \to \mathcal{B}(K)$ that is not completely positive such that $(\Phi \otimes I)\rho$ is not positive. By this criterion and the finite rank positive elementary operators constructed in [18], we construct a collection of finite rank entanglement witnesses.

By using these witnesses we obtain a rank-4 entanglement witness $W = |1\rangle|2'\rangle\langle 1|\langle 2'| - |1\rangle|1'\rangle\langle 2|\langle 2'| - |2\rangle|2'\rangle\langle 1|\langle 1'| + |2\rangle|1'\rangle\langle 2|\langle 1'|$ which is universal for pure states, that is, for a pure state ρ, ρ is separable if and only if $\mathrm{Tr}((U \otimes V)W(U^\dagger \otimes V^\dagger)\rho) \geq 0$ holds for all unitary operators U on H and V on K. In addition, for a mixed state ρ, if there exist unitary operators U_0 on H and V_0 on K such that $\mathrm{Tr}((U_0 \otimes V_0)W(U_0^\dagger \otimes V_0^\dagger)\rho) < 0$, then ρ is entangled and 1-distillable.

Another interesting result, maybe for the first time, gives a way of detecting the entanglement of a state in $H \otimes K$ by only a part entries of its density matrix. This method is simple, computable and practicable. Assume that ρ is a state on $H \otimes K$ and $n \leq \min\{\dim H, \dim K\}$ is a positive integer. If there exist permutations π and σ of $(1, 2, \cdots, n)$ with $\pi(i) \neq \sigma(i)$ for any $i = 1, 2, \cdots, n$ such that

$$(n-2)\sum_{i=1}^{n}\langle \pi(i), i'|\rho|\pi(i), i'\rangle + \sum_{i=1}^{n}\langle \sigma(i), i'|\rho|\sigma(i), i'\rangle - \sum_{1 \leq i \neq j \leq n}\langle \pi(i), i'|\rho|\pi(j), j'\rangle < 0,$$

then ρ is entangled. Thus we provide a way of detecting the entanglement of a state by finite suitably chosen entries of its matrix representation with respect to some product basis. As an illustration how to use this method, some new examples of entangled states that can be recognized by this way are proposed, which also provides some new entangled states that can not be detected by the PPT criterion and the realignment criterion.

References

[1] I. Bengtsson, K. Zyczkowski, Cambridge University Press, Cambridge, 2006.

[2] D. Bruß, J. Math. Phys. 43 (2002) 4237.

[3] K. Chen, L. Wu, Quant. Inf. Comput 3 (2003) 193.

[4] D. Chruściński and A. Kossakowski, Open Systems and Inf. Dynamics 14 (2007) 275; D. Chruściński and A. Kossakowski, J. Phys. A: Math. Theor. 41 (2008) 145301.

[5] J. Dixmier, Von Neumann Algebras, North-Holland Publishing Com., Amsterdan, New York, Oxford, 1981.

[6] Y. Guo, J. Hou, arXiv:1009.0116v1.

[7] M. Horodecki, P. Horodecki, R. Horodecki, Phys. Lett. A 223 (1996) 1.

[8] R. Horodecki, M. Horodecki, P. Horodecki, Phys. Lett. A 222 (1996) 21.

[9] M. Horodecki, P. Horodecki, R. Horodecki, Phys. Rev. Lett. 80 (1998) 5239.

[10] R. Horodecki, P. Horodecki, M. Horodecki, Rev. Mod. Phys. 81 (2009) 865.

[11] J. Hou, Sci. in China (ser.A), 36(9) (1993), 1025-1035.

[12] J. Hou, J. Operator Theory, 39 (1998), 43-58.

[13] J. Hou, J. Phys. A: Math. Theor. 43 (2010) 385201; arXiv[quant-ph]: 1007.0560v1.

[14] J. Hou, X. Qi, Phys. Rev. A 81 (2010) 062351.

[15] M. A. Jafarizadeh, N. Behzadi, Y. Akbari, Eur. Phys. J. D 55 (2009) 197.

[16] M. A. Nielsen, I. L. Chuang, Cambridge University Press, Cambridge, 2000.

[17] A. Peres, Phys. Lett. A 202 (1996) 16.

[18] X. Qi, J. Hou, Positive finite rank elementary operators and characterizing entanglement of states, arXiv:1008.3682v2

[19] S. Simon, S. P. Rajagopalan, R. Simon, Pramana-Journal of Physics, 73(3) (2009) 471-483.

[20] G. Tóth, O. Gühne, Phys. Rev. Lett. 94 (2005) 060501.

[21] R. F. Werner, Phys. Rev. A 40 (1989) 4277.

(Xiaofei Qi) Department of Mathematics, Shanxi University , Taiyuan 030006, P. R. of China;

E-mail address: qixf1980@126.com

Department of Mathematics, Taiyuan University of Technology, Taiyuan 030024, P. R. of China

E-mail address: jinchuanhou@yahoo.com.cn