Time-dependent coupled oscillator model for charged particle motion in the presence of a time-varying magnetic field

Salah Menouar[1*], **Mustapha Maamache**[1], **and Jeong Ryeol Choi**[2†]

[1]*Laboratoire de Physique Quantique et Systèmes Dynamiques, Département de Physique, Faculté des Sciences, Université Ferhat Abbas de Sétif, Sétif 19000, Algeria*

[2]*Division of Semiconductor and Display Engineering, College of IT Engineering, Kyungpook National University, 1370 Sankguk-dong, Buk-gu, Daegu 702-701, Republic of Korea*

Abstract

The dynamics of time-dependent coupled oscillator model for the charged particle motion subjected to a time-dependent external magnetic field is investigated. We used canonical transformation approach for the classical treatment of the system, whereas unitary transformation approach is used when managing the system in the framework of quantum mechanics. For both approaches, the original system is transformed to a much more simple system that is the sum of two independent harmonic oscillators which have time-dependent frequencies. We therefore easily identified the wave functions in the transformed system with the help of invariant operator of the system. The full wave functions in the original system is derived from the inverse unitary transformation of the wave functions associated to the transformed system.

Keywords: *charged particle motion; unitary transformation; canonical transformation; time-dependent coupled oscillator*
PACC numbers: 0365G, 0365D, 4190

[1*]E-mail: menouar_salah@yahoo.fr
[2†]Corresponding author, E-mail: choiardor@hanmail.net

1 Introduction

The time-dependent harmonic oscillators have attracted considerable interest in the literature thanks to their usefulness in describing the dynamics of many physical systems. After the Bateman's[1] proposition concerning the use of time-dependent harmonic oscillator model in describing dissipative systems, much attention was paid to quantum behavior of nonconservative and nonlinear systems.

In the meantime, coupled oscillators have emerged to become powerful modeling tools and, consequently, are frequently used in modeling wide range of physical phenomena. With the progress of research, one may be interested in what would happen if two-dimensional harmonic oscillator is elaborated through the coupling of two additive potentials? As far as we know, dealing with such an issue was set thirty years ago by Kim et al. [2-5]. Abdalla demonstrated how to treat the time-dependent coupled oscillators in the context of quantum mechanics[6]. The propagator for a time-dependent coupled and driven harmonic oscillators with time-varying frequencies and masses is investigated by Benamira [7] using path integral methods.

Among various systems that can be modeled by time-dependent coupled oscillators, the dynamics of charged particle motion in the presence of time-varying magnetic fields has played an important role in condensed matter physics and plasma physics. There are plenty of applications for this system such as magnetoresistance[8], the Aharonov-Bohm effect[9], magnetic confinement devices for fusion plasmas[10], electromagnetic lenses with variable magnetic fields[11], cyclotron resonance[12], and entanglement of a two-qubit Heisenberg XY model[13]. Though all of these problems are interesting, we can find their exact analytic solutions only for a few special cases due to their complex mathematical structures.

The quantum properties of a free electron, which have a time-dependent effective mass under the influence of external magnetic field, are investigated in both the Landau and the symmetric gauges [14, 15]. Laroze and Rivera[16] studied the dynamical behavior of electrons in the presence of a uniform time-dependent magnetic field and they presented the time evolution of the corresponding wave functions for the case that the initial state is a superposition of Landau levels. The propagators of a charged particle subjected to a time-dependent magnetic field are studied using the linear and the quadratic invariants [17].

Kim et al. [2-5] proposed a problem that what actually would take place if two harmonic oscillators are coupled so that the potential becomes $V(X_1, X_2) = \frac{1}{2}\left(c_1 X_1^2 + c_2 X_2^2 + c_3 X_1 X_2\right)$ where c_3 is a coupling constant. They studied the corresponding density matrix in order to establish the Wigner function. In this work, we are interested in the problem of Hamiltonian that involves the coupling term X_1X_2 in the presence of magnetic field. This system can be regarded as the generalization of the Hamiltonian model given in Refs. [14] and [18]. Though the coupling of two or more oscillators is among the most basic concepts in dealing with gyroscopic motions, interactions, and complex structures, the related theory has been scarcely developed so far. This class of coupled harmonic oscillators can be used to describe numerous physical systems. Some of them are the Bogoliubov transformation model of superconductivity [19], two-mode squeezed light [20], and the Lee model in quantum field theory [21]. One of the main focuses of research carried out by Zhang *et al.* in connection with time-dependent coupled oscillators including X_1X_2 term are some specific problems of time-dependent coupled electronic circuits[22, 23].

We will use the invariant methods[24, 25] in order to derive the exact wave functions for time-dependent coupled oscillators in a variable magnetic field. The invariant operator method in describing the quantum features of time-dependent harmonic oscillators is firstly introduced by Lewis[24] and now became a very useful tool in developing quantum theory for the case where the Hamiltonian of the system is explicitly dependent on time.

In Sec. 2, we formulate our problem by introducing a general time-dependent Hamiltonian describing the complicated motion of a charged particle in the presence of an arbitrary time-dependent magnetic field. Classical treatment of the system is presented in Sec. 3, on the basis of the canonical transformation method. Quantum analysis of the system is carried out in Sec. 4 using unitary transformation approach. The unitary transformation enables us to transform the original Hamiltonian (that is somewhat complicated) to that of a more simple system such as ordinary harmonic oscillator. We derive the quantum solutions of the system in Sec. 5 starting from the invariant operator associated to the transformed system described in Sec. 4. Finally, we give concluding remarks in the last section.

2 Formulation of the problem

For the dynamical system of our interest, the Hamiltonian has the form:

$$H(X_1, X_2, t) = \frac{\Pi_1^2}{2m_1(t)} + \frac{\Pi_2^2}{2m_2(t)} + \frac{1}{2}\left(C_1(t)X_1^2 + C_2(t)X_2^2 + C_3(t)X_1X_2\right), \tag{1}$$

where Π_1 and Π_2 are the conjugate momenta. Note that Π_1 and Π_2 can be simplified by choosing an appropriate gauge. Actually, in the symmetric gauge with $\overrightarrow{A}\left(\frac{-B(t)}{2}X_2, \frac{B(t)}{2}X_1, 0\right)$, they are given by

$$\Pi_1 = P_1 - \frac{eB(t)}{2}X_2 \ , \ \Pi_2 = P_2 + \frac{eB(t)}{2}X_1. \tag{2}$$

The parameters $m_1(t)$, $m_2(t)$, $C_1(t)$, $C_2(t)$, and $C_3(t)$ are arbitrary functions of time, (X_1, X_2) are the pair of position variables, and (P_1, P_2) are the canonical conjugate momentum variables.

The main difference of our study from that of Ref. [16] is that we considered the coupling term X_1X_2 in the Hamiltonian. Regarding the expressions of Π_1 and Π_2, the Hamiltonian in Eq. (1) can be recasted into

$$\begin{aligned} H(X_1, X_2, t) = & \frac{P_1^2}{2m_1(t)} + \frac{P_2^2}{2m_2(t)} + \frac{1}{2}\left(c_1(t)X_1^2 + c_2(t)X_2^2 + c_3(t)X_1X_2\right) \\ & + \frac{1}{2}\left(\omega_{2c}(t)P_2X_1 - \omega_{1c}(t)P_1X_2\right), \end{aligned} \tag{3}$$

where the new time-dependent functions $c_1(t)$, $c_2(t)$ and $c_3(t)$ are read

$$c_1(t) = C_1(t) + m_2(t)\frac{\omega_{2c}^2(t)}{4}, \quad c_2(t) = C_2(t) + m_1(t)\frac{\omega_{1c}^2(t)}{4}, \quad c_3(t) = C_3(t), \tag{4}$$

with the cyclotron frequencies

$$\omega_{1c}(t) = \frac{eB(t)}{m_1(t)}, \qquad \omega_{2c}(t) = \frac{eB(t)}{m_2(t)}. \tag{5}$$

3 Classical treatment

The time-dependent canonical transformation approach is in fact very powerful in investigating the properties of dynamical systems described by a time-dependent Hamiltonian. In many cases, we can convert a given Hamiltonian into a simple and desired one by means of the canonical transformation. Therefore, in order to recast the solutions of this problem into a more soluble form, it is convenient to use the canonical transformation method. To simplify the Hamiltonian given in Eq. (3), let us transform the variables (X_1, X_2, P_1, P_2) to the new variables $(x_1, x_2, p_{x_1}, p_{x_2})$ such that

$$x_1 = \left(\frac{m_1(t)}{m_2(t)}\right)^{1/4} X_1, \qquad x_2 = \left(\frac{m_2(t)}{m_1(t)}\right)^{1/4} X_2, \tag{6}$$

$$p_{x_1} = \left(\frac{m_2(t)}{m_1(t)}\right)^{1/4} P_1, \quad p_{x_2} = \left(\frac{m_1(t)}{m_2(t)}\right)^{1/4} P_2. \tag{7}$$

Replacing all of the canonical variables in Eq. (3) with the above ones, we have

$$\begin{aligned} H(x_1, x_2, t) = & \frac{1}{2m(t)} \left(p_{x_1}^2 + p_{x_2}^2\right) + \frac{1}{2} \left(d_1(t)x_1^2 + d_2(t)x_2^2 + d_3(t)x_1x_2\right) \\ & + \frac{\omega_c(t)}{2} \left(x_1 p_{x_2} - x_2 p_{x_1}\right), \end{aligned} \tag{8}$$

where $d_1 - d_3$ are new time-dependent functions of the form

$$d_1(t) = c_1(t) \left(\frac{m_2(t)}{m_1(t)}\right)^{1/2} = \left(C_1(t) + \frac{1}{4} m_2(t)\omega_{2c}^2(t)\right) \left(\frac{m_2(t)}{m_1(t)}\right)^{1/2}, \tag{9}$$

$$d_2(t) = c_2(t) \left(\frac{m_1(t)}{m_2(t)}\right)^{1/2} = \left(C_2(t) + \frac{1}{4} m_1(t)\omega_{1c}^2(t)\right) \left(\frac{m_1(t)}{m_2(t)}\right)^{1/2}, \tag{10}$$

$$d_3(t) = c_3(t) = C_3(t), \tag{11}$$

with the unique mass $m(t) = (m_1(t)m_2(t))^{1/2}$ and the cyclotron frequency $\omega_c(t) = (\omega_{1c}(t)\omega_{2c}(t))^{1/2} = eB(t)/m(t)$.

To simplify the Hamiltonian of Eq. (8), we perform the following canonical transformation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\phi(t) & \sin\phi(t) \\ -\sin\phi(t) & \cos\phi(t) \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{12}$$

$$\begin{pmatrix} p_{x_1} \\ p_{x_2} \end{pmatrix} = \begin{pmatrix} \cos\phi(t) & \sin\phi(t) \\ -\sin\phi(t) & \cos\phi(t) \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \tag{13}$$

where

$$\phi(t) = -\frac{1}{2} \int \omega_c(t) dt. \tag{14}$$

If (q_1, q_2, p_1, p_2) are canonical coordinates, there should exist a new Hamiltonian $H(q_1, q_2, t)$ which is determined by only in terms of the Hamiltonian given in Eq. (8) with the aid of the linear transformation shown in Eqs. (12) and (13). The variables $(x_1, x_2, p_{x_1}, p_{x_2})$ and (q_1, q_2, p_1, p_2) in two representations must satisfy the following relation [27]

$$(p_1\dot{q}_1 + p_2\dot{q}_2 - H(q_1, q_2, t) = p_{x_1}\dot{x}_1 + p_{x_2}\dot{x}_2 - H(x_1, x_2, t) + \frac{\partial F_1}{\partial t}, \tag{15}$$

where F_1 is a time-dependent generating function in phase space, which should be determined afterwards.

From the fundamental equations known in classical mechanics [27]

$$p_{x_1} = \frac{\partial}{\partial x_1} F_1\left(x_1, x_2, p_1, p_2, t\right), \qquad q_1 = \frac{\partial}{\partial p_1} F_1\left(x_1, x_2, p_1, p_2, t\right), \quad (16)$$

$$p_{x_2} = \frac{\partial}{\partial x_2} F_1\left(x_1, x_2, p_1, p_2, t\right), \qquad q_2 = \frac{\partial}{\partial p_2} F_1\left(x_1, x_2, p_1, p_2, t\right), \quad (17)$$

the generating function associated with the transformation is found to be

$$F_1\left(x_1, x_2, p_1, p_2 t\right) = \left(p_1 \cos\phi + p_2 \sin\phi\right) x_1 + \left(-p_1 \sin\phi + p_2 \cos\phi\right) x_2, \quad (18)$$

$$\frac{\partial F_1}{\partial t} = -\dot{\phi}(t)\left(x_1 p_{x_2} - x_2 p_{x_1}\right) = -\frac{\varpi_c(t)}{2}\left(x_1 p_{x_2} - x_2 p_{x_1}\right). \quad (19)$$

In terms of the new conjugate variables (q_1, q_2, p_1, p_2), the Hamiltonian of Eq. (8) becomes

$$H(q_1, q_2, t) = \frac{1}{2m(t)}\left(p_1^2 + p_2^2\right) + \frac{1}{2}\left(\lambda_1(t) q_1^2 + \lambda_2(t) q_2^2 + \lambda_3(t) q_1 q_2\right), \quad (20)$$

where

$$\lambda_1(t) = d_1(t)\cos^2\phi + d_2(t)\sin^2\phi - d_3(t)\sin\phi\cos\phi, \quad (21)$$

$$\lambda_2(t) = d_2(t)\cos^2\phi + d_1(t)\sin^2\phi + d_3(t)\sin\phi\cos\phi, \quad (22)$$

$$\lambda_3(t) = 2\left(d_1(t) - d_2(t)\right)\sin\phi\cos\phi + d_3(t)\left(\cos^2\phi - \sin^2\phi\right). \quad (23)$$

To eliminate the coupling term $q_1 q_2$, we now perform the following canonical transformation [7, 22, 23]

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{\sqrt{m(t)}} \begin{pmatrix} \cos\frac{\theta(t)}{2} & \sin\frac{\theta(t)}{2} \\ -\sin\frac{\theta(t)}{2} & \cos\frac{\theta(t)}{2} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \quad (24)$$

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \sqrt{m(t)} \begin{pmatrix} \cos\frac{\theta(t)}{2} & \sin\frac{\theta(t)}{2} \\ -\sin\frac{\theta(t)}{2} & \cos\frac{\theta(t)}{2} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} - \begin{pmatrix} \frac{\dot{m}(t)}{2} & 0 \\ 0 & \frac{\dot{m}(t)}{2} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \quad (25)$$

where $\theta(t)$ is an arbitrary function of time. Note that Eqs. (24) and (25) do not always represent the canonical transformation [27] between variables $(q_i, p_i)\,[i = 1, 2]$ and (Q_i, P_i). If (Q_i, P_i) are canonical coordinates, there should exist a new Hamiltonian which is determined only by the Hamiltonian of Eq. (20) and the linear transformation given in Eqs. (24) and (25). The relation between variables (q_i, p_i) and (Q_i, P_i) in the two representations are [27]

$$\sum_{i=1}^{2} P_i \dot{Q}_i - H_Q = \sum_{i=1}^{2} p_i \dot{q}_i - H_q + \frac{\partial F}{\partial t}, \quad (26)$$

where F is an another time-dependent generating function in phase space.

Using the basic equations

$$p_i = \frac{\partial}{\partial q_i} F\left(q_1, q_2, P_1, P_2, t\right), \quad Q_i = \frac{\partial}{\partial P_i} F\left(q_1, q_2, P_1, P_2, t\right), \tag{27}$$

where $i = 1, 2$, we see that the generating function is given by

$$\begin{aligned} F\left(q_1, q_2, P_1, P_2, t\right) = & \sqrt{m(t)}\left(P_1 \cos\frac{\theta(t)}{2} + P_2 \sin\frac{\theta(t)}{2}\right) q_1 \\ & + \sqrt{m(t)}\left(-P_1 \sin\frac{\theta(t)}{2} + P_2 \cos\frac{\theta(t)}{2}\right) q_2 \\ & - \frac{1}{4}\dot{m}(t)\left(q_1^2 + q_2^2\right). \end{aligned} \tag{28}$$

Then, in terms of the new conjugate variables (Q_i, P_i), the Hamiltonian can be represented in the form

$$\begin{aligned} H_Q(Q_1, Q_2, t) = & \frac{1}{2}\left(P_1^2 + P_2^2\right) + \frac{1}{2}\Omega_1^2(t) Q_1^2 + \frac{1}{2}\Omega_2^2(t) Q_2^2 \\ & + \frac{\dot{\theta}(t)}{2}\left[P_1 Q_2 - P_2 Q_1\right] + \delta(t) Q_1 Q_2. \end{aligned} \tag{29}$$

Here, the time-dependent coefficients $\Omega_1(t), \Omega_2(t)$ and $\delta(t)$ are given by

$$\Omega_1(t) = \left(\tilde{\omega}_1^2(t)\cos^2\frac{\theta(t)}{2} + \tilde{\omega}_2^2(t)\sin^2\frac{\theta(t)}{2} - \frac{\lambda_3(t)\sin\theta(t)}{m(t)}\right)^{1/2}, \tag{30}$$

$$\Omega_2(t) = \left(\tilde{\omega}_1^2(t)\sin^2\frac{\theta(t)}{2} + \tilde{\omega}_2^2(t)\cos^2\frac{\theta(t)}{2} + \frac{\lambda_3(t)\sin\theta(t)}{m(t)}\right)^{1/2}, \tag{31}$$

$$\delta(t) = \frac{1}{2}\left(\tilde{\omega}_1^2(t) - \tilde{\omega}_2^2(t)\right)\sin\theta(t) + \frac{\lambda_3(t)\cos\theta(t)}{m(t)}, \tag{32}$$

where

$$\tilde{\omega}_1^2(t) = \frac{\lambda_1(t)}{m(t)} + \frac{1}{4}\left(\frac{\dot{m}^2(t)}{m^2(t)} - 2\frac{\ddot{m}(t)}{m(t)}\right), \tag{33}$$

$$\tilde{\omega}_2^2(t) = \frac{\lambda_2(t)}{m(t)} + \frac{1}{4}\left(\frac{\dot{m}^2(t)}{m^2(t)} - 2\frac{\ddot{m}(t)}{m(t)}\right). \tag{34}$$

If we take the choice $\theta(t) = \text{Const}$, the terms $P_1 Q_2$ and $P_2 Q_1$ in Eq. (29) are canceled out so that the Hamiltonian becomes

$$H_Q(Q_1, Q_2, t) = \frac{1}{2}\left(P_1^2 + P_2^2\right) + \frac{1}{2}\Omega_1^2(t) Q_1^2 + \frac{1}{2}\Omega_2^2(t) Q_2^2 + \delta(t) Q_1 Q_2. \tag{35}$$

Notice that, with the above canonical transformation, the coupling $\delta(t)$ is a functional on the parameters of the original system. It is hence clear that the separation of variables in Eq. (35) requires that $\delta(t) = 0$, i.e.

$$\lambda_3(t) = \left(\tilde{\omega}_2^2(t) - \tilde{\omega}_1^2(t)\right) m(t) \tan\theta, \tag{36}$$

and consequently

$$\tan\theta = \frac{\lambda_3(t)}{m(t)\left(\tilde{\omega}_2^2(t) - \tilde{\omega}_1^2(t)\right)}. \tag{37}$$

By taking into account Eq. (36), the Hamiltonian in Eq. (35) is rewritten as

$$H_Q(Q_1, Q_2, t) = \frac{1}{2}\left(P_1^2 + P_2^2\right) + \frac{1}{2}\Omega_1^2(t)Q_1^2 + \frac{1}{2}\Omega_2^2(t)Q_2^2. \tag{38}$$

Then, Eq.(38) represents the sum of two independent Hamiltonians of the simple harmonic oscillators with the time-dependent frequencies $\Omega_1(t)$ and $\Omega_2(t)$.

4 Quantum treatment

The canonical transformations in classical mechanics, treated in the previous section, is the analogous of the unitary transformations in quantum mechanics. Now we are going to demonstrate this relationship between the two transformations and confirm how to obtain the quantum-mechanical Hamiltonian from the classical one. To manage the system in the context of quantum physics, we replace the canonical variables (X_1, X_2) in Eq. (3) by quantum operators $(\hat{X}_1, \hat{X}_2)$. Then the corresponding Hamiltonian has the form

$$\begin{aligned}\hat{H}(\hat{X}_1, \hat{X}_2, t) = & \frac{\hat{P}_1^2}{2m_1(t)} + \frac{\hat{P}_2^2}{2m_2(t)} + \frac{1}{2}\left(c_1(t)\hat{X}_1^2 + c_2(t)\hat{X}_2^2 + c_3(t)\hat{X}_1\hat{X}_2\right) \\ & + \frac{1}{2}\left(\omega_{2c}(t)\hat{P}_2\hat{X}_1 - \omega_{1c}(t)\hat{P}_1\hat{X}_2\right). \end{aligned} \tag{39}$$

In this quantum case, the pair of momentum operators are given by $(\hat{P}_1 = -i\hbar\partial/\partial X_1,\ \hat{P}_2 = -i\hbar\partial/\partial X_2)$. The Schrödinger equation in the original system is

$$i\hbar\frac{\partial}{\partial t}\Psi(X_1, X_2, t) = \hat{H}(\hat{X}_1, \hat{X}_2, t)\Psi(X_1, X_2, t). \tag{40}$$

To simplify the Hamiltonian in Eq. (39), we perform the unitary transformation such that

$$\Psi(X_1, X_2, t) = \hat{U}_1(t)\psi(X_1, X_2, t), \tag{41}$$

where $\hat{U}_1(t)$ is a time-dependent unitary operator of the form

$$\hat{U}_1(t) = \exp\frac{i}{2\hbar}\left[(\hat{P}_1\hat{X}_1 + \hat{X}_1\hat{P}_1)\ln\left(\frac{m_1(t)}{m_2(t)}\right)^{1/4}\right] \times \exp\frac{i}{2\hbar}\left[(\hat{P}_2\hat{X}_2 + \hat{X}_2\hat{P}_2)\ln\left(\frac{m_2(t)}{m_1(t)}\right)^{1/4}\right]. \tag{42}$$

In this case, the Hamiltonian, Eq. (39), can be rewritten as

$$\hat{H}_1(\hat{X}_1, \hat{X}_2, t) = \frac{1}{2m(t)}\left(\hat{P}_1^2 + \hat{P}_2^2\right) + \frac{1}{2}\left(d_1(t)\hat{X}_1^2 + d_2(t)\hat{X}_2^2 + d_3(t)\hat{X}_1\hat{X}_2\right) + \frac{\omega_c(t)}{2}\left(\hat{P}_2\hat{X}_1 - \hat{P}_1\hat{X}_2\right). \tag{43}$$

It is easy to confirm that the commutation relations, $[\hat{L}_Z\ ,\ \hat{X}_1^2 + \hat{X}_2^2] = 0$ and $[\hat{L}_z\ ,\ \hat{P}_1^2 + \hat{P}_2^2] = 0$, are hold where $\hat{L}_Z$ is the angular momentum operator. This implies that there are common eigenfunctions between $\hat{L}_Z$ and $\hat{X}_1^2 + \hat{X}_2^2$, and between $\hat{L}_Z$ and $\hat{P}_1^2 + \hat{P}_2^2$. However, $\hat{L}_Z$ does not commutes with $\hat{X}_1\hat{X}_2$: $[\hat{L}_Z\ ,\ \hat{X}_1\hat{X}_2] \neq 0$, and consequently $[\hat{L}_Z\ ,\ \hat{H}] \neq 0$. If we regard that $\hat{L}_Z$ and $\hat{H}$ do not have the same eigenfunctions, it is not possible to simplify the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(X_1, X_2, t) = \hat{H}_1(\hat{X}_1, \hat{X}_2, t)\psi(X_1, X_2, t), \tag{44}$$

by decomposing it. However, we can overcome this difficulty through the transformation of the Hamiltonian of Eq. (39) into a simple form by introducing an appropriate unitary transformation operators. In the first step, we perform the following unitary transformation

$$\psi(X_1, X_2, t) = \hat{U}_2(t)\varphi(X_1, X_2, t), \tag{45}$$

where

$$\hat{U}_2(t) = \exp\left(-\frac{i}{2\hbar}\left(\hat{P}_2\hat{X}_1 - \hat{P}_1\hat{X}_2\right)\int \varpi_c(t)dt\right) = \exp\left(-\frac{i\hat{L}_Z}{2\hbar}\int \varpi_c(t)dt\right). \tag{46}$$

Under this transformation, the Schrödinger equation (41) is mapped into

$$i\hbar\frac{\partial}{\partial t}\varphi(X_1,X_2,t)=\hat{H}_2(\hat{X}_1,\hat{X}_2,t)\varphi(X_1,X_2,t), \tag{47}$$

where the new Hamiltonian $\hat{H}_2(\hat{X}_1,\hat{X}_2,t)$ has the form

$$\hat{H}_2(\hat{X}_1,\hat{X}_2,t)=\frac{1}{2m(t)}\left(\hat{P}_1^2+\hat{P}_2^2\right) \\ +\frac{1}{2}\left(\lambda_1(t)\hat{X}_1^2+\lambda_2(t)\hat{X}_2^2+\lambda_3(t)\hat{X}_1\hat{X}_2\right). \tag{48}$$

Now the term involving $\hat{L}_Z$ has disappeared in Eq. (48). This means that the magnetic field term is removed in the new frame rotating with the time-dependent phase $\phi(t)=-\frac{1}{2}\int\varpi_c(t)dt$.

To decouple the Hamiltonian of Eq. (48), we take another unitary transformation such that

$$\varphi(X_1,X_2,t)=\hat{V}(t)\chi(X_1,X_2,t), \tag{49}$$

where the unitary operator $\hat{V}(t)$ is given by

$$\hat{V}(t)=\hat{V}_1(t)\hat{V}_2(t)\hat{V}_3(t), \tag{50}$$

with

$$\hat{V}_1(t)=\exp\frac{i}{2\hbar}\left[(\hat{P}_1\hat{X}_1+\hat{X}_1\hat{P}_1)\ln\sqrt{m(t)}\right] \\ \times\exp\frac{i}{2\hbar}\left[(\hat{P}_2\hat{X}_2+\hat{X}_2\hat{P}_2)\ln\sqrt{m(t)}\right], \tag{51}$$

$$\hat{V}_2(t)=\exp\left[-\frac{i}{\hbar}\frac{\theta}{2}(\hat{P}_2\hat{X}_1-\hat{P}_1\hat{X}_2)\right], \tag{52}$$

$$\hat{V}_3(t)=\exp-\frac{i}{4\hbar}\dot{m}(t)\left(\hat{X}_1^2+\hat{X}_2^2\right). \tag{53}$$

Some algebra with the substitution of Eqs. (48) and (49) into Eq.(47) yields a transformed Hamiltonian that represents the sum of two uncoupled simple harmonic oscillators having frequencies $\Omega_1(t)$ and $\Omega_2(t)$ and the unit mass:

$$\hat{H}_3(\hat{X}_1,\hat{X}_2,t)=\hat{V}^{-1}(t)\hat{H}_2(\hat{X}_1,\hat{X}_2,t)\hat{V}(t)-i\hbar\hat{V}^{-1}(t)\frac{\partial}{\partial t}\hat{V}(t) \\ =\frac{1}{2}\left(\hat{P}_1^2+\hat{P}_2^2\right)+\frac{1}{2}\Omega_1^2(t)\hat{X}_1^2+\frac{1}{2}\Omega_2^2(t)\hat{X}_2^2. \tag{54}$$

At this stage, it is possible to confirm that the classically transformed Hamiltonian given in Eq. (38) is right, since the above equation is consistent with it. Note that $\hat{U}_1(t)$ and $\hat{V}_1(t)$ given in Eqs. (42) and (51) are the squeeze operators, whereas $\hat{U}_2(t)$ and $\hat{V}_2(t)$ given in Eqs. (46) and (52) are the rotation operators characterized by the time-varying angles $\phi(t)$ and $\frac{\theta(t)}{2}$, respectively.

5 Quantum solutions

It can be seen that there exists invariant for the harmonic oscillator with time-dependent mass and/or frequency[24]. In our case, the transformed system consists of the two independent harmonic oscillators which have time-dependent frequency. It is easy to verify, from Liouville-von Neumann equation for the invariant $\hat{I}$

$$\frac{d\hat{I}}{dt} = \frac{\partial \hat{I}}{\partial t} + \frac{1}{i\hbar}[\hat{I}, \hat{H}_3] = 0, \tag{55}$$

that the invariant associated to the Hamiltonian of two-dimensional harmonic oscillator is given by

$$\begin{aligned} \hat{I}(\hat{X}_1, \hat{X}_2, t) &= \hat{I}(\hat{X}_1, t) + \hat{I}(\hat{X}_2, t) \\ &= \frac{1}{2}\left[\left(\frac{\hat{X}_1}{\rho_1}\right)^2 + \left(\rho_1 \dot{\hat{X}}_1 - \dot{\rho}_1 \hat{X}_1\right)^2\right] \\ &\quad + \frac{1}{2}\left[\left(\frac{\hat{X}_2}{\rho_2}\right)^2 + \left(\rho_2 \dot{\hat{X}}_2 - \dot{\rho}_2 \hat{X}_2\right)^2\right], \end{aligned} \tag{56}$$

where $\rho_1(t)$ and $\rho_2(t)$ are c-number quantities obeying the auxiliary equations

$$\ddot{\rho}_1 + \Omega_1^2(t)\rho_1 = 1/\rho_1^3, \tag{57}$$
$$\ddot{\rho}_2 + \Omega_2^2(t)\rho_2 = 1/\rho_2^3. \tag{58}$$

To guarantee the Hermiticity of Eq. (56) ($\hat{I}^\dagger = \hat{I}$), we choose only the real solutions of the above two equations. It is clear that $\hat{I}(\hat{X}_1, \hat{X}_2, t)$ satisfies the Liouville-Von Neumann equation. We now derive a complete orthonormal set of eigenfunctions $\xi_{n_1 n_2}(X_1, X_2, t)$ of $\hat{I}(\hat{X}_1, \hat{X}_2, t)$ form the eigenvalue equation

$$\hat{I}(\hat{X}_1, \hat{X}_2, t)\xi_{n_1 n_2}(X_1, X_2, t) = \lambda_{n_1 n_2}\xi_{n_1 n_2}(X_1, X_2, t), \tag{59}$$

where $\lambda_{n_1 n_2}$ are time-*in*dependent eigenvalues. Through a straightforward evaluation after inserting Eq. (56) into the above equation, we get the eigenvalues and the eigenfunctions such that

$$\lambda_{n_1 n_2} = \hbar\left(n_1 + \frac{1}{2}\right) + \hbar\left(n_2 + \frac{1}{2}\right), \tag{60}$$

$$
\begin{aligned}
\xi_{n_1 n_2}(X_1, X_2, t) = & \left[\frac{1}{\pi\hbar n_1! n_2! 2^{n_1+n_2}\rho_1\rho_2}\right]^{1/2} \\
& \times H_{n_1}\left(\frac{X_1}{\hbar^{1/2}\rho_1}\right) H_{n_2}\left(\frac{X_2}{\hbar^{1/2}\rho_2}\right) \\
& \times \exp\left[\frac{i}{2\hbar}\left(\frac{\dot{\rho}_1}{\rho_1}+\frac{i}{\rho_1^2}\right)X_1^2 + \frac{i}{2\hbar}\left(\frac{\dot{\rho}_2}{\rho_2}+\frac{i}{\rho_2^2}\right)X_2^2\right], \qquad (61)
\end{aligned}
$$

where H_{n_1} and H_{n_2} are the usual Hermite polynomial of order n_1 and n_2 respectively.

The solutions of the Schrödinger equation

$$
i\hbar\frac{\partial\chi_{n_1 n_2}(X_1, X_2, t)}{\partial t} = \hat{H}_3(\hat{X}_1, \hat{X}_2, t)\chi_{n_1 n_2}(X_1, X_2, t), \qquad (62)
$$

can be written as

$$
\chi_{n_1 n_2}(X_1, X_2, t) = e^{i\alpha_{n_1 n_2}(t)}\xi_{n_1 n_2}(X_1, X_2, t), \qquad (63)
$$

where the phase functions $\alpha_{n_1 n_2}(t)$ satisfy the equation

$$
\frac{\partial}{\partial t}\alpha_{n_1 n_2}(t) = \frac{1}{\hbar}\left\langle \xi_{n_1 n_2}(X_1, X_2, t)\right| \frac{\partial}{\partial t} - \hat{H}_3(\hat{X}_1, \hat{X}_2, t)\left|\xi_{n_1 n_2}(X_1, X_2, t)\right\rangle . \qquad (64)
$$

According to Eqs. (61) and (63), the solutions $\chi_{n_1 n_2}(X_1, X_2, t)$ of the Schrödinger equation (62), in the transformed system, becomes

$$
\begin{aligned}
\chi_{n_1 n_2}(X_1, X_2, t) = & e^{i\alpha_{n_1 n_2}(t)}\left[\frac{1}{\pi\hbar n_1! n_2! 2^{n_1+n_2}\rho_1\rho_2}\right]^{1/2} \\
& \times H_{n_1}\left(\frac{X_1}{\hbar^{1/2}\rho_1}\right) H_{n_2}\left(\frac{X}{\hbar^{1/2}\rho_2}\right) \\
& \times \exp\left[\frac{i}{2\hbar}\left(\frac{\dot{\rho}_1}{\rho_1}+\frac{i}{\rho_1^2}\right)X_1^2 + \frac{i}{2\hbar}\left(\frac{\dot{\rho}_2}{\rho_2}+\frac{i}{\rho_2^2}\right)X_2^2\right], \qquad (65)
\end{aligned}
$$

where the time-dependent phase functions are given by

$$
\alpha_{n_1 n_2}(t) = -\left(n_1+\frac{1}{2}\right)\int_0^t \frac{dt'}{\rho_1^2(t')} - \left(n_2+\frac{1}{2}\right)\int_0^t \frac{dt'}{\rho_2^2(t')}. \qquad (66)
$$

The relation between the wave functions, $\Psi_{n_1 n_2}(X_1, X_2, t)$, in the original system described by the Hamiltonian of Eq. (3) and the wave functions $\chi_{n_1 n_2}(X_1, X_2, t)$ in the transformed system is

$$
\begin{aligned}
\Psi_{n_1 n_2}(X_1, X_2, t) &= \hat{U}_1(t)\hat{U}_2(t)\hat{V}(t)\chi_{n_1 n_2}(X_1, X_2, t) \\
&= \hat{U}_1(t)\hat{U}_2(t)\hat{V}_1(t)\hat{V}_2(t)\hat{V}_3(t)\chi_{n_1 n_2}(X_1, X_2, t). \qquad (67)
\end{aligned}
$$

Using Eqs. (42), (46), (50) and (65), we derive the full wave functions in the form

$$
\begin{aligned}
\Psi_{n_1 n_2}(X_1, X_2, t) = & \left[\frac{\sqrt{m_1 m_2}}{\pi\hbar n_1! n_2! 2^{n_1+n_2}\rho_1\rho_2}\right]^{1/2} \\
& \times H_{n_1}\left(\frac{\sqrt{m_1}\cos(\phi+\theta/2)\,X_1 - \sqrt{m_2}\sin(\phi+\theta/2)\,X_2}{\hbar^{1/2}\rho_1}\right) \\
& \times H_{n_2}\left(\frac{\sqrt{m_1}\sin(\phi+\theta/2)\,X_1 + \sqrt{m_2}\cos(\phi+\theta/2)\,X_2}{\hbar^{1/2}\rho_2}\right) \\
& \times \exp\frac{im_1}{2\hbar}\left(\frac{\gamma}{2}+\frac{\beta}{2}+\left(\frac{\beta}{2}-\frac{\gamma}{2}\right)\sin(\theta+2\phi)\right)X_1^2 \\
& \times \exp\frac{im_2}{2\hbar}\left(\frac{\gamma}{2}+\frac{\beta}{2}-\left(\frac{\beta}{2}-\frac{\gamma}{2}\right)\sin(\theta+2\phi)\right)X_2^2 \\
& \times \exp\frac{i}{2\hbar}\sqrt{m_1 m_2}\,((\beta-\gamma)\cos(\theta+2\phi))\,X_1 X_2 \\
& \times \exp i\left[-\left(n_1+\frac{1}{2}\right)\int_0^t \frac{dt'}{\rho_1^2(t')} - \left(n_2+\frac{1}{2}\right)\int_0^t \frac{dt'}{\rho_2^2(t')}\right], \qquad (68)
\end{aligned}
$$

where the time-dependent coefficients $\gamma(t)$ and $\beta(t)$ are given as

$$\gamma(t) = \left(\frac{\dot{\rho}_1}{\rho_1} + \frac{i}{\rho_1^2} - \frac{1}{2}\frac{d}{dt}\sqrt{m_1 m_2}\right), \qquad (69)$$

$$\beta(t) = \left(\frac{\dot{\rho}_2}{\rho_2} + \frac{i}{\rho_2^2} - \frac{1}{2}\frac{d}{dt}\sqrt{m_1 m_2}\right). \qquad (70)$$

The full solutions in the original system, given in Eq. (68), are exact since we did not use approximation or perturbation methods. Though these solutions are somewhat complicated, they are very useful in predicting the quantum behavior of the system. A merit of such analytical solutions is that they can be employed in deriving the evolution of the probability distribution, regardless of the change of the system's parameters. However, the numerical solutions in this field, such as the one obtained from FDTD (finite difference time domain) method[28], are somewhat inconvenient as inputs to further analyses, since one should recalculate the results whenever the parameters of the system changes. Using Eq. (68), one can easily take a complete description of the charged particle motion even when the parameters of the system vary from time to time provided that the classical solutions of Eqs. (57) and (58) are known.

6 Conclusion

We investigated the quantal problem of the time-dependent coupled oscillator model associated to the charged particle motion in the presence of time-dependent magnetic field. Though the behavior of charged particle in magnetic field drew great concern in both quantum and classical view point, researches in this line are rather concentrated on static problems that can be modeled by time-*in*dependent harmonic oscillator.

The system we treated in this work is however a more generalized one. It is summarized as follows:
(i) We supposed that the effective mass of the charged particle varies explicitly with time under the influence of the time-dependent magnetic field. If electrons or holes in the condensed matter interact with environment or various excitations such as pressure, energy, temperature, and stress, their effective mass may naturally vary with time[14]. Moreover, the random changes of the external field in the heterojunctions and solid solutions give rise to the variation of effective mass in accordance with the fluctuation of the composition in the system[29].
(ii) We let the external magnetic field $B(t)$ be an *arbitrary* function of time. Therefore, the application of our theory is not confined in a special system that has a specific class of time-dependence for $B(t)$. In fact, we can apply it in wide range of practical systems with the flexible choice of the type of $B(t)$.
(iii) Our system is further generalized by adding a coupling term X_1X_2 in the Hamiltonian.

Through these generalization, the system became a somewhat complicated one that is described in terms of time-dependent Hamiltonian. Since the treatment of the original Hamiltonian system is not an easy task in this case, we transformed our system to that of a much more simplified one using two different techniques. In the first one, we carried out canonical transformations in order to simplify the problem relevant to the original classical Hamiltonian given in Eq. (1). After the transformation, the Hamiltonian reduced to a simple form associated to two uncoupled harmonic oscillators that each have time-dependent frequencies $\Omega_1(t)$ and $\Omega_2(t)$. In the second technique we used an alternative approach on the basis of the unitary transformation method. With the choice of unitary operators $\hat{U}_1(t)$, $\hat{U}_2(t)$ and $\hat{V}(t)$, the quantum Hamiltonian (39) has been transformed to an equally simple one as that of the canonical transformation previously performed, but within the realm of quantum mechanics.

Since the Hamiltonian in the transformed system is very simple, we easily constructed dynamical invariant operator $\hat{I}(\hat{X}_1, \hat{X}_2, t)$ associated to the transformed system, as given in Eq. (55). The eigenstates $\xi_{n_1n_2}(X_1, X_2, t)$ of this invariant operator are represented in terms of the Hermite polynomial. The Schrödinger solutions $\chi_{n_1n_2}(X_1, X_2, t)$ in the transformed system are the same as $\xi_{n_1n_2}(X_1, X_2, t)$ except for the time-dependent phase factor $e^{i\alpha_{n_1n_2}(t)}$. From the inverse transformation of $\chi_{n_1n_2}(X_1, X_2, t)$ with the unitary operators, we derived the full wave functions (quantum solutions) in the original system [see Eq. (68)]. The quantum solutions are expressed in terms of ρ_1 and ρ_2 that are the two independent solutions of the classical equation of motion given in Eqs. (56) and (57), respectively. Even if we represented the quantum solutions in terms of the classical solutions associated with the *transformed system*, it is also possible to represent them in terms of the classical solutions associated with *original system*. The wave functions given in Eq. (68) can be used to investigate various quantum properties of the system such as the fluctuations of canonical variables, the evolution of quantum energy, and probability densities, even when the parameters of the system vary from time to time. This is the advantage of such analytical solutions over numerical solutions obtained, for example, using the FDTD method[28].

Acknowledgements
The work of J. R. Choi was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (No. 2010-0016914).

References

[1] Bateman H 1931 *Phys. Rev.* **38** 815.
[2] Kim Y S, Noz M E and Oh S H 1979 *Am. J. Phys.* **47** 892
[3] Han D, Kim Y S and Noz M E 1989 *Phys. Lett. A* **144** 111
[4] Kim Y S and Wigner E P 1990 *Phys. Lett. A* **147** 343
[5] Han D, Kim Y S and Noz M E 1995 *J. Math. Phys.* **36** 3940
[6] Abdalla M S 1996 *J. Phys A: Math. Gen.* **29** 1997
[7] Benamira F and Ghechi L 2003 *Czechoslovac J. Phys.* **9** 717
[8] Bykov A A, Gusev G M, Leite J R, Bakarov A K, Goran A V, Kudryashev V M and Toropov A I 2001 *Phys. Rev. B* **65** 035302
[9] van der Wiel W G, Nazarov Yu V, De Franceschi S, Fujisawa T, Elzerman J M, Huizeling E W G M, Tarucha S and Kouwenhoven L P 2003 *Phys. Rev. B* **67** 033307
[10] Varma R K 2003 *Phys. Rep.* **378** 301
[11] Calvo M 2004 *Ultramicroscopy.* **99** 179
[12] Kennedy T A, Wagner R, McCombe B and Tsui D 1975 *Phys. Rev. Lett.* **35** 1031
[13] Sadiek G, Lashin E I and Abdalla M S 2009 *Physica B: Condens. Matter.* **404** 1719
[14] Choi J R 2003 *J. Phys: Condens. Matter.* **15** 823

[15] S. Menouar, M. Maamache, and J. R. Choi, Gaussian wave packet for time-dependent harmonic oscillator model of a charged particle in a variable magnetic field, unpublished.
[16] Laroze D and Rivera R 2006 *Phys. Lett. A* **355** 348
[17] Abdalla M S and Choi J R 2007 *Ann. Phys.* **322** 2795
[18] Schrade G, Man'ko V I, Schleich W P and Glauber R J 1995 *Quantum Semiclass. Opt.* **7** 307
[19] Han D, Kim Y S and Noz M E 1990 *Phys. Rev. A* **41** 6233
[20] Caves C M and Schumaker B L 1985 *Phys. Rev. A* **31** 3068
[21] Schweber S. S 1961 "An Introduction to Relativistic Quantum Field Theory"(Row-Peterson, Elmsford, New York)
[22] Zhang S, Choi J R, Um C I and Yeon K H 2001 *Phys. Lett. A* **289** 257
[23] Zhang S, Choi J R, Um C I and Yeon K H 2002 *Phys. Lett. A* **294** 319
[24] Lewis H R Jr 1968 *Math.Phys,* **9** 1976
[25] Lewis H R Jr. and Riesenfeld W B 1969 *J. Math. Phys.* **10** 1458
[26] Lewis H R Jr 1967 *Phys. Rev.Lett.***18** 510
[27] Goldstein H 1980 Classical Mechanics (Addison-Wesley, Reading, MA)
[28] Sudiarta I W and Geldart D J W 2008 *Phys. Lett. A* **372** 3145
[29] Gevorkian Z S and Lozovik Y E 1988 *J. Phys. C: Solid State Phys.* **21** 2181