

# Optimal unambiguous comparison of two unknown squeezed vacua

Stefano Olivares,<sup>1,2,\*</sup> Michal Sedlák,<sup>3,4,†</sup> Peter Rapčan,<sup>3</sup> Matteo G. A. Paris,<sup>2,1</sup> and Vladimír Bužek<sup>3,5</sup>

<sup>1</sup>*CNISM UdR Milano Università, I-20133 Milano, Italy*

<sup>2</sup>*Dipartimento di Fisica, Università degli Studi di Milano, I-20133 Milano, Italy*

<sup>3</sup>*Research Centre for Quantum Information, Institute of Physics,  
Slovak Academy of Sciences, Dúbravská cesta 9, 845 11 Bratislava, Slovak Republic*

<sup>4</sup>*QUIT group, Dipartimento di Fisica “A. Volta”, via Bassi 6, I-27100 Pavia, Italy*

<sup>5</sup>*Faculty of Informatics, Masaryk University, Botanická 68a, 602 00 Brno, Czech Republic*

We propose a scheme for unambiguous state comparison (USC) of two unknown squeezed vacuum states of an electromagnetic field. Our setup is based on linear optical elements and photon-number detectors, and achieves optimal USC in an ideal case of unit quantum efficiency. In realistic conditions, i.e., for non-unit quantum efficiency of photodetectors, we evaluate the probability of getting an ambiguous result as well as the reliability of the scheme, thus showing its robustness in comparison to previous proposals.

PACS numbers: 03.67.-a,42.50.Ex

## I. INTRODUCTION

The possibility of creating physical systems with identical properties is crucial for any physical theory that is verifiable by experiments. Comparison of preparators – a procedure of determining whether they prepare the same objects or not – is one of the basic experiments we would like to do when testing a theory, because it allows us to operationally define equivalence of such devices for their further use. In the framework of classical physics, we can in principle measure and determine the state of the system perfectly without disturbing it. Thus, to compare states of two systems it suffices to measure each system separately. However, in quantum theory, due to its statistical nature, we cannot make deterministic conclusions/predictions even for the simplest experimental situations. Therefore, the comparison of quantum states is different compared to the classical situation.

Imagine we are given two independently prepared quantum systems of the same physical nature (e.g., two photons or two electrons). We would like to determine unambiguously whether the (internal) states of these two systems are the same or not. If we have just a single copy of each of the states and we possess no further information about the preparation then a measurement performed on each system separately cannot determine the states precisely enough to allow an error-free comparison. In this case, also all other strategies would fail, because our knowledge about the states is insufficient [1], e.g., if each of the systems can be in an arbitrary mixed state, then it is impossible to unambiguously test whether the states are equal or not. However, there are often situations in which we have some additional *a priori* information on the states we want to compare. For example, we might know that each system has been prepared in a pure state. This kind of scenario has been considered in Ref. [2] for two qudits and in Ref. [3] for the comparison of a larger number of systems. Thereafter, the comparison of coherent states and its applica-

tion to quantum cryptography has been addressed in Ref. [4]. Sedlák *et al.* [5] analyzed the comparison with more copies of the two systems and proposed an optimal comparator for coherent states, which, on this subset, outperforms the optimal universal comparator [2] working for all pure states.

In the present paper we analyze the unambiguous quantum state comparison (USC) of two unknown squeezed vacuum states, that is, we would like to unambiguously determine whether two unknown squeezed-vacuum states are the same or not. The conclusion has to be drawn from a procedure using only a single copy of the states. At the end of the procedure, using only the outcome of the measurement we have to decide whether the two states given to us have been the same, different, or that we don't know which of the former conclusions is true. We thrive to find an optimal procedure, i.e., one maximizing the probability of correctly judging the equivalence of the compared squeezed states.

Our proposal relies on the interference of two squeezed states at a beam splitter and on the subsequent measurement of the difference between the number of detected photons at the two output ports. In Ref. [4], unambiguous comparison of coherent states has been considered in detail and a short remark is devoted to the comparison of squeezed vacua. In the setup of Ref. [4], after interference at a beam splitter, one needs to measure the parity of the detected number of photons: a detection of an odd number of photons indicates the difference between the inputs. As a consequence, the quantum efficiency of the detectors is a critical parameter and plays a crucial role in the robustness of the scheme. As we will show, this problem is less relevant in our case, since our setup requires the measurement of the difference of the detected number of photons. Our configuration also allows us to prove optimality of our setup.

The plan of the paper is as follows. In Section II we introduce our scheme to compare two squeezed vacuum states, whereas the proof of the optimality of the setup is given in Section III. The performances of our scheme, also in the presence of imperfections at the detection stage, are investigated in Section IV, together with its reliability in the presence of noise. Section V closes the paper with concluding remarks.

\*Electronic address: stefano.olivares@mi.infn.it

†Electronic address: fyzimsed@savba.sk; On leave from Bratislava.

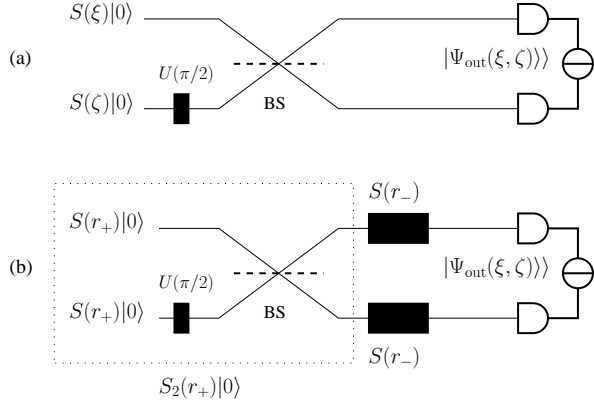


FIG. 1: Schematic diagrams of two equivalent setups for USC of squeezed vacuum states. Here  $S(\xi)|0\rangle$  and  $S(\zeta)|0\rangle$  are squeezed vacuum states,  $\arg(\xi) = \arg(\zeta)$ , and  $r_{\pm} = (\xi \pm \zeta)/2$ , respectively. See the text for details.

## II. COMPARISON OF SQUEEZED VACUUM STATES

Our goal is the comparison of two squeezed vacuum states  $|\xi\rangle \equiv S(\xi)|0\rangle$  and  $|\zeta\rangle \equiv S(\zeta)|0\rangle$ , where  $S(\gamma) = \exp[\frac{1}{2}\gamma(a^\dagger)^2 - \frac{1}{2}\gamma^*a^2]$  is the single-mode squeezing operator,  $\xi, \zeta, \gamma \in \mathbb{C}$  [6]. We let  $\xi = r e^{i\psi}$  and  $\zeta = s e^{i\varphi}$ , where  $r = |\xi|$ ,  $\psi = \arg(\xi)$ ,  $s = |\zeta|$ ,  $\varphi = \arg(\zeta)$ , and, for the sake of simplicity, we consider the scenario in which the states have the same phase, that is  $\arg(\xi) = \arg(\zeta) = \varphi$ . We remind that a comparator is a measuring device with two systems at the input and two or more possible outcomes, aimed at determining whether the two systems have been prepared in the same state. The setup we propose for the comparison of the two squeezed vacuum states is composed of a phase shifter, beam splitter and photon-counting detectors, and can be implemented with a current technology. The basic idea is sketched in Fig. 1 (a): we start from the two squeezed vacuum states we wish to compare,  $S(\xi)|0\rangle$  and  $S(\zeta)|0\rangle$ . At the first stage of our protocol, one of the two states, say  $S(\xi)|0\rangle$ , undergoes a phase shift  $U(\pi/2)$ , i.e.  $U(\pi/2)S(\xi)|0\rangle = S(-\xi)|0\rangle$ ; then we mix the states, having now orthogonal phases, at a balanced beam splitter (BS). If  $\xi = \zeta$ , i.e., the input states are equal, then the output state is the two-mode squeezed vacuum state of radiation (twin-beam state, TWB) [7], namely:

$$|\Psi_{\text{out}}(\xi, \xi)\rangle = U_{\text{BS}}S(\xi) \otimes S(-\xi)|0\rangle \equiv S_2(\xi)|0\rangle \quad (1)$$

$$= \sqrt{1 - |\lambda(\xi)|^2} \sum_{n=0}^{\infty} \lambda(\xi)^n |n\rangle|n\rangle, \quad (2)$$

where  $|n\rangle|n\rangle \equiv |n\rangle \otimes |n\rangle$ ,  $U_{\text{BS}}$  is the unitary operator describing the action of the BS,  $S_2(\xi) = \exp(\xi a^\dagger b^\dagger - \xi^* ab)$  is the two-mode squeezing operator acting on the two modes  $a$  and  $b$ , respectively, and  $\lambda(\gamma) = e^{i\arg(\gamma)} \tanh|\gamma|$ . One finds perfect correlations in the photon number of the two beams, that can be detected, e.g., by measuring the difference between the number of photons at the outputs (see Fig. 1), that, in this case, is always equal to zero. On the contrary, if  $\xi \neq \zeta$ , a different number of photons can be detected in the two beams, as we

are going to show in the following.

The same result of the evolution as in Fig. 1 (a) can be obtained considering the setup displayed in Fig. 1 (b) (see Appendix A). Here the two input states with squeezing parameters  $\xi$  and  $\zeta$  are substituted with two squeezed vacuum states having the same squeezing parameter amplitude  $r_+ = (\xi + \zeta)/2$ ; now, after the mixing at the BS, the outgoing modes undergo two local squeezing operations with amplitude  $r_- = (\xi - \zeta)/2$ . In formula [recall that  $\arg(\xi) = \arg(\zeta)$ ]:

$$U_{\text{BS}} S(\xi) \otimes S(-\zeta)|0\rangle = S(r_-) \otimes S(r_-) S_2(r_+)|0\rangle. \quad (3)$$

Since  $S_2(r_+)|0\rangle = \sqrt{1 - |\lambda(r_+)|^2} \sum_n \lambda(r_+)^n |n\rangle|n\rangle$ , we have:

$$|\Psi_{\text{out}}(\xi, \zeta)\rangle = \sqrt{1 - |\lambda(r_+)|^2} \sum_{n=0}^{\infty} \lambda(r_+)^n |\psi_n\rangle|\psi_n\rangle, \quad (4)$$

where we defined the new basis  $|\psi_n\rangle = S(r_-)|n\rangle$ . Finally, the probability of measuring  $h$  and  $k$  photons in the two beams, respectively, is given by:

$$p(h, k) = |\langle h| \langle k| \Psi_{\text{out}}(\xi, \zeta)\rangle|^2, \quad (5)$$

with:

$$\langle h| \langle k| \Psi_{\text{out}}(\xi, \zeta)\rangle = \sqrt{1 - |\lambda(r_+)|^2} \sum_n \lambda(r_+)^n [S(r_-)]_{hn} [S(r_-)]_{kn}, \quad (6)$$

where  $[S(r_-)]_{lm} = \langle l| S(r_-)|m\rangle$  are the matrix elements of the squeezing operator, whose analytical expressions are given e.g., in Ref. [8]. If  $\xi = \zeta$  and  $h \neq k$ , then  $\langle h| \langle k| \Psi_{\text{out}}(r, r)\rangle = 0$  and  $p(h, k) = 0$ , as one can see from Eq. (2). Thus, the probability  $p(h, k)$ , for  $h \neq k$ , can be non-zero only if  $\xi \neq \zeta$ , that is only if the input states are different.

In the ideal case (unit quantum efficiency of the detectors) the measurement apparatus we want to use gives two possible outcomes: zero or non-zero photon-counting difference. Thus, the POVM performed is defined by the effects  $E_0$  and  $E_D$ , corresponding to the “zero” and “non-zero” photon-counting events, respectively, given by:

$$E_0 = \sum_{n=0}^{\infty} |n\rangle\langle n| \otimes |n\rangle\langle n|, \quad E_D = \mathbb{I} - E_0. \quad (7)$$

The occurrence of the “D” event implies that the incident squeezed-vacuum states could not have been identical [see Eqs. (4) and (6)]. The occurrence of the “O” event, on the other hand, implies nothing, as each possible pair of squeezed-vacuum states leads to a non-zero overlap with any of the states  $|n\rangle|n\rangle$ . Thus, event “D” unambiguously indicates the difference of the compared squeezed states, whereas “O” is an inconclusive outcome.

## III. PROOF OF THE OPTIMALITY OF THE SETUP

Let us denote by  $\mathcal{S}^\varphi \equiv \{S(re^{i\varphi})|0\rangle; r \in \mathbb{R}\}$  the set of squeezed states from which we randomly chose the states to

be compared. We also define the sets  $\mathcal{S}_S^\varphi \equiv \{S(re^{i\varphi})|0\rangle \otimes S(re^{i\varphi})|0\rangle; r \in \mathbb{R}\}$ ,  $\mathcal{S}_D^\varphi \equiv \mathcal{S}^\varphi \otimes \mathcal{S}^\varphi \setminus \mathcal{S}_S^\varphi$ , composed by pairs of identical and different squeezed vacuum states, respectively. We assume a generic measurement with three outcomes ("same", "different" and "don't know") described by the POVM  $\Pi_S + \Pi_D + \Pi_0 = \mathbb{I}$  and we optimize the overall probability:

$$P = z_S \int_{\mathcal{S}_S^\varphi} d\Phi p_S(\Phi) \langle \Phi | \Pi_S | \Phi \rangle + z_D \int_{\mathcal{S}_D^\varphi} d\Phi p_D(\Phi) \langle \Phi | \Pi_D | \Phi \rangle, \quad (8)$$

where  $z_D$  and  $z_S = 1 - z_D$  are the a priori probability of being different or the same,  $p_S(\Phi)$ ,  $p_D(\Phi)$  are probability densities of choosing  $|\Phi\rangle$  from  $\mathcal{S}_S^\varphi$ ,  $\mathcal{S}_D^\varphi$ , respectively. We also impose the no-error constraints:

$$\text{Tr}(\Pi_S |\Phi\rangle \langle \Phi|) = 0, \quad \forall |\Phi\rangle \in \mathcal{S}_D^\varphi, \quad (9a)$$

$$\text{Tr}(\Pi_D |\Phi\rangle \langle \Phi|) = 0, \quad \forall |\Phi\rangle \in \mathcal{S}_S^\varphi, \quad (9b)$$

which guarantee the unambiguity of the results. From the mathematical point of view, the constraints (9) restrict the support of the operators  $\Pi_S$  and  $\Pi_D$ . The fact that the possible states in  $\mathcal{S}^\varphi$  form a continuous subset of pure states, is responsible for the impossibility to unambiguously confirm that the compared states are identical. The proof of this statement can be found in Appendix B and essentially states that, due to the no-error conditions (9), we must have  $\Pi_S = 0$ . Thus, the measurement actually has only two outcomes, the effective POVM is given by  $\Pi_D$ ,  $\Pi_0 = \mathbb{I} - \Pi_D$ , and it is clear that increasing the eigenvalues of  $\Pi_D$  without changing its support increases the figure of merit and leaves the no-error conditions satisfied. This is true independently of the distribution  $p_D$  and thus the optimal measurement is formed by  $\Pi_D$  being a projector onto the biggest support allowed by the no-error condition (9) and  $\Pi_0$  being a projector onto the orthocomplement. Moreover, the quantity that completely characterizes the behavior of the squeezed-states comparator is  $p(D|r, s) = \langle \Phi | \Pi_D | \Phi \rangle$ , i.e., the conditional probability of obtaining the outcome  $\Pi_D$  if different squeezed states  $|\Phi\rangle = S(re^{i\varphi})|0\rangle \otimes S(se^{i\varphi})|0\rangle$  ( $r \neq s$ ) are sent to the comparator. Summarizing, in order to find an optimal comparator of squeezed states from  $\mathcal{S}^\varphi$  we need to refine the definition of the largest allowed support of  $\Pi_D$  hidden in the no-error condition (9b). To do this we equivalently rewrite Eq. (9b) as:

$$\text{Tr}(W\Pi_D W^\dagger W |\Phi\rangle \langle \Phi| W^\dagger) = 0 \quad \forall |\Phi\rangle \in \mathcal{S}_S^\varphi, \quad (10)$$

which, by denoting  $E_D \equiv W\Pi_D W^\dagger$  and choosing  $W$  to be the unitary transformation performed by the proposed setup from Fig. 1 (a), becomes:

$$\text{Tr}(E_D |\Psi_{\text{out}}(r, r)\rangle \langle \Psi_{\text{out}}(r, r)|) = 0 \quad \forall r \in \mathbb{R}. \quad (11)$$

The optimality of the proposed setup is proved by showing that the biggest support allowed by the previous condition coincides with the support of the projective measurement  $E_D$  we use, see Eq. (7).

From the expression of  $|\Psi_{\text{out}}(r, r)\rangle$ , Eq. (2), it is clear that for any operator  $E_D$  with the support orthogonal to the span of  $|n\rangle|n\rangle$ , with  $n \in \mathbb{N}$ , the unambiguous no-error condition (11) holds. Hence, if any such operator  $E_D$  is a part of a POVM, then the emergence of the outcome related to it unambiguously indicates the difference of the squeezing parameters. We now proceed to show that the support of such  $E_D$  cannot be further enlarged. Now let us assume that a vector that a vector  $|v\rangle = \sum_{h,k=0}^{\infty} d_{hk} |h\rangle |k\rangle$  with at least one non-zero coefficient  $d_{ii}$  is in the support of  $E_D$ . As a consequence of the required no-error condition (11) the overlap

$$\langle v | \Psi_{\text{out}}(r, r) \rangle = \sqrt{1 - |\lambda(r)|^2} \sum_{n=0}^{\infty} d_{nn}^* \lambda(r)^n \quad (12)$$

has to be vanishing for all values of  $r$ . Eq. (12) is vanishing if and only if

$$\frac{\langle v | \Psi_{\text{out}}(r, r) \rangle}{\sqrt{1 - |\lambda(r)|^2}} = \sum_{n=0}^{\infty} d_{nn}^* \lambda(r)^n \quad (13)$$

vanishes for all  $r$ . The sum on the right-hand side of Eq. (13) can be seen as a polynomial in  $\lambda(r)$  and should vanish for all possible values of  $\lambda(r)$ , i.e. for all  $|\lambda(r)| < 1$ . Polynomials of this type on a finite interval form a vector space with linearly independent basis vectors  $\lambda(r)^k$ , with  $k \in \mathbb{N}$ . Thus the sum in Eq. (13) vanishes  $\forall r \in (0, \infty)$  only if  $d_{nn} = 0, \forall n \in \mathbb{N}$ . This is in contradiction with our assumption about the vector  $|v\rangle$  and therefore the largest support an operator  $E_D$ , unambiguously indicating the difference of the squeezing parameters, can have is the orthocomplement of the span of vectors  $|n\rangle|n\rangle$ , with  $n \in \mathbb{N}$ . This concludes the proof.

In the case of compared states with arbitrary phases of the complex squeezing parameters, the proof can be done in the same way as before, up to defining accordingly the set of pairs of same/different states.

#### IV. PERFORMANCES OF THE SETUP

In this section we give a thorough analysis of the statistics of our setup also in the presence of non-unit quantum efficiency at the detection stage in order to assess its reliability in Section IV C.

##### A. Probability of revealing the difference

The conditional probability of revealing the difference of compared states with  $\xi \neq \zeta$  [but  $\arg(\xi) = \arg(\zeta) = \varphi$ ], that is the probability to obtain a  $E_D$  outcome, reads:

$$p(D|\xi, \zeta) = 1 - p(0|\xi, \zeta) \quad (14)$$

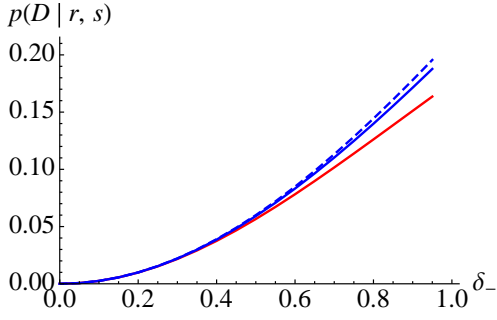


FIG. 2: (Color online) Plot of the conditional probability of revealing the difference of two squeezed vacuum states  $|\psi_1\rangle = S(r)|0\rangle$  and  $|\psi_2\rangle = S(s)|0\rangle$  in the ideal case ( $\eta = 1$ ) as a function of  $\delta_- = |r - s|$ . Solid lines, from top to bottom, correspond to the optimal squeezed-states comparator (blue) and the universal comparator (red line). The dashed line is the upper bound on the probability in the case of only two possible squeezings. See the section IV C for details.

with:

$$\begin{aligned}
 p(0|\xi, \zeta) &= \langle\langle \Psi_{\text{out}}(\xi, \zeta) | E_0 | \Psi_{\text{out}}(\xi, \zeta) \rangle\rangle \\
 &= [1 - |\lambda(r_+)|^2] \sum_{n,m=0}^{\infty} [\lambda(r_+)]^n [\lambda^*(r_+)]^m \\
 &\quad \times \sum_{k=0}^{\infty} \{[S(r_-)]_{kn}\}^2 \{[S^\dagger(r_-)]_{mk}\}^2, \quad (15)
 \end{aligned}$$

where  $|\Psi_{\text{out}}(\xi, \zeta)\rangle\rangle$  is given in Eq. (4). For  $\xi \rightarrow \zeta$  we correctly obtain  $p(0|\xi, \xi) = 1$ . By noting that [8]:

$$[S(\gamma)]_{hk} \propto \begin{cases} \exp\{i(\frac{h-k}{2})\theta\} & \text{for } h, k \text{ odd or even,} \\ 0 & \text{otherwise,} \end{cases} \quad (16)$$

where  $\gamma = |\gamma|e^{i\theta}$ , it is straightforward to see that Eq. (15) does not depend on the (equal) phase  $\varphi$  of  $\xi$  and  $\zeta$ . Thus, in order to investigate the performances of the optimal squeezed-states comparator, we may set  $\varphi = 0$  and let  $\xi = r$  and  $\zeta = s$ , with  $r, s \in \mathbb{R}$ , without loss of generality. Furthermore, it is possible to show by numerical means that the probability  $p(D|r, s)$  does not depend on the sum of the squeezing parameter  $\delta_+ = r + s$ , but only on the difference  $\delta_- = |r - s|$ . In Fig. 2 we plot the probability  $p(D|r, s)$  given in Eq. (15) as a function of  $\delta_- = |r - s|$ , and we compare it with the possible use of the universal comparator [2], which works unambiguously for all pure states leading to

$$p_{\text{UC}}(D|\omega) = \frac{1}{2}(1 - \omega^2), \quad (17)$$

where  $\omega = |\langle\psi_1|\psi_2\rangle| = (\cosh \delta_-)^{-1/2}$  is the overlap between the two squeezed vacuum states.

### B. Influence of non-ideal detectors

In a realistic scenario, in which the photon-number resolving detectors have non-unit quantum efficiency  $\eta$ , we should

modify the POVM by replacing to the projectors  $|n\rangle\langle n|$  in Eq. (7) by the following operators [9, 10]:

$$\Pi_n(\eta) = \eta^n \sum_{k=n}^{\infty} (1 - \eta)^{k-n} \binom{k}{n} |k\rangle\langle k|, \quad (18)$$

namely (we assume that the two detectors have the same quantum efficiency):

$$E_0(\eta) = \sum_{n=0}^{\infty} \Pi_n(\eta) \otimes \Pi_n(\eta), \quad (19a)$$

$$E_D(\eta) = \mathbb{I} - E_0(\eta). \quad (19b)$$

The performance of this kind of detector and its reliability to resolve up to tens of photons have been recently investigated in Ref. [11]. The conditional probability now reads:

$$p_\eta(D|\xi, \zeta) = 1 - p_\eta(0|\xi, \zeta), \quad (20)$$

with:

$$\begin{aligned}
 p_\eta(0|\xi, \zeta) &= \langle\langle \Psi_{\text{out}}(\xi, \zeta) | E_0(\eta) | \Psi_{\text{out}}(\xi, \zeta) \rangle\rangle \\
 &= [1 - |\lambda(r_+)|^2] \sum_{n,l,m=0}^{\infty} \eta^{2n} [\lambda(r_+)]^l [\lambda^*(r_+)]^m \\
 &\quad \times \sum_{h,k=n}^{\infty} (1 - \eta)^{h+k-2n} \binom{h}{n} \binom{k}{n} \\
 &\quad \times [S(r_-)]_{kl} [S(r_-)]_{hl} [S^\dagger(r_-)]_{mk} [S^\dagger(r_-)]_{mh}, \quad (21)
 \end{aligned}$$

that, in the case of  $\xi = \zeta$ , reduces to:

$$\begin{aligned}
 p_\eta(0|\xi, \xi) &= \langle\langle \Psi_{\text{out}}(\xi, \xi) | E_0(\eta) | \Psi_{\text{out}}(\xi, \xi) \rangle\rangle \\
 &= [1 - |\lambda(\xi)|^2] \sum_{n=0}^{\infty} \eta^{2n} |\lambda(\xi)|^{2n} \\
 &\quad \times {}_2F_1[1 + n, 1 + n, 1, (1 - \eta)^2 |\lambda(\xi)|^2], \quad (22)
 \end{aligned}$$

where  ${}_2F_1$  are hypergeometric functions and  $[S(r_-)]_{lm}$  are the matrix elements of the squeezing operator as in Eq. (6). Because of Eq. (16), the probabilities (20) and (21) are still independent on  $\varphi$ , thus, from now on, we set  $\varphi = 0$  and put  $\xi = r$  and  $\zeta = s$ , with  $r, s \in \mathbb{R}$ , without loss of generality. In Fig. 3 we plot  $p_\eta(0|r, r)$  and  $p_\eta(D|r, r)$  for different values of  $\eta$ . If  $r \ll 1$ , then Eq. (22) can be expanded up to the second order in  $r$ , obtaining:

$$p_\eta(0|\xi, \xi) \approx 1 - 2\eta(1 - \eta)r^2. \quad (23)$$

### C. Reliability of the setup

In order to assess the reliability of our setup, we address the scenario in which only two squeezing parameters for each of the squeezed vacua are possible. In such case one knows that the two squeezing parameters are either  $\{(r, r), (s, s)\}$  or  $\{(r, s), (s, r)\}$  with the same prior probability. Our squeezed-states comparator may not be optimal in this case. However,

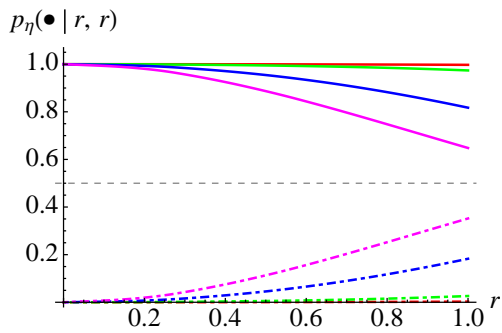


FIG. 3: (Color online) Plot of  $p_\eta(0|r, r)$  (solid) and  $p_\eta(D|r, r)$  (dot-dashed lines) as functions of  $r$  for different values of the efficiency  $\eta$ ; from top to bottom (solid) and from bottom to top (dot-dashed lines):  $\eta = 0.999$  (red), 0.99 (green), 0.90 (blue), 0.50 (magenta).

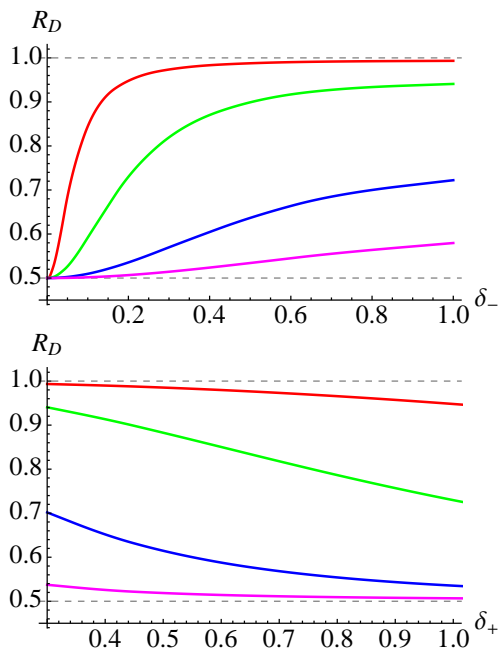


FIG. 4: (Color online) Top: Reliability  $R_D(\eta; r, s)$  (reliability) as a function of  $\delta_-$  for fixed  $\delta_+ = 1.0$  and different values of the efficiency. Bottom: Reliability  $R_D(\eta; r, s)$  as a function of  $\delta_+$  for difference  $\delta_- = 0.2$  and different values of the efficiency. In both plots, from top to bottom:  $\eta = 0.999$  (red), 0.99 (green), 0.90 (blue), 0.50 (magenta).

as one can see in Fig. 2, the performance of our setup is nearly as good as if it was optimized also for this restricted scenario. In particular, the dashed line in Fig. 2 refers to the optimal measurement, unambiguously detecting the difference in the case of only two possible squeezing parameters, in formula [12]:

$$p_{\max}(D|\omega) = \frac{1 - \omega^2}{1 + \omega^2}. \quad (24)$$

We define the reliability  $R_D$  of the scheme in revealing the difference of the squeezing parameters  $r$  and  $s$  as the conditional probability of the two squeezed vacuum states being

different if the outcome  $E_D$  is found, i.e., (we assume equal prior probabilities):

$$R_D(\eta; r, s) = \frac{p_\eta(D|r, s) + p_\eta(D|s, r)}{\sum_{u,v=r,s} p_\eta(D|u, v)}. \quad (25)$$

In the ideal case, i.e.,  $\eta = 1$ , we have  $p_\eta(D|r, r) = 0$  and, thus,  $R_D(\eta; r, s) = 1$ , which is guaranteed by the construction of the setup. On the other hand, if  $\eta < 1$ , then  $p_\eta(D|r, r) \neq 0$  and the actual value of  $R_D$  can be numerically calculated starting from Eq.s (20) and (21). The reliability  $R_D(\eta; r, s)$  is plotted in the upper panel of Fig. 4 as a function of  $\delta_- = |r - s|$ . Note that differently from the case  $\eta = 1$ , for  $\eta < 1$  the probability  $p_\eta(D|r, s)$  depends not only on the difference  $\delta_- = |r - s|$  but also on the sum  $\delta_+ = r + s$ . The dependence on  $\delta_+$  is shown in the lower panel of Fig. 4, where we plot  $R_D(\eta; r, s)$  as a function of  $\delta_+$  for fixed difference  $\delta_- = 0.2$ .

## V. CONCLUDING REMARKS

In this paper we have addressed the comparison of two squeezed vacuum states of which we have a single copy available. We have suggested an optical setup based on a beam splitter, a phase shifter and two photodetectors which is feasible with the current technology. We have proved the optimality of our scheme in the ideal case and analyzed its performances and reliability also in the presence of non-unit quantum efficiency at the detection stage. As one may expect, the detection efficiency strongly affects the reliability; nevertheless we have shown that, for small energies and not too low quantum efficiency, the setup is still robust. Our scheme may be employed not only for the comparison of two squeezed vacua, but for a more general scenario in which the input states  $|\xi\rangle$  and  $|\zeta\rangle$  are known to be transformed by two *fixed known* local unitaries  $U$  and  $V$ , respectively (namely,  $U|\xi\rangle \otimes V|\zeta\rangle$ ) or by any *fixed known* global unitary transformation  $W$  ( $W|\xi\rangle \otimes |\zeta\rangle$ ): now it is enough to apply the inverse of the transformation before processing the state with our setup.

## Acknowledgments

Fruitful discussions with M. Ziman are acknowledged. This work has been supported by the project INQUEST APVV SK-IT-0007-08 within the ‘‘Executive programme of scientific and technological co-operation between Italy and Slovakia’’, by the European Union projects Q-ESSENCE 248095, HIP 221889, and partially supported by the CNR-CNISM agreement.

## Appendix A: Proof of the equivalence of the two schemes

In this Appendix we show the equivalence between the schemes in Fig. 1 (a) and 1 (b). Since the squeezed states are Gaussian states and all operations involved in the schemes

(phase shift and beam splitter mixing) preserve the Gaussian character, we use the phase-space description of the system evolution [13]. For the sake of simplicity we focus on the case of real squeezing parameters, i.e.,  $\xi = r$  and  $\zeta = s$ , with  $r, s \in \mathbb{R}$ . The symplectic transformation associated with the squeezing operator  $S(r)$  is:

$$\mathbf{S}(r) = \begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix}, \quad (\text{A1})$$

while the symplectic transformation associated with the balanced beam splitter operator  $U_{\text{BS}}$  is:

$$\mathbf{S}_{\text{BS}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1}_2 & -\mathbb{1}_2 \\ \mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix}, \quad (\text{A2})$$

where  $\mathbb{1}_2$  is a  $2 \times 2$  identity matrix. The covariance matrix of the outgoing Gaussian state in the scheme Fig. 1 (a) [for the sake of simplicity we used  $U(\pi/2)S(s) = S(-s)$  and we do not write explicitly the symplectic transformation of the phase shift]:

$$|\Psi_{\text{out}}(r, s)\rangle\rangle = U_{\text{BS}}\mathbf{S}(r) \otimes S(-s) |0\rangle, \quad (\text{A3})$$

is, thus, given by:

$$\boldsymbol{\sigma}_{\text{out}} = \mathbf{S}_{\text{BS}} \mathbf{L}\mathbf{S}(r, -s) \boldsymbol{\sigma}_0 \mathbf{L}\mathbf{S}(r, -s)^T \mathbf{S}_{\text{BS}}^T, \quad (\text{A4})$$

where  $\boldsymbol{\sigma}_0 = \frac{1}{2} \mathbb{1}_4$ ,

$$\mathbf{L}\mathbf{S}(r, -s) = \begin{pmatrix} \mathbf{S}(r) & \mathbf{0} \\ \mathbf{0} & \mathbf{S}(-s) \end{pmatrix}, \quad (\text{A5})$$

represents the two local squeezing operations  $\mathbf{L}\mathbf{S}(r, -s) = \mathbf{S}(r) \otimes \mathbf{S}(-s)$ . The explicit form of (A4) reads:

$$\boldsymbol{\sigma}_{\text{out}} = \frac{1}{2} \begin{pmatrix} f(r, -s) & 0 & g(r, -s) & 0 \\ 0 & f(-r, s) & 0 & g(-r, s) \\ g(r, -s) & 0 & f(r, -s) & 0 \\ 0 & g(-r, s) & 0 & f(-r, s) \end{pmatrix}, \quad (\text{A6})$$

where:

$$f(x, y) = \frac{e^{2x} + e^{2y}}{2} \quad \text{and} \quad g(x, y) = \frac{e^{2x} - e^{2y}}{2}. \quad (\text{A7})$$

Note that by setting  $s = r$  one obtains the covariance matrix of the TWB in Eq. (1).

It is now straightforward to verify that the same result of the evolution as in Fig. 1 (a), corresponding to the covariance matrix in Eq. (A6), may be obtained considering the setup displayed in Fig. 1 (b). Here two input states with same squeezing parameter amplitude  $r_+ = (r + s)/2$  are mixed after a phase shift at the BS and the outgoing modes undergo two local squeezing operations with amplitude  $r_- = (r - s)/2$ ; in formula:

$$\boldsymbol{\sigma}' = \mathbf{L}\mathbf{S}(r_-, r_-) \mathbf{S}_2(r_+) \boldsymbol{\sigma}_0 \mathbf{S}_2(r_+)^T \mathbf{L}\mathbf{S}(r_-, r_-)^T, \quad (\text{A8})$$

where  $\mathbf{S}_2(r_+) = \mathbf{S}_{\text{BS}} \mathbf{L}\mathbf{S}(r_+, -r_+)$  is the symplectic transformation associated with  $\mathbf{S}_2(r)$  defined in Eq. (1). By performing the calculation one finds  $\boldsymbol{\sigma}' = \boldsymbol{\sigma}_{\text{out}}$ , and, since Gaussian states are completely characterized by their covariance matrix (and first moments), one can conclude that the final states are the same.

### Appendix B: No unambiguous detection of sameness of two states

In this Appendix we show that the no-error condition given in Eq. (9b), together with continuity of the involved mappings, imply that we cannot unambiguously detect the sameness of two states. Let us consider a state  $|\Phi\rangle = S(re^{i\varphi})|0\rangle \otimes S(se^{i\varphi})|0\rangle \in \mathcal{S}_D^\varphi$  with  $r \neq s$ . The no-error condition (9b) demand that:

$$\text{Tr}(\Pi_S |\Phi\rangle\langle\Phi|) = 0, \quad \forall r \neq s. \quad (\text{B1})$$

Let us now take the limit  $s \rightarrow r$ . Thanks to continuity of the trace and the chosen parameterization of the set of states, we conclude that  $\forall r$ :

$$\langle 0 | S^\dagger(re^{i\varphi}) \otimes S^\dagger(re^{i\varphi}) \Pi_S S(re^{i\varphi}) \otimes S(re^{i\varphi}) | 0 \rangle = 0. \quad (\text{B2})$$

It follows that Eq. (B1) has to hold for arbitrary  $r$  and  $s$ . Since  $\Pi_S$  is a positive operator, it should be zero on the relevant part of the Hilbert space spanned by  $\mathcal{S}^\varphi \otimes \mathcal{S}^\varphi$ , i.e., all the possible pairs of the compared states. Hence, without loss of generality, we can choose  $\Pi_S = 0$  on the whole Hilbert space.

[1] I. Jex, E. Andersson and A. Chefles, *J. Mod. Opt.* **51**, 505 (2004).  
 [2] S. M. Barnett, A. Chefles and I. Jex, *Phys. Lett. A* **307** 189, (2003).  
 [3] A. Chefles, E. Andersson and I. Jex, *J. Phys. A: Math. Gen.* **37**, 7315 (2004).  
 [4] E. Andersson, M. Curty and I. Jex, *Phys. Rev. A* **74**, 022304 (2006).  
 [5] M. Sedláč, M. Ziman, V. Bužek and M. Hillery, *Phys. Rev. A* **77**, 042304 (2008).  
 [6] R. Loudon and P. L. Knight, *J. Mod. Opt.* **34**, 709 (1987).  
 [7] M. G. A. Paris, *Phys. Lett. A* **225**, 28 (1997).

[8] P. Marian, *Phys. Rev. A* **44**, 3325 (1991).  
 [9] P. L. Kelley, and W. H. Kleiner, *Phys. Rev.* **136**, 316 (1964).  
 [10] W. H. Louisell, *Quantum Statistical properties of Radiation*, (Wiley, 1973).  
 [11] A. Allevi, A. Andreoni, M. Bondani, M. G. Genoni and S. Olivares, *Phys. Rev. A* **82**, 013816 (2010).  
 [12] M. Kleinmann, H. Kampermann, D. Bruss, *Phys. Rev. A* **72**, 032308 (2005).  
 [13] A. Ferraro, S. Olivares and M. G. A. Paris, *Gaussian States in Quantum Information* (Bibliopolis, Napoli, 2005).