Zeno Paradox for Bohmian Trajectories: The Unfolding of the Metatron

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Abstract

We study an analogue of the quantum Zeno paradox for the Bohm trajectory of a sharply located particle (or a system of particles). We show that a continuously observed Bohm trajectory is the classical trajectory predicted by Hamiltonian mechanics.

1 Introduction

Einstein writes to Bohm in 1954,

> *I am glad that you are deeply immersed seeking an objective description of the phenomena and that you feel the task is much more difficult as you felt hitherto. You should not be depressed by the enormity of the problem. If God had created the world his primary worry was certainly not to make its understanding easy for us. I feel it strongly since fifty years.*[10]

When David Bohm completed his book, "Quantum Theory" [3], which was an attempt to present a clear account of Bohr's actual position, he became dissatisfied with the overall approach [5]. The reason for this dissatisfaction was the fact that the theory had no place in it for an adequate notion of an independent actuality, that is of an actual movement or activity by which one physical state could pass over into another.

In a meeting with Einstein, ostensibly to discuss the content of his book, the conversation eventually turned to the possibility of whether a deterministic extension of quantum mechanics could be found. Later while exploring the WKB approximation, Bohm realised that this approximation was giving an essentially deterministic approach. Surely truncating a series cannot turn a probabilistic theory into a deterministic theory. Thus by retaining all the terms in the series, Bohm found that one could, indeed, obtain a deterministic description of quantum phenomena. To carry this through, he had to assume that a quantum particle actually *had* a well defined but unknown position and momentum and followed a well-defined trajectory.

In Bohm's approach, the Schrödinger equation can be cast into a form that brings out its close relationship to the classical Hamilton-Jacobi theory, the only difference being an additional term which can be regarded as a new quality of energy, called the 'quantum potential energy'. It is the properties of this energy that enables us to account for all quantum phenomena such as, for example, the two-slit interference effect where the trajectories are shown to undergo a non-classical behaviour [32].

In this paper we will show that if we continuously observe a Bohm trajectory, it becomes a classical trajectory. Thus, in a sense, continuous observation "dequantizes" quantum trajectories. This property is, of course, essentially a consequence of the quantum Zeno effect, which has been shown to inhibit the decay of unstable quantum systems when under continuous observation (see [7, 11, 17, 18]).

The idea lying behind the Bohm approach (Bohm and Hiley [7], Hiley [21], Hiley and collaborators [24, 25], Holland [26]) is the following: let $\Psi = \Psi(\mathbf{r}, t)$ be a wavefunction solution of Schrödinger's equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla_{\mathbf{r}}^2 + V(\mathbf{r})\right]\Psi.$$

Writing Ψ in polar form $e^{iS/\hbar}\sqrt{\rho}$ Schrödinger's equation is equivalent to the coupled systems of partial differential equations:

$$\frac{\partial S}{\partial t} + \frac{(\nabla_{\mathbf{r}}S)^2}{2m} + V(\mathbf{r}) + Q^{\Psi}(\mathbf{r}, t) = 0 \tag{1}$$

where

$$Q^{\Psi} = -\frac{\hbar^2}{2m}\frac{\nabla_{\mathbf{r}}^2\sqrt{|\Psi|}}{\sqrt{|\Psi|}}. \tag{2}$$

is Bohm's quantum potential (equation (1) is thus mathematically a Hamilton-Jacobi equation), and

$$\frac{\partial\rho}{\partial t} + \nabla_{\mathbf{r}}\left(\rho\frac{\nabla_{\mathbf{r}}S}{m}\right) = 0 \tag{3}$$

which is an equation of continuity. The trajectory of the particle is determined by the equation

$$m\dot{\mathbf{r}}^{\Psi} = \nabla_{\mathbf{r}} S(\mathbf{r}^{\Psi}, t) \quad , \quad \mathbf{r}^{\Psi}(t_0) = \mathbf{r}_0 \tag{4}$$

where $\mathbf{r}_0$ is the initial position.

Since the quantum potential depends only on the wave function, and the latter is ultimately a property of the *metaplectic* representation [13, 14, 16], we proposed in [15] to call the entity whose motion is governed by the equation (4) a *metatron*. We chose this name because the 'particle', rather than being a classical object, is essentially an excitation induced by the metaplectic representation of the underlying Hamiltonian evolution.

The question we will answer in this paper is the following:

> *What do we see if we perform a continuous observation of the metatron's trajectory ?*

If the observed trajectory is smooth, we will see the *classical* trajectory determined by the Hamiltonian function

$$H(\mathbf{r}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}).$$

Does this mean that Bohmian trajectories are therefore not "real", that they are "surrealistic"? *No*, they are not surreal simply because we are making a distinction between what *is*, and what is *observed* by a *physical* measurement. For example, in the two-slit experiment referred to above, we find that if we observe the motion of the 'particle' as it passes through one of the slits we will see no wave-like behaviour, but a classical trajectory showing on interference effects.

We are often asked if Bohm believed that there was an actual classical point-like particle following these quantum trajectories. For Bohm there was no solid 'particle' but instead, at the fundamental level, there was a basic process or activity so that the "track" left in, say, a bubble chamber could be explained by an *enfolding–unfolding* of this process [6]. Thus rather than seeing the track as the continuous movement of a material particle, it can be regarded as the continuity of a "quasi-local, semi-stable autonomous form" evolving within the unfolding process [21]. As we will see, this is exactly what happens when we observe continuously the unfolding process, and we can regard the visible track as arising from the evolution of the *metatron*.

2 Bohmian Trajectories Are Hamiltonian

We will show that in the particular case of a metatron initially localized at a point, the Bohm trajectory is Hamiltonian (the general case is slightly more subtle; we refer to the papers by Holland [27, 28] for a thorough discussion of the interpretation of Bohmian trajectories from the Hamiltonian point of view).

We will consider systems of N material particles with the same mass m, and work in generalized coordinates $x = (q_1, ..., q_n)$ and $p = (p_1, ..., p_n)$, $n = 3N$. Suppose that this system is sharply localized at a point $x_0 = (q_{1,0}, ..., q_{n,0})$ at time t_0. The classical Hamiltonian function is

$$H(x,p) = \frac{p^2}{2m} + V(x) \tag{5}$$

hence the organising field of this system is the solution of the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla_x^2 + V(x)\right]\Psi \quad , \quad \Psi(x,t_0) = \delta(x - x_0) \tag{6}$$

where ∇_x is the n-dimensional gradient in the variables $q_1, ..., q_n$. The function Ψ is thus just the propagator $G(x, x_0; t, t_0)$ of the Schrödinger equation. We write G in polar form

$$G(x, x_0; t, t_0) = \sqrt{\rho(x, x_0; t, t_0)}e^{\frac{i}{\hbar}S(x,x_0;t,t_0)}.$$

The equation of motion (4) is in this case

$$m\dot{x}^\Psi = \nabla_x S(x^\Psi, x_0; t, t_0) \quad , \quad x^\Psi(t_0) = x_0. \tag{7}$$

2.1 Short-time estimates

We are going to give a short-time estimate for the function S. The interest of this estimate is two-fold: it will not only allow us to give a precise statement of the Zeno effect for Bohmian trajectories, but it will also allow us to prove in detail the Hamiltonian character of these trajectories.

We will assume that the potential V is at least twice continuously differentiable in the variables $q_1, ..., q_n$.

In [15], Chapter 7, we established the following short-time formulas for $t - t_0 \to 0$ (a similar formula has been obtained in [29, 30, 34, 35]):

$$S(x, x_0; t, t_0) = \sum_{j=1}^{n} \frac{m(q_j - q_{0,j})^2}{2(t - t_0)} - \tilde{V}(x, x_0)(t - t_0) + O((t - t_0)^2) \tag{8}$$

where $\tilde{V}(x,x')$ is the average value of the potential on the line segment $[x',x]$:

$$\tilde{V}(x,x_0)=\int_0^1 V(\lambda x+(1-\lambda)x_0)d\lambda. \tag{9}$$

We observe that the quantum potential is absent from formula (8); we would actually have obtained the same approximation if we had replaced S with the solution to the classical Hamilton–Jacobi equation

$$\frac{\partial S_{\mathrm{cl}}}{\partial t}+\frac{(\nabla_x S)^2}{2m}+V(x)=0$$

while S is a solution of the quantum Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t}+\frac{(\nabla_x S)^2}{2m}+V(x)+Q^{\Psi}(x,t)=0. \tag{10}$$

How can this be? The reason is that if we replace the propagator $G(x,x_0;t,t_0)$ by its "classical" approximation

$$G_{\mathrm{cl}}(x,x_0;t,t_0)=\sqrt{\rho_{\mathrm{cl}}(x,x_0;t,t_0)}e^{\frac{i}{\hbar}S_{\mathrm{cl}}(x,x_0;t,t_0)}$$

where ρ_{cl} is the Van Vleck density (i.e. the determinant of the matrix of second derivatives of S_{cl}) then we have

$$G(x,x_0;t,t_0)-G_{\mathrm{cl}}(x,x_0;t,t_0)=O((t-t_0)^2)$$

(cf. Lemma 241 in [15]) from which follows that

$$-\frac{\hbar^2}{2m}\frac{\nabla_x^2 G}{G}-\left(-\frac{\hbar^2}{2m}\right)\frac{\nabla_x^2 G_{\mathrm{cl}}}{G_{\mathrm{cl}}}=O((t-t_0)^2);$$

the difference between the two terms $O((t-t_0)^2)$ in is thus absorbed by the term (8). [We take the opportunity to remark that when the potential $V(x)$ is quadratic in the position variables $q_1,...,q_n$ then $G_{\mathrm{cl}}=G$; we will come back to this relation later in section 3.1].

Moreover, formula (8) can be twice continuously differentiated with respect to the variables q_j and $q_{0,j}$. It follows that the second derivatives of S are given by

$$\frac{\partial^2 S}{\partial q_j \partial q_{0,k}}=\frac{m}{t-t_0}\delta_{jk}+O(t-t_0)$$

and hence the Hessian matrix S_{x,x_0} (i.e. the matrix of mixed second derivatives) satisfies

$$\det S_{x,x_0}=\left(\frac{m}{t-t_0}\right)^n+O(t-t_0). \tag{11}$$

Formula (8) is the key to the following important asymptotic version of Bohm's equation (7):

$$\dot{x}^{\Psi} = \frac{x^{\Psi} - x_0}{t - t_0} - \frac{1}{2m}\nabla_x V(x_0)(t - t_0) + O((t - t_0)^2)). \tag{12}$$

Let us prove this formula. Using the expansion (8), formula (7) becomes

$$\dot{x}^{\Psi} = \frac{x^{\Psi} - x_0}{t - t_0} - \frac{1}{m}\nabla_x \tilde{V}(x^{\Psi}, x_0)(t - t_0) + O((t - t_0)^2). \tag{13}$$

Let us show that

$$\nabla_x \tilde{V}(x^{\Psi}, x_0)(x^{\Psi}, x_0) = \frac{1}{2}\nabla_x V(x_0) + O(t - t_0); \tag{14}$$

this will complete the proof of formula (12). We first note that (13) implies in particular that

$$\dot{x}^{\Psi} = \frac{x^{\Psi} - x_0}{t - t_0} + O(t - t_0)$$

and thus x^{Ψ} is given by

$$x^{\Psi}(t) = x_0 + \frac{p_0}{m}(t - t_0) + O((t - t_0)^2) \tag{15}$$

where p_0 is an arbitrary constant vector. In particular we have $O(x^{\Psi} - x_0) = O(t - t_0)$ and hence

$$\begin{aligned} \nabla_x \tilde{V}(x^{\Psi}, x_0) &= \nabla_x \tilde{V}(x_0, x_0) + O(x^{\Psi} - x_0) \\ &= \nabla_x \tilde{V}(x_0, x_0) + O(t - t_0) \end{aligned}$$

from which it follows that

$$\begin{aligned} \nabla_x \tilde{V}(x^{\Psi}, x_0)(x^{\Psi}, x_0) &= \int_0^1 \lambda \nabla_x V(\lambda x_0 + (1 - \lambda)x_0)d\lambda + O(t - t_0) \\ &= \frac{1}{2}\nabla_x V(x_0) + O(t - t_0) \end{aligned}$$

which is precisely the estimate (14).

2.2 The Hamiltonian character of Bohmian trajectories

Let $p_0 = (p_{1,0}, ..., p_{n,0})$ be an arbitrary momentum vector, and set

$$p_0 = -\nabla_{x_0} S(x, x_0; t, t_0). \tag{16}$$

In view of formula (11), the Hessian of S in the variables x and x_0 is invertible for small values of t, hence the implicit function theorem implies that (16) determines a function $x = x(t)$ (depending on x_0 and t_0 viewed as parameters), defined by

$$p_0 = -\nabla_{x_0} S(x(t), x_0; t, t_0). \tag{17}$$

Setting

$$p(t) = \nabla_x S(x(t), x_0; t, t_0) \tag{18}$$

we claim that the functions $x(t)$ and $p(t)$ thus defined are solutions of the Hamilton equations

$$\dot{x} = \nabla_p H^\Psi(x, p, t) \ \ , \ \ \dot{p} = -\nabla_x H^\Psi(x, p, t) \tag{19}$$

and that we have $x(t_0) = x_0$, $p(t_0) = p_0$. We are actually going to use classical Hamilton–Jacobi theory (see [2, 12, 15, 16] or any introductory text on analytical mechanics). For notational simplicity we assume that $n = 1$. The function S satisfies the equation

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V(x) - \frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\frac{\partial^2 \sqrt{\rho}}{\partial x^2} = 0; \tag{20}$$

introducing the quantum potential

$$Q^\Psi = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\frac{\partial^2 \sqrt{\rho}}{\partial x^2} \tag{21}$$

we set $H^\Psi = H + Q^\Psi$ so that (20) is just the quantum Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H^\Psi\left(x, \frac{\partial S}{\partial x}, t\right) = 0. \tag{22}$$

Differentiating the latter with respect to $p = \partial S/\partial x$ yields, using the chain rule,

$$\frac{\partial^2 S}{\partial x_0 \partial t} + \frac{\partial H^\Psi}{\partial p}\frac{\partial^2 S}{\partial x_0 \partial x} = 0 \tag{23}$$

and differentiating the equation (17) with respect to time yields

$$\frac{\partial^2 S}{\partial x_0 \partial t} + \frac{\partial^2 S}{\partial x \partial x_0}\dot{x} = 0. \tag{24}$$

Subtracting (24) from (23) we get

$$\frac{\partial^2 S}{\partial x \partial x_0}\left(\frac{\partial H^\Psi}{\partial p} - \dot{x}\right) = 0$$

which produces the first Hamilton equation (19) since it is assumed that we have $\partial^2 S/\partial x \partial x_0 \neq 0$. Let us next show that the second Hamilton equation (19) is satisfied as well. For this we differentiate the quantum Hamilton–Jacobi equation (22) with respect to x, which yields

$$\frac{\partial^2 S}{\partial x \partial t} + \frac{\partial H^\Psi}{\partial x} + \frac{\partial H^\Psi}{\partial p}\frac{\partial^2 S}{\partial x^2} = 0. \tag{25}$$

Differentiating the equality (18) with respect to t we get

$$\frac{\partial^2 S}{\partial t \partial x} = -\dot{p}(t) - \frac{\partial^2 S}{\partial x^2}\dot{x} \tag{26}$$

and hence the equation (25) can be rewritten

$$-\dot{p}(t) - \frac{\partial^2 S}{\partial x^2}\dot{x} + \frac{\partial H^\Psi}{\partial x} + \frac{\partial H^\Psi}{\partial p}\frac{\partial^2 S}{\partial x^2} = 0.$$

Taking into account the relation $\dot{x} = \partial H^\Psi/\partial p$ established above we have

$$-\dot{p}(t) - \frac{\partial H^\Psi}{\partial x} = 0$$

which is precisely the second Hamilton equation (19). There remains to show that we have $x(t_0) = x_0$ and $p(t_0) = p_0$. Recall that $x(t)$ is defined by the implicit equation

$$p_0 = -\nabla_{x_0} S(x(t), x_0; t, t_0)$$

(equation (17)); in view of the short-time estimate (8) this means that we have

$$p_0 = \frac{m(x(t) - x_0)}{t - t_0} + O(t - t_0)$$

and hence we must have $\lim_{t \to t_0} x(t) = x(t_0) = x_0$. This also implies that $p_0 = m\dot{x}(t_0) = p(t_0)$.

In conclusion we have thus shown that:

Bohm's equation of motion (7) is equivalent to Hamilton's equations (19).

To complete our discussion, we make two important observations:

- Even when the Hamiltonian function H does not depend explicitly on time, the function $H^{\Psi} = H + Q^{\Psi}$ is usually time-dependent (because the quantum potential generally is), so the flow (f_t^{Ψ}) it determines does not inherit the usual group property $f_t f_{t'} = f_{t+t'}$ of the flow determined by the classical Hamiltonian H. One has instead to use the "time-dependent flow" $(f_{t,t'}^{\Psi})$, which has a groupoid property in the sense that $f_{t,t'}^{\Psi} f_{t',t''}^{\Psi} = f_{t,t''}^{\Psi}$.

- The time-dependent flow $(f_{t,t'}^{\Psi})$ consists of canonical transformations; that is, the Jacobian matrix of $f_{t,t'}^{\Psi}$ calculated at any point (x,p) where it is defined by a symplectic matrix. This is an immediate consequence of the fact discussed above, namely, that the flow determined by *any* Hamiltonian function has this property.

We have seen that the Bohmian trajectory for a particle initially sharply localized at a point x_0 is Hamiltonian, and in fact governed by the Hamilton equations (19):

$$\dot{x} = \nabla_p H^{\Psi}(x,p,t) \quad , \quad \dot{p} = -\nabla_x H^{\Psi}(x,p,t). \tag{27}$$

The discussion of short-time solutions of Bohm's equation of motion allows us to give approximations to the solution. First, the solutions of the equation $\dot{x} = \nabla_p H^{\Psi}(x,p,t)$ are given by the simple relation

$$x^{\Psi}(t) = x_0 + \frac{p_0}{m}(t-t_0) + O((t-t_0)^2)$$

as was already noticed in (15). Then we proved that the momentum $p^{\Psi}(t) = m\dot{x}^{\Psi}(t)$ is given by equation (12):

$$m\dot{x}^{\Psi}(t) = \frac{m(x^{\Psi}(t)-x_0)}{t-t_0} - \frac{1}{2}\nabla_x V(x_0)(t-t_0) + O((t-t_0)^2). \tag{28}$$

However we cannot solve this equation by inserting the value of $x^{\Psi}(t)$ above since this would lead to an estimate modulo $O(t-t_0)$ not $O((t-t_0)^2)$. What we do is the following: differentiating both sides of the equation (28) with respect to t we get

$$\ddot{x}^{\Psi}(t) = \frac{x^{\Psi}(t)-x_0}{(t-t_0)^2} + \frac{\dot{x}^{\Psi}(t)}{t-t_0} - \frac{1}{2m}\nabla_x V(x_0) + O(t-t_0)$$

that is, replacing $\dot{x}^\Psi(t)$ by the value given by (28),

$$\dot{p}^\Psi(t) = m\ddot{x}^\Psi(t) = -\nabla_x V(x_0) + O(t - t_0).$$

Solving this equation we get

$$p^\Psi(t) = p_0 - \nabla_x V(x_0)(t - t_0) + O((t - t_0)^2).$$

Summarizing, the solutions of the Hamilton equations (27) for $H^\Psi = H + Q^\Psi$ are given by

$$\begin{aligned} x^\Psi(t) &= x_0 + \frac{p_0}{m}(t - t_0) + O((t - t_0)^2) \\ p^\Psi(t) &= p_0 - \nabla_x V(x_0)(t - t_0) + O((t - t_0)^2). \end{aligned}$$

The observant reader will have noticed that (up to the error term $O((t - t_0)^2))$ there is no trace of the quantum potential Q^Ψ in these short-time formulas. Had we replaced the function H^Ψ with the classical Hamiltonian H we would actually have obtained exactly the same solutions, up to the $O((t - t_0)^2)$ term.

3 Bohmian Zeno Effect

3.1 The case of quadratic potentials

Here is an easy case; it is in fact so easy that it is slightly misleading: the Bohmian trajectories are here classical trajectories from the beginning, because the quantum potential vanishes.

Let us assume that the potential $V(x)$ is a quadratic form in the position variables, that is

$$V(x) = \frac{1}{2}Mx \cdot x$$

where M is a symmetric matrix. Using the theory of the metaplectic representation [13, 14, 15, 16] it is well-known that the propagator G is given by the formula

$$G(x, x_0; t, t_0) = \left(\tfrac{1}{2\pi i\hbar}\right)^{n/2} i^{m(t,t_0)} \sqrt{|\rho(t, t_0)|} e^{\frac{i}{\hbar}W(x,x_0;t,t_0)} \tag{29}$$

where $W(x, x_0; t, t_0)$ is Hamilton's two-point characteristic function (see e.g. [2, 12]): it is a quadratic form

$$W = \frac{1}{2}Px \cdot x - Lx \cdot x_0 + \frac{1}{2}Bx_0 \cdot x_0$$

where $P = P(t,t_0)$ and $B = B(t,t_0)$ are symmetric matrices and $L = L(t,t_0)$ is invertible; viewed as function of x it satisfies the Hamilton–Jacobi equation

$$\frac{\partial W}{\partial t} + \frac{(\nabla_x W)^2}{2m} + \frac{1}{2}Mx \cdot x.$$

Moreover, $m(t,t_0)$ is an integer ("Maslov index") and $\rho(t,t_0)$ is the determinant of $L = L(t,t_0)$ (the Van Vleck density). Since $m(t,t_0)$ and $\rho(t,t_0)$ do not depend on x, it follows that the quantum potential Q^Ψ determined by the propagator (29) is zero. Since we have $H^\Psi = H + Q^\Psi$, we see immediately that the quantum motion is perfectly classical in this case: the quantum equations of motion (19) reduce to the ordinary Hamilton equations

$$\dot{x} = \frac{p}{m} \ , \ \dot{p} = -Mx \tag{30}$$

which can be easily integrated: in particular the flow (f_t) they determine is a true flow (because $H = H^\Psi$ is time-independent) and consists of symplectic matrices ([2, 15, 16, 12]). In fact,

$$f_t = e^{tX} \quad , \quad X = \begin{pmatrix} 0_{n\times n} & \frac{1}{m}I_{n\times n} \\ -M & 0_{n\times n} \end{pmatrix}.$$

Thus, in the case of quadratic potentials the Bohmian trajectories associated with the propagator are the usual Hamilton trajectories associated with the classical Hamiltonian function of the problem.

Suppose now that we observe "continuously" the time evolution of the metatron –which is so far "quantum"– and try to find out what is recorded by our observation process. Practically this is done by performing repeated position measurements at very short time intervals Δt. We assume that the recorded trajectory is, in the limit $\Delta t \to 0$, continuous and moreover smooth; by this we mean that we can assign at every point a velocity vector (we are thus excluding Brownian motion-type behavior). Let us choose a time interval $[0,t]$ (typically $t = 1\,\mathrm{s}$) and subdivide it in a sequence of N intervals

$$[0,\Delta t]\ [\Delta t, 2\Delta t]\ [2\Delta t, 3\Delta t] \cdots [(N-1)\Delta t, N\Delta t]$$

with $\Delta t = t/N$; the integer N is assumed to be very large (for instance $N = 10^{18}$). Assume that a measurement at time $t_0 = 0$ localizes the particle at a point x_0 it will be detected at a point x_1 after time Δt; its momentum is p_1 and we have $(x_1, p_1) = f_{\Delta t}(x_0, p_0)$. We now repeat the procedure, replacing x_0 by x_1; since the observed trajectory is assumed to be smooth the initial momentum will be p_1 and after time Δt a new measurement is performed, and we find the particle at x_2 with momentum p_2 such that $(x_2, p_2) = f_{\Delta t}(x_1, p_1) = f_{\Delta t} f_{\Delta t}(x_0, p_0)$. Repeating the same process until time $t = N\Delta t$ we find a series of points in space which the particle takes as positions one after another[1] that $(x_N, p_N) = (f_{\Delta t})^N (x_0, p_0)$. But in view of the group property $f_t f_{t'} = f_{t+t'}$ of the flow we have $(f_{\Delta t})^N = f_{N\Delta t} = f_t$ and hence $(x_N, p_N) = f_t(x_0, p_0)$. The observed Bohmian trajectory is thus the classical trajectory predicted by Hamilton's equations.

3.2 The general case

In generalizing the discussion above to arbitrary potentials, $V(x)$, there are two difficulties. The first is that we do not have exact equations for the Bohmian trajectory, but only short-time approximations. The second is that the Hamilton equations for x^Ψ and p^Ψ no longer determine a flow having a group property because the Hamiltonian H^Ψ is time-dependent. Nevertheless the material we have developed so far is actually sufficient to show that the observed trajectory is the classical one.

The key will be the theory of Lie–Trotter algorithms which is a powerful method for constructing exact solutions from short-time estimates. The method goes back to early work of Trotter [37] elaborating on Sophus Lie's proof of the exponential matrix formula $e^{A+B} = \lim_{N\to\infty} \left(e^{A/N} e^{B/N}\right)^N$; see Chorin et al. [9] for a detailed and rigorous study; we have summarized the main ideas in the Appendix B of [15]); also see Nelson [31]. (We mention that there exists an operator variant of this procedure, called the Trotter–Kato formula.

Let us begin by introducing some notation. We have seen that the datum of the propagator $G_0 = G(x, x_0; t, t_0)$ determines a quantum potential Q^Ψ and thus Hamilton equations (19) associated with $H^\Psi = H + Q^\Psi$. We now choose $t_0 = 0$ and denote the corresponding quantum potential by Q^0 and set $H^0 = H + Q^0$. After time Δt we make a position measurement and find that the particle is located at x_1. The future evolution of the particle is now governed by the new propagator $G_1 = G(x, x_1; t, t_0)$, leading to a new quantum potential Q^1 and to a new Hamiltonian H^1; repeating this until time t we thus have a sequence of points $x_0, x_1, ..., x_N = x$ and a corresponding sequence of Hamiltonian functions $H^0, H^1, ..., H^N$ determined by the quantum potentials $Q^0, Q^1, ..., Q^N$. We denote by (f^0_{t,t_0}), (f^1_{t,t_1}),...,$(f^{N-1}_{t,t_{N-1}})$ the time dependent flows determined by the Hamiltonian functions $H^0, H^1, ..., H^N$;

[1] In conformity with W. Heisenberg's statement: "By path we understand a series of points in space which the electron takes as 'positions' one after another" [19]

we have set here $t_1 = t_0 + \Delta t$, $t_2 = t_1 + \Delta t$ and so on.

Repeating the observation procedure explained in the case of quadratic potentials, we get in this case a sequence of successive equalities

$$\begin{aligned}(x_1, p_1) &= f^0_{t_1,t_0}(x_0, p_0)\\ (x_2, p_2) &= f^1_{t_2,t_1}(x_1, p_1)\\ &\cdots\cdots\cdots..\\ (x, p) &= f^{N-1}_{t,t_{N-1}}(x_{N-1}, p_{N-1})\end{aligned}$$

which implies that the final point $x = x_N$ observed at time t is expressed in terms of the initial point x_0 by the formula

$$(x, p) = f^{N-1}_{t,t_{N-1}} \cdots f^1_{t_2,t_1} f^0_{t_1,t_0}(x_0, p_0).$$

Denote now by (g^0_{t,t_0}), (g^1_{t,t_1}),...,$(g^{N-1}_{t,t_{N-1}})$ the approximate flows determined by the equations

$$\begin{aligned}(x_1, p_1) &= (x_0 + \frac{p_0}{m}\Delta t, p_0 - \nabla_x V(x_0)\Delta t)\\ (x_2, p_2) &= (x_1 + \frac{p_1}{m}\Delta t, p_1 - \nabla_x V(x_1)\Delta t)\\ &\cdots\cdots\cdots\cdot\\ (x, p) &= (x_{N-1} + \frac{p_{N-1}}{m}\Delta t, p_{N-1} - \nabla_x V(x_{N-1})\Delta t).\end{aligned}$$

Invoking the Lie–Trotter formula, the sequence of estimates

$$f^0_{t_k,t_{k-1}}(x_{k-1}, p_{k-1}) - g^0_{t_k,t_{k-1}}(x_{k-1}, p_{k-1}) = O(\Delta t^2)$$

implies that we have

$$\lim_{N\to\infty} g^{N-1}_{t,t_{N-1}} \cdots g^1_{t_2,t_1} g^0_{t_1,0}(x_0, p_0) = \lim_{N\to\infty} f^{N-1}_{t,t_{N-1}} \cdots f^1_{t_2,t_1} f^0_{t_1,0}(x_0, p_0)$$

The argument goes as follows (for a detailed proof see [15]): since we have $g^k_{t_k,t_{k-1}} = f^k_{t_k,t_{k-1}} + O(\Delta t^2)$ the product is approximated by

$$g^{N-1}_{t,t_{N-1}} \cdots g^1_{t_2,t_1} g^0_{t_1,t_0} = f^{N-1}_{t,t_{N-1}} \cdots f^1_{t_2,t_1} f^0_{t_1,t_0} + NO(\Delta t^2)$$

and since $\Delta t = t/N$ we have $NO(\Delta t^2) = O(\Delta t)$ which goes to zero when $N \to \infty$.

Now, recall our remark that the quantum potential is absent from the approximate flows $g^k_{t_k,t_{k-1}}$; using again the Lie–Trotter formula together with short-time approximations to the Hamiltonian flow (f_t) determined by the classical Hamiltonian H, we get

$$\lim_{N\to\infty} g^{N-1}_{t,t_{N-1}} \cdots g^{1}_{t_2,t_1} g^{0}_{t_1,0}(x_0,p_0) = f_t$$

and hence

$$\lim_{N\to\infty} f^{N-1}_{t,t_{N-1}} \cdots f^{1}_{t_2,t_1} f^{0}_{t_1,0}(x_0,p_0) = f_t$$

which shows that the observed trajectory is the classical one.

4 Conclusion.

We have shown that if a quantum particle is watched continuously, it will follow a classical trajectory. In terms of the Bohm model, what this implies is that the quantum potential is forced to remain zero so that no quantum effects can occur. This result supports the conclusions reached for the transition in an Auger-like particle discussed in Bohm and Hiley [7]. There it was shown that the perturbed wave function, which is proportional to t for times less that $1/\Delta E$, (ΔE is the energy released in the transition) will never become large and therefore cannot make a significant contribution to the quantum potential. For this reason no transition will take place.

From these results we see that in the Bohm approach, it is the magnitude of the quantum potential energy that distinguishes the quantum behaviour from the classical. Indeed this conclusion is quite obvious if we examine equation (1) since when Q is negligible compared with the kinetic energy, the equation is simply the classical Hamilton-Jacobi equation. Hiley and Aziz Mufti [20] give an interesting demonstration of how the quantum potential can become negligible over time. They give a simplified cosmological example of how, in an inflationary scenario, quantum behaviour can become classical at later stages of the inflation.

This last example provides us with a very different way of arriving at the classical limit than the prevailing view based on decoherence. In our view the main difficulty in using decoherence is that it merely destroys the off-diagonal elements of the density matrix but it does not give rise to the classical equations of motion. It continues to describe classical objects by wave functions, a criticism that has already been made by Primas [33]. Furthermore it does not show how the Schrödinger equation becomes Hamilton's equations of motion. Our method shows that it is the relation between the symplectic and metaplectic representations that shows how the classical is related to the quantum. It is when the global properties of the covering group become unimportant that the classical world emerges. As has been pointed out by Hiley [22] [23], the Bohm approach is much closer to the Moyal approach using a deformation Poisson algebra. In the Moyal approach the classical limit emerges in a very simple way, namely, in those situations where the deformation parameter can be considered to be small.

References

[1] Abraham, R. and Marsden, J.E.: Foundations of Mechanics. The Benjamin/Cummings Publishing Company, 2nd edition, (1978)

[2] Arnold, V.I.: Mathematical Methods of Classical Mechanics. Graduate Texts in Mathematics, second edition, Springer-Verlag, (1989)

[3] Bohm, D., *Quantum Theory* Prentice-Hall, Englewood Cliffs, N. J., 1951.

[4] Bohm, D.: A Suggested Interpretation of the Quantum Theory in Terms of Hidden Variables. Phys. Rev. 85 (1952), 166–179, 180–193

[5] Bohm, D., Hidden Variables and the Implicate Order, in Hiley, B. J. and Peat, D. F., *Quantum Implications: Essays in Honour of David Bohm*, Routledge, London, 1987.

[6] Bohm, D., *Wholeness and the Implicate Order*, Routledge, London, 1980.

[7] Bohm, D., Hiley, B.: The Undivided Universe: An Ontological Interpretation of Quantum Theory. London & New York: Routledge (1993)

[8] Bohm, D., Hiley, B.: An Ontological Basis for the Quantum Theory: I-Nonrelativistic Particle Systems. Phys. Reports 144 (1987), 323–348

[9] Chorin, A.J., Hughes T.J.R, McCracken, M.F., Marsden, J.E.: Product formulas and numerical algorithms. Comm. Pure and Appl. Math. 31(2) (1978), 205–256

[10] Einstein, A., from a letter to Bohm in São Paulo, February 10 1954.

[11] Facchi, P. and Pascazio, S.: Quantum Zeno dynamics: mathematical and physical aspects. J. Phys. A 41(49) (2008), 493001–493005

[12] Goldstein, H.: Classical Mechanics. Addison–Wesley, (1950), 2nd edition, (1980), 3d edition, (2002).

[13] de Gosson, M.: Maslov Classes, Metaplectic Representation and Lagrangian Quantization. Research Notes in Mathematics 95, Wiley–VCH, Berlin, 1997.

[14] de Gosson, M.: On the classical and quantum evolution of Lagrangian half-forms in phase space. Ann. Inst. H. Poincaré, 70(6) (1999), 547–573

[15] de Gosson, M.: The Principles of Newtonian and Quantum Mechanics: The need for Planck's constant, h. With a foreword by Basil Hiley. Imperial College Press (2001)

[16] de Gosson, M.: Symplectic Geometry and Quantum Mechanics. Birkhäuser, Basel, series "Operator Theory: Advances and Applications" (subseries: "Advances in Partial Differential Equations"), Vol. 166 (2006)

[17] Gustafson, K.: A Zeno story. arXiv:quant-ph/0203032

[18] Hannabuss, K.C.: An introduction to quantum theory. Oxford graduate texts in mathematics; 1. Oxford (1997)

[19] Heisenberg, W.: Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, *Zeit. fur Phsik.*, **43**,(1927) 172-98.

[20] Hiley, B.J. and Aziz Mufti, A.H., The Ontological Interpretation of Quantum Field Theory applied in a Cosmological Context, *Fundamental Problems in Quantum Physics*, ed., M. Ferrero and A. van der Merwe, pp.141-156, Kluwer, Dordrecht. (1995)

[21] Hiley, B.J.: Non-Commutative Geometry, the Bohm Interpretation and the Mind-Mater Relationship. In Proc. CASYS'2000, Liège, Belgium, Aug. 7–12, 2000

[22] Hiley, B.J., Phase Space Descriptions of Quantum Phenomena, *Proc. Int. Conf. Quantum Theory: Reconsideration of Foundations 2*, 267-86, ed. Khrennikov, A., Växjö University Press, Växjö, Sweden, 2003

[23] Hiley, B.J., On the Relationship between the Wigner-Moyal and Bohm Approaches to Quantum Mechanics: A step to a more General Theory. *Found. Phys.*, **40** (2010) 365-367. DOI 10.1007/s10701-009-9320-y

[24] Hiley, B.J. and Callaghan, R.E.: Delayed-choice experiments and the Bohm approach. Phys. Scr. 74 (2006), 336–348

[25] Hiley, B.J., Callaghan, R.E., Maroney O.J.E.: Quantum trajectories, real, surreal or an approximation to a deeper process? Arxiv preprint quant-ph/0010020, (2000)

[26] Holland, P.R.: The quantum theory of motion. An account of the de Broglie-Bohm causal interpretation of quantum mechanics. Cambridge University Press, Cambridge (1995)

[27] Holland, P.: Hamiltonian theory of wave and particle in quantum mechanics I: Liouville's theorem and the interpretation of the de Broglie-Bohm theory. Nuovo Cimento B 116 (2001), 1043–1070

[28] Holland, P.: Hamiltonian theory of wave and particle in quantum mechanics II: Hamilton-Jacobi theory and particle back-reaction. Nuovo Cimento B 116, 1143–1172 (2001)

[29] Makri. N., Miller W.H.: Correct short time propagator for Feynman path integration by power series expansion in Δt. Chemical Phys. Lett. 151(1) (1988), 1–8

[30] Makri. N., Miller W.H.: Exponential power series expansion for the quantum time evolution operator. J. Chem. Phys. 90(2) (1989), 904–911

[31] Nelson, E.: Topics in Dynamics I: Flows. Mathematical Notes, Princeton University Press (1969)

[32] Philippidis, C., Dewdney, C. and Hiley, B. J., Quantum Interference and the Quantum Potential, *Nuovo Cimento*, **52B**, (1979), 15-28.

[33] Primas, H., *Chemistry, Quantum Mechanics and Reductionism,* Springer, Berlin (1983)

[34] Schiller, R.: Quasi-Classical Theory of the Nonspinning Electron. Phys. Rev. 125 (1962), 1100–1108

[35] Schiller, R.: Quasi-Classical Transformation Theory. Phys. Rev. 125 (1962), 1109–1115

[36] Schrödinger, E.: The Interpretation of Quantum Mechanics. Dublin (1949–1955) and other unpublished essays. Edited and with Introduction by Michel Bitbol. Ox Bow Press, Woodbridge, CT (1995)

[37] Trotter, H.F.: On the product of semi-groups of operators. Proc. Amer. Math. Soc. 10 (1959) 545–551