Analyzing the spin-bath model without simulations

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On the basis of a lemma designed to decide whether a discrete system decoheres or not with no need of computer simulations, in this paper we analyze the well-known spin-bath model. The lemma allows us to predict the decoherence of the system by analytical means.

I. INTRODUCTION

In previous papers we have developed a *general theoretical framework* for decoherence ([1], [2], [3], [4]), which can be applied to open and closed systems. The conceptually relevant step in this framework is the selection of the relevant observables, relative to which the question about decoherence is posed. Then, decoherence can be explained in three steps:

1. **First step:** The space $\mathcal{O}_R$ of relevant observables is defined.
2. **Second step:** The expectation value $\langle O_R\rangle_{\rho(t)}$, for any $O_R \in \mathcal{O}_R$, is obtained.
3. **Third step:** It is proved that $\langle O_R\rangle_{\rho(t)} = \langle O_R\rangle_{\rho_R(t)}$ reaches a final equilibrium value:

$$\lim_{t\to\infty}\langle O_R\rangle_{\rho(t)} = \lim_{t\to\infty}\langle O_R\rangle_{\rho_R(t)} = \langle O_R\rangle_{\rho_*} = \langle O_R\rangle_{\rho_{R*}} \qquad \forall O_R \in \mathcal{O}_R \tag{1}$$

As it is explained in another paper of this issue ([5]), this general framework strictly applies when the limit of eq. (1) exists, and this happens when the Riemann-Lebesgue theorem is valid. But the Riemann-Lebesgue theorem strictly applies only in cases of continuous energy spectrum. Nevertheless, it can be shown that the validity conditions of the discrete analogue of the Riemann-Lebesgue theorem are expressed by the following lemma:

Lemma 1 *Let $\{x_i\}$ be a set of points uniformly distributed, and $f(x_i) : \mathbb{R} \to \mathbb{R}$ be a discrete function defined over $\{x_i\}$, such that:*

- *$i \in [0, N]$ and $N \gg 1$.*
- *$\exists G \in \mathbb{N}, \exists P \in \mathbb{N}$ such that $P \gg 1$ and $\{x_i\} = \bigcup_{k=1}^{G}\left\{x_{(k-1)(P+1)+1}, ..., x_{k(P+1)}\right\} = \bigcup_{k=1}^{G} X_k$. In this case we will say that the set $\{x_i\}$ is quasi-continuous of class 1.*
- *$\forall X_k$, $f(x_{r_k}) \cong C_k$, with $x_{r_k} \in X_k$. In this case we will say that $f(x_i) \in \mathcal{L}_1$.*

then,

$$\lim_{t\longrightarrow t_P/2}\sum_{i=0}^{N}\frac{1}{N} f\left(x_i\right) e^{ix_i t} \cong 0 \tag{2}$$

where t_P is the recurrence or Poincaré time.

The general aim of this paper is to apply this lemma to the spin-bath model, in order to show that this method allows us to predict decoherence by analytical means. For this purpose, the paper is organized as follows. In Section 2 we will explain how the general framework for decoherence applies to open systems, in particular, to models traditionally treated by means of the environment-induced decoherence (EID) approach ([6]-[11]). In Section 3 we will present the spin-bath model, showing which the relevant observables are in this case and how the expectation values have to be computed. Section 4 will be devoted to compare two methods for analyzing the model: the standard method, based on computer simulations, and the analytical method, based on our lemma. Finally, in Section 5 we will draw our conclusions.

II. EID FROM THE GENERAL FRAMEWORK

In the case of the EID approach, the three steps of the general framework for decoherence are usually not explicit in the formalism. However, the theory can be rephrased in such a way that it can be analyzed from that framework. In this section we will undertake this task in order to apply our just introduced lemma in a following section.

1. **First step:** Let us consider a closed system U that can be decomposed into a proper system S and its environment E. Let the Hilbert space of U be $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$, where $\mathcal{H}_S$ is the Hilbert space of S and $\mathcal{H}_E$ the Hilbert space of E. The corresponding von Neumann-Liouville space of U is $\mathcal{L} = \mathcal{H} \otimes \mathcal{H} = \mathcal{L}_S \otimes \mathcal{L}_E$, where $\mathcal{L}_S = \mathcal{H}_S \otimes \mathcal{H}_S$ and $\mathcal{L}_E = \mathcal{H}_E \otimes \mathcal{H}_E$. A generic observable belonging to $\mathcal{L}$ reads

$$O = \sum_I O_S^{(I)} \otimes O_E^{(I)} \ \in \mathcal{L}, \quad \text{with } O_S^{(I)} \in \mathcal{L}_S \text{ and } O_E^{(I)} \in \mathcal{L}_E \tag{3}$$

i.e. O is an observable with coordinates $(O^{(I)}_{i\alpha j\beta}) = (\sum_I O^{(I)}_{ij} O^{(I)}_{\alpha\beta})$, where $i, j, ...$ are the indices corresponding to $\mathcal{H}_S$, and $\alpha, \beta, ...$ are the indices corresponding to $\mathcal{H}_E$. The relevant observables are those having the following form:

$$O_R = O_S \otimes I_E \in \mathcal{O}_R, \quad \text{with coordinates } (O_{ij}\delta_{\alpha\beta}) \tag{4}$$

where I_E is the identity operator in $\mathcal{L}_E$. Therefore, $\mathcal{O}_R \subset \mathcal{L}$ is the subspace of the relevant observables, in this EID case those essentially corresponding to the proper system S.

2. **Second step:** The expectation value of any observable $O_R \in \mathcal{O}_R$ in the state ρ of U reads

$$\langle O_R \rangle_\rho = Tr\,(\rho O_R) = \sum_{ij\alpha\beta} \rho^*_{i\alpha j\beta}\, O_{ij}\, \delta_{\alpha\beta} = \sum_{ij} O_{ij} \sum_{\alpha\beta} \rho^*_{i\alpha j\beta}\, \delta_{\alpha\beta} = \sum_{ij} O_{ij} \sum_{\alpha} \rho^*_{i\alpha j\alpha} \tag{5}$$

The reduced density operator ρ_R is defined by tracing over the environmental degrees of freedom,

$$\rho_S = Tr_E\, \rho \in \mathcal{L}'_S, \quad \text{with coordinates } \left(\sum_\alpha \rho_{i\alpha j\alpha} \right) = (\rho_{ij}) \tag{6}$$

where $\mathcal{L}'_S$ is the dual space of $\mathcal{L}_S$. Therefore, the expectation value $\langle O_R \rangle_{\rho(t)}$ can be expressed as

$$\langle O_R \rangle_{\rho(t)} = Tr\ (\rho(t)\, O_R) = Tr\ (\rho(t)(O_S \otimes I_E)) = Tr\,(\rho_S(t)\, O_S) = \langle O_S \rangle_{\rho_S(t)} \tag{7}$$

3. **Third step:** The EID approach studies the time evolution of the reduced density operator $\rho_S(t)$ governed by an effective master equation. For many physical models where the space $\mathcal{O}_R$ has a finite number of dimensions, this approach shows that, for $t \to \infty$, $\rho_S(t)$ reaches an equilibrium state ρ_{S*}:

$$\rho_S(t) \longrightarrow \rho_{S*} \tag{8}$$

Since ρ_{S*} is obviously diagonal in its eigenbasis, the system S decoheres in the eigenbasis of ρ_{S*}, which turns out to be the final decoherence basis. But if we take into account the definition of ρ_S as a partial trace (see eq. (6)), we can obtain the limit of the expectation values of eq. (7) as

$$\lim_{t\to\infty} \langle O_S \rangle_{\rho_S(t)} = \lim_{t\to\infty} \langle O_R \rangle_{\rho(t)} = \langle O_S \rangle_{\rho_{S*}} = \langle O_R \rangle_{\rho_*} \tag{9}$$

where ρ_* is such that ρ_{S*} results from the projection of ρ_* onto $\mathcal{O}_R$. Therefore, for any observable $O_R \subset \mathcal{O}_R$,

$$\lim_{t\to\infty} \langle O_R \rangle_{\rho(t)} = \langle O_R \rangle_{\rho_*} \tag{10}$$

This result can also be expressed as a weak limit

$$W - \lim_{t\to\infty} \rho(t) = \rho_* \tag{11}$$

If the just obtained eq. (10) is compared with eq. (1), it turns out to be clear that the EID approach can also be formulated from the viewpoint of the closed composite system U and, from this perspective, it can be explained in the context of the general framework introduced in the Introduction. In other words, the split of the closed system into a proper open system and an environment is just a way of selecting the relevant observables of the closed system.

The limit of eqs. (8) and (9) can be computed by means of two different strategies:

. By solving the unitary evolution equation for $\rho(t)$, computing $Tr\ (\rho(t)\, O_R)$ and finding the limit.

. By solving the non-unitary evolution equation for $\rho_S(t) = Tr_E\,(\rho(t))$, computing $Tr\,(\rho_S(t)\, O_S)$ and finding the limit.

Of course, the two strategies give the same result: although the second is the usual method in the EID literature, the first may lead to a simpler solution, as in the model developed in the next section.

III. THE SPIN-BATH MODEL FROM THE GENERAL FRAMEWORK

The spin-bath model is a very simple model that has been exactly solved in previous papers (see [8]). Here we will study it from the general framework applied to the EID approach, as presented in the previous section. This task will allow us to compare the method traditionally used in the literature for solving the model with the method based on our lemma.

Let us consider a closed system $U = P \cup P_1 \cup \ldots \cup P_N = P \cup (\cup_{i=1}^{N} P_i)$, where (i) P is a spin-1/2 particle represented in the Hilbert space $\mathcal{H}_P$, and (ii) each P_i is a spin-1/2 particle represented in its Hilbert space $\mathcal{H}_i$. The Hilbert space of the composite system U is, then,

$$\mathcal{H} = \mathcal{H}_P \otimes \left(\bigotimes_{i=1}^{N} \mathcal{H}_i \right) \tag{12}$$

In the particle P, the two eigenstates of the spin operator $S_{P,\overrightarrow{v}}$ in direction $\overrightarrow{v}$ are $|\Uparrow\rangle, |\Downarrow\rangle$:

$$S_{P,\overrightarrow{v}}\,|\Uparrow\rangle = \frac{1}{2}\,|\Uparrow\rangle \qquad S_{P,\overrightarrow{v}}\,|\Downarrow\rangle = -\frac{1}{2}\,|\Downarrow\rangle \tag{13}$$

In each particle P_i, the two eigenstates of the corresponding spin operator $S_{i,\overrightarrow{v}}$ in direction $\overrightarrow{v}$ are $|\uparrow_i\rangle, |\downarrow_i\rangle$:

$$S_{i,\overrightarrow{v}}\,|\uparrow_i\rangle = \frac{1}{2}\,|\uparrow_i\rangle \qquad S_{i,\overrightarrow{v}}\,|\downarrow_i\rangle = -\frac{1}{2}\,|\downarrow_i\rangle \tag{14}$$

Therefore, a pure initial state of U reads

$$|\psi_0\rangle = (a\,|\Uparrow\rangle + b\,|\Downarrow\rangle) \otimes \left(\bigotimes_{i=1}^{N} (\alpha_i|\uparrow_i\rangle + \beta_i|\downarrow_i\rangle) \right) \tag{15}$$

where $|a|^2 + |b|^2 = 1$ and $|\alpha_i|^2 + |\beta_i|^2 = 1$ (for a generalization with M spins $\{\Uparrow, \Downarrow\}$ and N spins $\{\uparrow, \downarrow\}$, see [2]). If the self-Hamiltonians H_P of P and H_i of P_i are taken to be zero, and there is no interaction among the P_i, then the total Hamiltonian H of the composite system U is given by the interaction between the particle P and each particle P_i (see [8], [12]):

$$H = \frac{1}{2}\,(|\Uparrow\rangle\langle\Uparrow| - |\Downarrow\rangle\langle\Downarrow|) \otimes \sum_{i=1}^{N} \left[g_i\,(|\uparrow_i\rangle\langle\uparrow_i| - |\downarrow_i\rangle\langle\downarrow_i|) \otimes \left(\bigotimes_{j\neq i}^{N} \mathbb{I}_j \right) \right] \tag{16}$$

where $\mathbb{I}_j = |\uparrow_j\rangle\langle\uparrow_j| + |\downarrow_j\rangle\langle\downarrow_j|$ is the identity operator on the subspace $\mathcal{H}_j$. Under the action of H, the state $|\psi_0\rangle$ evolves into

$$|\psi(t)\rangle = a\,|\Uparrow\rangle\,|\mathcal{E}_{\Uparrow}(t)\rangle + b\,|\Downarrow\rangle\,|\mathcal{E}_{\Downarrow}(t)\rangle \tag{17}$$

where

$$|\mathcal{E}_{\Uparrow}(t)\rangle = |\mathcal{E}_{\Downarrow}(-t)\rangle = \bigotimes_{i=1}^{N} \left(\alpha_i\, e^{-ig_i t/2}\ |\uparrow_i\rangle + \beta_i\, e^{ig_i t/2}\ |\downarrow_i\rangle \right) \tag{18}$$

A. Computing the expectation values

The space $\mathcal{O}$ of the observables of the composite system U can be obtained as $\mathcal{O} = \mathcal{O}_P \otimes (\otimes_{i=1}^{N} \mathcal{O}_i)$, where $\mathcal{O}_P$ is the space of the observables of the particle P and $\mathcal{O}_i$ is the space of the observables of the particle P_i. Then, an observable $O \in \mathcal{O} = \mathcal{H} \otimes \mathcal{H}$ can be expressed as

$$O = O_P \otimes (\bigotimes_{i=1}^{N} O_i) \tag{19}$$

where

$$O_P = s_{\Uparrow\Uparrow} |\Uparrow\rangle \langle\Uparrow| + s_{\Uparrow\Downarrow} |\Uparrow\rangle \langle\Downarrow| + s_{\Downarrow\Uparrow} |\Downarrow\rangle \langle\Uparrow| + s_{\Downarrow\Downarrow} |\Downarrow\rangle \langle\Downarrow| \ \in \mathcal{O}_P \tag{20}$$

$$O_i = \epsilon^{(i)}_{\uparrow\uparrow} |\uparrow_i\rangle\langle\uparrow_i| + \epsilon^{(i)}_{\downarrow\downarrow} |\downarrow_i\rangle\langle\downarrow_i| + \epsilon^{(i)}_{\downarrow\uparrow} |\downarrow_i\rangle\langle\uparrow_i| + \epsilon^{(i)}_{\uparrow\downarrow} |\uparrow_i\rangle\langle\downarrow_i| \ \in \mathcal{O}_i \tag{21}$$

Since the operators O_P and O_i are Hermitian, the diagonal components $s_{\Uparrow\Uparrow}$, $s_{\Downarrow\Downarrow}$, $\epsilon^{(i)}_{\uparrow\uparrow}$, $\epsilon^{(i)}_{\downarrow\downarrow}$ are real numbers, and the off-diagonal components are complex numbers satisfying $s_{\Uparrow\Downarrow} = s^*_{\Downarrow\Uparrow}$, $\epsilon^{(i)}_{\uparrow\downarrow} = \epsilon^{(i)*}_{\downarrow\uparrow}$. Then, the expectation value of the observable O in the state $|\psi(t)\rangle$ of eq. (17) can be computed as

$$\langle O \rangle_{\psi(t)} = (|a|^2 s_{\Uparrow\Uparrow} + |b|^2 s_{\Downarrow\Downarrow})\, \Gamma_0(t) + 2\,\mathrm{Re}\,[ab^*\, s_{\Downarrow\Uparrow}\, \Gamma_1(t)] \tag{22}$$

where (see [12])

$$\Gamma_0(t) = \prod_{i=1}^{N} \left[|\alpha_i|^2 \epsilon^{(i)}_{\uparrow\uparrow} + |\beta_i|^2 \epsilon^{(i)}_{\downarrow\downarrow} + 2\,\mathrm{Re}(\alpha_i\, \beta_i^* \epsilon^{(i)}_{\downarrow\uparrow} e^{i g_i t}) \right] \tag{23}$$

$$\Gamma_1(t) = \prod_{i=1}^{N} \left[|\alpha_i|^2 \epsilon^{(i)}_{\uparrow\uparrow} e^{i g_i t} + |\beta_i|^2 \epsilon^{(i)}_{\downarrow\downarrow} e^{-i g_i t} + 2\,\mathrm{Re}(\alpha_i\, \beta_i^* \epsilon^{(i)}_{\downarrow\uparrow}) \right] \tag{24}$$

B. Selecting the relevant observables

In the typical situation studied by the EID approach, the open system S is the particle P, and the remaining particles P_i play the role of the environment E: $S = P$ and $E = \cup_{i=1}^{N} P_i$. Therefore, the relevant observables O_R of the closed system U are those corresponding to the particle P, and they are obtained from eqs. (19), (20) and (21), by making $\epsilon^{(i)}_{\uparrow\uparrow} = \epsilon^{(i)}_{\downarrow\downarrow} = 1$ and $\epsilon^{(i)}_{\uparrow\downarrow} = 0$:

$$O_R = O_S \otimes \mathbb{I}_E = \left(\sum_{s,s'=\Uparrow,\Downarrow} s_{ss'} |s\rangle\langle s'| \right) \otimes \left(\bigotimes_{i=1}^{N} \mathbb{I}_i \right) \tag{25}$$

The expectation value of these observables in the state $|\psi(t)\rangle$ of eq. (17) is given by

$$\langle O_R \rangle_{\psi(t)} = |a|^2\, s_{\Uparrow\Uparrow} + |b|^2\, s_{\Downarrow\Downarrow} + 2\,\mathrm{Re}[ab^*\, s_{\Downarrow\Uparrow}\, r(t)] \tag{26}$$

where

$$r(t) = \langle \mathcal{E}_{\Downarrow}(t) | \mathcal{E}_{\Uparrow}(t) \rangle = \prod_{i=1}^{N} \left(|\alpha_i|^2\, e^{-i g_i t} + |\beta_i|^2\, e^{i g_i t} \right) \tag{27}$$

This means that, in eq. (22), $\Gamma_0(t) = 1$ and $\Gamma_1(t) = r(t)$.

IV. SIMULATION VERSUS PREDICTION

A. Simulation: the usual method

In order to know the time-behavior of the expectation value of eq. (26), the time-behavior of $r(t)$ has to be computed. From eq. (26) $|r(t)|^2$ results

$$|r(t)|^2 = \prod_{i=1}^{N} (|\alpha_i|^4 + |\beta_i|^4 + 2|\alpha_i|^2 |\beta_i|^2 \cos 2 g_i t) \tag{28}$$

If $|\alpha_i|^2$ and $|\beta_i|^2$ are taken as random numbers in the closed interval $[0,1]$, such that $|\alpha_i|^2+|\beta_i|^2=1$, then

$$\max_t(|\alpha_i|^4+|\beta_i|^4+2|\alpha_i|^2|\beta_i|^2\cos 2g_it)=\left(\left(|\alpha_i|^2+|\beta_i|^2\right)^2\right)=1$$
$$\min_t\left(|\alpha_i|^4+|\beta_i|^4+2\left|\alpha_i\right|^2\left|\beta_i\right|^2\cos\left(2g_it\right)\right)=\left(\left(|\alpha_i|^2-|\beta_i|^2\right)^2\right)=\left(2\left|\alpha_i\right|^2-1\right)^2 \tag{29}$$

Therefore, $(|\alpha_i|^4+|\beta_i|^4+2|\alpha_i|^2|\beta_i|^2\cos 2g_it)$ is a random number which, if $t\neq 0$, fluctuates between 1 and $\left(2\left|\alpha_i\right|^2-1\right)^2$.

In order to obtain the limit of $r(t)$ for $t\to\infty$, different numerical simulations are performed and presented in the literature, where the aleatory numbers $|\alpha_i|^2$ and $|\beta_i|^2$ are obtained from a generator of aleatory numbers: the generator fixed the value of $|\alpha_i|^2$, and the $|\beta_i|^2$ is computed as $|\beta_i|^2=1-|\alpha_i|^2$. The value of the g_i and the time interval $[0,t_0]$ for the computations is usually stipulated. In general, the model is studied and the conclusions about decoherence are drawn by means of this kind of numerical simulations ([8], [3], [12], [4]).

B. Prediction: using the lemma

The first step for applying the lemma is to express the expectation values in the energy eigenbasis. In this model the energy eigenbasis coincides with the eigenbasis of the spin in direction z, which is obtained in terms of the tensorial product of the eigenstates of the spin in direction z for all the particles. In other words, the eigenvectors of H, which form a basis of $\mathcal{H}$, are

$$\begin{aligned}&|\Uparrow\rangle|\uparrow_1\rangle...|\uparrow_k\rangle...|\uparrow_{N-1}\rangle|\uparrow_N\rangle\\&|\Uparrow\rangle|\uparrow_1\rangle...|\uparrow_k\rangle...|\uparrow_{N-1}\rangle|\downarrow_N\rangle\\&...\\&|\Downarrow\rangle|\downarrow_1\rangle...|\downarrow_k\rangle...|\downarrow_{N-1}\rangle|\downarrow_N\rangle\end{aligned} \tag{30}$$

Then, it is easy to see that in eq. (16) H is written in its diagonal form, and that the expectation value of eq. (26) is expressed in the energy eigenbasis.

In order to simplify the expressions, we will introduce a particular arrangement into the set of those eigenvectors by calling them $|\mathcal{A}_i\rangle$: the set $\{|\mathcal{A}_i\rangle\}$ is an eigenbasis of H with 2^{N+1} elements. The $|\mathcal{A}_i\rangle$ will be ordered in terms of their eigenvalues, which depend on the number of particles of E having spin $|\downarrow\rangle$ in any state. Then, we have:

- Two states where all the particles of E have spin $|\uparrow\rangle$:

$$|\mathcal{A}_1\rangle=|\Uparrow,\uparrow,...,\uparrow,\uparrow\rangle\Longrightarrow H|\mathcal{A}_1\rangle=\frac{1}{2}\left(\sum_{i=1}^N g_i\right)|\mathcal{A}_1\rangle$$
$$|\mathcal{A}_{-1}\rangle=|\Downarrow,\uparrow,...,\uparrow,\uparrow\rangle\Longrightarrow H|\mathcal{A}_{-1}\rangle=-\frac{1}{2}\left(\sum_{i=1}^N g_i\right)|\mathcal{A}_{-1}\rangle \tag{31}$$

- $2N$ states where only one particle of E has spin $|\downarrow\rangle$:

$$|\mathcal{A}_j\rangle=|\Uparrow,\uparrow,...,\uparrow,\downarrow,\uparrow,...,\uparrow,\uparrow\rangle\Longrightarrow H|\mathcal{A}_j\rangle=\frac{1}{2}\left(\sum_{i=1}^N g_i-g_k\right)|\mathcal{A}_j\rangle$$
$$|\mathcal{A}_{-j}\rangle=|\Downarrow,\uparrow,...,\uparrow,\downarrow,\uparrow,...,\uparrow,\uparrow\rangle\Longrightarrow H|\mathcal{A}_{-j}\rangle=-\frac{1}{2}\left(\sum_{i=1}^N g_i-g_k\right)|\mathcal{A}_{-j}\rangle$$
$$\text{with}\quad j=2,3,...,N+1\text{ and }k=1,2,...,N \tag{32}$$

- $(N-1)N$ states where two particles have spin $|\downarrow\rangle$:

$$|\mathcal{A}_j\rangle=|\Uparrow,\uparrow,,...,\uparrow,\downarrow,\uparrow,,...,\uparrow,\downarrow,\uparrow,...,\uparrow,\uparrow\rangle\Longrightarrow H|\mathcal{A}_j\rangle=\frac{1}{2}\left(\sum_{i=1}^N g_i-g_k-g_l\right)|\mathcal{A}_j\rangle$$
$$|\mathcal{A}_{-j}\rangle=|\Uparrow,\uparrow,,...,\uparrow,\downarrow,\uparrow,,...,\uparrow,\downarrow,\uparrow,...,\uparrow,\uparrow\rangle\Longrightarrow H|\mathcal{A}_{-j}\rangle=-\frac{1}{2}\left(\sum_{i=1}^N g_i-g_k-g_l\right)|\mathcal{A}_{-j}\rangle$$
$$\text{with}\quad j=N+2,N+3,...,N+1+\frac{(N-1)N}{2}\text{ and }k,l=1,2,...,N \tag{33}$$

- For the remaining particles with more spins $|\downarrow\rangle$, the procedure is analogous.

Let us consider the two extreme cases. If the coupling coefficients g_i are random numbers, in principle all eigenvectors are different. If the coupling coefficients are $g_i = g$, then there are

$$\begin{aligned}
&2 \text{ eigenvectors with eigenvalue } \frac{N}{2}g \\
&2N \text{ eigenvectors with eigenvalue } \frac{N-2}{2}g \\
&\vdots \\
&2\frac{N!}{(N-l)!l!} \text{ eigenvectors with eigenvalue } \frac{N-2l}{2}g
\end{aligned} \tag{34}$$

with $l = 0, 1, ...N$. In this case, H is degenerate: it has 2^{N+1} eigenvectors but only $2N$ different eigenvalues. In both cases, random g_i or equal g_i, the number of different possible energies is large when N is large enough.

On the other hand, eq. (26) can be written as

$$\langle O_R \rangle_{\psi(t)} = \sum_i \rho_i O_i \, + \sum_\nu \rho_\nu^* O_\nu \, e^{i\omega_\nu t} \tag{35}$$

where

$$\sum_i \rho_i O_i = |a|^2 \, s_{\Uparrow\Uparrow} + |b|^2 \, s_{\Downarrow\Downarrow} \tag{36}$$

$$\sum_\nu \rho_\nu^* O_\nu \, e^{i\omega_\nu t} = 2 \, \mathrm{Re}[ab^* \, s_{\Downarrow\Uparrow} \prod_{i=1}^{N} \left(|\alpha_i|^2 \, e^{-ig_i t} + |\beta_i|^2 \, e^{ig_i t} \right)] \tag{37}$$

The r.h.s. of this last expression includes a binomial product which can be rewritten as a sum by means of the following strategy. First we define the index ν that establishes the number of the term of the sum: since in eq. (37) there are 2^N terms, then $\nu = 0, 1, \cdots 2^N - 1$. Then we define the number $p_{\nu,i}$ as the i digit of the number ν written in the binary system. Moreover, each term of the sum is a product of N exponentials of the form $e^{-ig_i t}$, which can be grouped into a single exponential $e^{i\omega_\nu t}$. The ω_ν are all possible additions and subtractions between the coefficients g_i; so, a generic ω_ν can be computed as

$$\omega_\nu = \left(\textstyle\sum_{i=1}^{N} (-1)^{p_{\nu,i}} \, g_i \right) \tag{38}$$

Precisely

$$\begin{aligned}
&\nu = 0 = 0 \cdots 000_b \longrightarrow \omega_0 = \left(\textstyle\sum_{i=1}^{N} g_i \right) \\
&\nu = 1 = 0 \cdots 001_b \longrightarrow \omega_1 = \left(\textstyle\sum_{i=1}^{N-1} g_i - g_N \right) \\
&\nu = 2 = 0 \cdots 010_b \longrightarrow \omega_2 = \left(\textstyle\sum_{i=1}^{N-2} g_i - g_{N-1} + g_N \right) \\
&\nu = 3 = 0 \cdots 011_b \longrightarrow, \omega_3 = \left(\textstyle\sum_{i=1}^{N-2} g_i - g_{N-1} - g_N \right) \\
&\nu = 4 = 0 \cdots 100_b \longrightarrow \omega_4 = \left(\textstyle\sum_{i=1}^{N-2} g_i - g_{N-2} + g_{N-1} + g_N \right) \\
&\vdots \\
&\nu = 2^N - 1 = 1 \cdots 1_b \longrightarrow \omega_{2^N} = \left(-\textstyle\sum_{i=1}^{N} g_i \right)
\end{aligned} \tag{39}$$

On the basis of this strategy, we can define the discrete function $f_d(\omega_\nu)$ as

$$f_d(\omega_\nu) = \prod_{k=1}^{N} |\gamma_{\nu,k}|^2 \tag{40}$$

where $\gamma_{\nu,k} = (\alpha_k - \beta_k)\, p_{\nu,k} + \beta_k$, which is equal to α_k if $p_{\nu,k} = 1$ and is equal to β_k if $p_{\nu,k} = 0$. Then, the binomial product of eq. (37) results

$$\prod_{i=1}^{N} \left(|\alpha_i|^2\, e^{-ig_i t} + |\beta_i|^2\, e^{ig_i t} \right) = \sum_{\nu=0}^{2^N-1} f_d(\omega_\nu) e^{-i\omega_\nu t} \tag{41}$$

then,

$$\langle O_R \rangle_{\psi(t)} = \sum_i \rho_i O_i \; + 2\,\mathrm{Re}[ab^*\, s_{\Downarrow\Uparrow} \sum_{\nu=0}^{2^N-1} f_d(\omega_\nu)\, e^{-i\omega_\nu t}] \tag{42}$$

In order to apply our lemma to eq. (42), it is necessary that $f_d(\omega_\nu) \in \mathcal{L}_1$. On the one hand, since $\{\omega_\nu\}$ has 2^N elements, then for $N \gg 1$ the set $\{\omega_\nu\}$ is quasi-continuous of class 1. On the other hand, since $f_d(\omega_\nu)$ is defined in eq. (40), where $0 < |\gamma_{\nu,k}|^2 < 1$, then $f_d(\omega_\nu)$ is the product of N numbers lower than 1. Therefore, if $N \gg 1$,

$$\begin{aligned} |\gamma_{\nu,k}|^2 < 1 \Rightarrow f_d(\omega_\nu) = \prod_{k=1}^{N} |\gamma_{\nu,k}|^2 \ll 1 \text{ if } N \gg 1 \quad \Rightarrow \quad 0 < f_d(\omega_\nu) < \varepsilon \ll 1 \\ \Rightarrow \max_{\mu,\nu} \left(|f_d(\omega_\mu) - f_d(\omega_\nu)| \right) \ll 1 \end{aligned} \tag{43}$$

But this is precisely the condition for $f_d(\omega_\nu) \in \mathcal{L}_1$. As a consequence, according to the lemma,

$$\sum_\nu \rho_\nu^* O_\nu\, e^{i\omega_\nu t} \to 0 \quad \Longrightarrow \quad \langle O_R \rangle_{\psi(t)} \to \sum_i \rho_i O_i \tag{44}$$

and we can conclude, with no need of computer simulations, that the system decoheres.

V. CONCLUSIONS

In another paper of this issue, a discrete analogue of the Riemann-Lebesgue theorem is presented and, on this basis, a lemma relevant for discrete models is introduced: such a lemma provides a criterion for deciding whether or not the system decoheres with no need of numerical simulations. In order to present an example of how these results can be usefully exploited for the study of decoherence in discrete models, in this paper we have applied that lemma to the well-known spin-bath model, and we have shown that the conclusion drawn from that application agrees with the results obtained by means of computer simulations in the previous literature.

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