On the concurrence of superpositions of many states

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In this paper we use the *concurrence vector*, as a measure of entanglement, and investigate lower and upper bounds on the concurrence of a superposition of bipartite states as a function of the concurrence of the superposed states. We show that the amount of entanglement quantified by the concurrence vector is exactly the same as that quantified by *I-concurrence*, so that our results can be compared to those given in [Phys. Rev. A **76**, 042328 (2007)]. We obtain a tighter lower bound in the case that two superposed states are orthogonal. We also show that when the two superposed states are not necessarily orthogonal, both lower and bounds are, in general, tighter than the bounds given in terms of the I-concurrence. An extension of the results to the case with more than two states in the superpositions is also given.

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I. INTRODUCTION

Quantum entanglement is one of the most challenging feature of quantum mechanics which has recently attracted much attention in view of its connection with the theory of quantum information and computation. It turns out that quantum entanglement provides a fundamental potential resource for communication and information processing, therefore, both characterization and quantification of the entanglement are important tasks in the theory of quantum information.

Although entanglement is a global property of a state of bipartite system, originating from the superposition of different states, but the entanglement of a superposition of pure states cannot be simply expressed as a function of the entanglement of the individual states in the superposition. Recently, Linden, Popescu and Smolin [1] have raised the following problem: Given a bipartite quantum state $|\Gamma\rangle$ and a certain decomposition of it as a superposition of two others

$$|\Gamma\rangle = \alpha|\psi\rangle + \beta|\phi\rangle, \tag{1}$$

what is the relation between the entanglement of $|\Gamma\rangle$ and those of the two states in the superposition? They found an upper bound on the entanglement of $|\Gamma\rangle$ in terms of the entanglement of $|\psi\rangle$ and $|\phi\rangle$, using the von Neumann entropy of the reduced state of either of the parties as a measure of entanglement [2].

In order to have motivation on this problem, it is worth considering two simple examples of a two-qubit system [1]. Let us first consider a state of two-qubit system as $|\gamma\rangle = \frac{1}{\sqrt{2}}|0\rangle|0\rangle + \frac{1}{\sqrt{2}}|1\rangle|1\rangle$. As we know, each term in the superposition is unentangled, yet the superposition is a maximally entangled state of the two-qubit system. On the other hand, consider $|\gamma'\rangle = \frac{1}{\sqrt{2}}|\Phi^+\rangle + \frac{1}{\sqrt{2}}|\Phi^-\rangle$, where $|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$. In this case, it is evident that each of the terms in the superposition is maximally entangled, yet the superposition itself is unentangled. It follows therefore that the entanglement of a superposition may be very different from the entanglement of the superposed states.

Several authors have generalized the results of [1] to include different measures of entanglement [3–6], superpositions of more than two states [7], and superpositions of multipartite states [8, 9].

The aim of this paper is to use the notion of *concurrence vector* [10], as a measure of entanglement, and provide tight lower and upper bounds on the entanglement of superpositions of more than two pure states of a bipartite system. Investigation of the entanglement of the superposition of two states, by using generalized concurrence as a measure of entanglement, are given recently in [3, 5]. We shall see that our definition of the generalized concurrence, based on the concurrence vector, is completely equivalent to the generalized concurrence introduced by Rungta *et al.* [11], known as *I-concurrence*, so that we can directly compare our results with those of Ref. [5]. We show that when the two superposed states are orthogonal, our lower bound is tighter than that given in [5], and also in the case that the two superposed states are not necessarily orthogonal, both lower and upper bounds are, in general, tighter than those given [5]. An extension of the problem to include superpositions of more than two states is also presented. In the case that the states being superposed are biorthogonal, we obtain an exact solution for the concurrence of superpositions. In other situations, where the states in the superposition are one-sided orthogonal, orthogonal, and arbitrary, we present tight lower and upper bounds on the concurrence of superposition.

The paper is organized as follows. First, in section II, we give a brief review on the concurrence vector, and show that this definition is equivalent to the so-called

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I-concurrence. In this section we also provide some definitions and useful lemmas, which will be used repeatedly in the subsequent sections. Section III is devoted to provide bounds on the concurrence of superpositions of two terms. In section IV, we extend our results and investigate the concurrence of superpositions of more than two states. The paper is concluded in section V, with a brief conclusion.

II. CONCURRENCE VECTOR

We begin by introducing the generalized concurrence, based on the definition of the concurrence vector. The generalized concurrence for a bipartite pure state $|\psi\rangle \in \mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ is given by [10]

$$C(\psi) = \sqrt{\sum_\alpha \sum_\beta |C_{\alpha\beta}(\psi)|^2}, \tag{2}$$

where $C_{\alpha\beta}(\psi)$, with $\alpha = 1, \cdots, N_1(N_1-1)/2$, $\beta = 1, \cdots, N_2(N_2-1)/2$ are the components of the concurrence vector $\mathbf{C}(\psi)$, defined by

$$C_{\alpha\beta}(\psi) = \langle\psi|S_{\alpha\beta}|\psi^*\rangle. \tag{3}$$

Here $S_{\alpha\beta} = L_\alpha \otimes L_\beta$, where L_α and L_β are generators of $SO(N_1)$ and $SO(N_2)$, respectively.

For further use, we extend the above definition of the concurrence of a pure state to the concurrence of two pure states. Let $|\psi\rangle$ and $|\phi\rangle$ be two pure states of $\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$. We define $C(\psi, \phi)$ as the concurrence of $|\psi\rangle$ and $|\phi\rangle$ by

$$C(\psi, \phi) = \sqrt{\sum_\alpha \sum_\beta |C_{\alpha\beta}(\psi, \phi)|^2}, \tag{4}$$

where $C_{\alpha\beta}(\psi, \phi)$ are the components of the concurrence vector $\mathbf{C}(\psi, \phi)$, defined by

$$\begin{aligned} C_{\alpha\beta}(\psi, \phi) &= \langle\psi|S_{\alpha\beta}|\phi^*\rangle \\ &= \langle\phi|S_{\alpha\beta}|\psi^*\rangle = C_{\alpha\beta}(\phi, \psi). \end{aligned} \tag{5}$$

The second line indicates that this definition is symmetric in its arguments, i.e. $\mathbf{C}(\psi, \phi) = \mathbf{C}(\phi, \psi)$. Notice that according to the above definition, we have $\mathbf{C}(\psi, \psi) = \mathbf{C}(\psi)$.

Before starting our study, it is worth making observation that the definition (2) for the concurrence is equivalent to the definition of the I-concurrence given by Rungta *et al.* [11]. To do so, note that in Eq. (2) we need to calculate the expectation value of the term $\sum_\alpha \sum_\beta S_{\alpha\beta}(|\psi^*\rangle\langle\psi^*|)S_{\alpha\beta}$. Motivated by this, we define the superoperator $\mathcal{S}$ such that its action on an arbitrary operator σ is given by

$$\mathcal{S}(\sigma) = \sum_\alpha \sum_\beta S_{\alpha\beta}(\sigma)^T S_{\alpha\beta}. \tag{6}$$

Now, invoking the definition of $S_{\alpha\beta}$, we find that Eq. (6) can be written as

$$\mathcal{S}(\sigma) = \mathrm{Tr}(\sigma) I_A \otimes I_B - \sigma_A \otimes I_B - I_A \otimes \sigma_B + \sigma. \tag{7}$$

This equation resembles the definition of the universal invertor superoperator, introduced in Ref. [11]. Using this, we can rewrite the concurrence of Eq. (2) as

$$\begin{aligned} C(\psi) &= \sqrt{\sum_\alpha \sum_\beta \langle\psi|S_{\alpha\beta}(|\psi^*\rangle\langle\psi^*|)S_{\alpha\beta}|\psi\rangle} \\ &= \sqrt{\langle\psi|\mathcal{S}(|\psi\rangle\langle\psi|)|\psi\rangle} = \sqrt{2\,(1 - \mathrm{Tr}[\rho_A^2])}, \end{aligned} \tag{8}$$

which is exactly the same as the I-concurrence introduced by Rungta *et al.* [11].

Now we give below the definition of biorthogonal and one-sided orthogonal states.

Definition 1: Biorthogonal and one-sided orthogonal states. Two bipartite pure states $|\psi\rangle$ and $|\phi\rangle$ are called biorthogonal if satisfy the following conditions

$$\mathrm{Tr}_A[\rho_\psi^A \rho_\phi^A] = 0 \Longleftrightarrow \langle e_i^\psi | e_j^\phi\rangle = 0, \tag{9}$$

and

$$\mathrm{Tr}_B[\rho_\psi^B \rho_\phi^B] = 0 \Longleftrightarrow \langle f_i^\psi | f_j^\phi\rangle = 0, \tag{10}$$

where $\rho_\psi^A = \mathrm{Tr}_B[|\psi\rangle\langle\psi|] = \sum_i (\lambda_i^\psi)^2 |e_i^\psi\rangle\langle e_i^\psi|$ and $\rho_\psi^B = \mathrm{Tr}_A[|\psi\rangle\langle\psi|] = \sum_i (\lambda_i^\psi)^2 |f_i^\psi\rangle\langle f_i^\psi|$ are the reduced density matrices of $|\psi\rangle$ on the A and B subsystems, respectively, with $|e_i^\psi\rangle$ and $|f_i^\psi\rangle$ as the local Schmidt basis of $|\psi\rangle$, with the corresponding Schmidt numbers λ_i^ψ. Similar definitions are hold for ρ_ϕ^A, ρ_ϕ^B, $|e_j^\phi\rangle$ and $|f_j^\phi\rangle$.

On the other hand, we say that $|\psi\rangle$ and $|\phi\rangle$ are one-sided orthogonal states if they satisfy one of the conditions (9) and (10). In the following, without loss of generally, we assume that one-sided orthogonal states satisfy the first condition (9), but not necessarily the condition (10).

The following lemma hold for the concurrence vectors of a pair of biorthogonal or one-sided orthogonal states, and will be used in the following sections.

Lemma 1. Let $|\psi\rangle$ and $|\phi\rangle$ be either biorthogonal or one-sided orthogonal states. Then the inner product of any pair of the concurrence vectors $\mathbf{C}(\psi)$, $\mathbf{C}(\phi)$ and $\mathbf{C}(\psi, \phi)$ are zero.

Proof. Using Eqs (3), (5) and (7) we get

$$\begin{aligned} \mathbf{C}(\psi)\cdot\mathbf{C}(\phi) &= 2(\langle\psi|\phi\rangle)^2 - \mathrm{Tr}_A[(\mathrm{Tr}_B[|\phi\rangle\langle\psi|])^2] \\ &- \mathrm{Tr}_B[(\mathrm{Tr}_A[|\phi\rangle\langle\psi|])^2], \end{aligned} \tag{11}$$

and

$$\mathbf{C}(\psi)\cdot\mathbf{C}(\psi, \phi) = 2\langle\psi|\phi\rangle - 2\mathrm{Tr}_A[(\mathrm{Tr}_B[|\phi\rangle\langle\psi|])\rho_\psi^A] \tag{12}$$

and a similar relation for $\mathbf{C}(\phi)\cdot\mathbf{C}(\psi, \phi)$. Now by writing $|\psi\rangle$ and $|\phi\rangle$ in their Schmidt forms, it is straightforward to check that every term in the right-hand side of the Eqs. (11) and (12) can be written as a sum of terms including $\langle e_i^\psi|e_j^\phi\rangle\langle f_k^\psi|f_l^\phi\rangle$. But it follows from condition (9) or (10) that in these cases $\langle e_i^\psi|e_j^\phi\rangle = 0$ or $\langle f_i^\psi|f_j^\phi\rangle = 0$, which completes the proof.

The following lemma is a generalization of the above lemma to the case of more than two biorthogonal or one-sided orthogonal states.

Lemma 2. Let $\{|\psi_i\rangle\}$ denotes a set of either biorthogonal or one-sided orthogonal states. Then the set of concurrence vectors $\{\mathbf{C}(\psi_i,\psi_j)\}$ are orthogonal, i.e. $\mathbf{C}(\psi_i,\psi_j)\cdot\mathbf{C}(\psi_{i'},\psi_{j'}) = 0$ unless $\mathbf{C}(\psi_i,\psi_j) = \mathbf{C}(\psi_{i'},\psi_{j'})$.

Proof. The proof is similar to that given for lemma 1.

Lemma 3. Let $|\psi\rangle$ and $|\phi\rangle$ be two pure states of the bipartite system. Then for the concurrence $C(\psi,\phi)$ we have the following relations.

1. If $|\psi\rangle$ and $|\phi\rangle$ are biorthogonal, then $C(\psi,\phi) = 1$.
2. If $|\psi\rangle$ and $|\phi\rangle$ are one-sided orthogonal, then $C(\psi,\phi) \le 1$.
3. If $|\psi\rangle$ and $|\phi\rangle$ are orthogonal, then $C(\psi,\phi) \le 1$.
4. If $|\psi\rangle$ and $|\phi\rangle$ are arbitrary states, then $C(\psi,\phi) \le \sqrt{1+|\langle\psi|\phi\rangle|^2}$.

Proof. Using Eqs (4) and (7), we find

$$C^2(\psi,\phi) = 1 + |\langle\psi|\phi\rangle|^2 - \mathrm{Tr}_A[\rho_\psi^A\rho_\phi^A] - \mathrm{Tr}_B[\rho_\psi^B\rho_\phi^B]. \quad (13)$$

Next, by using the conditions on each class, we obtain the results of the lemma.

Armed with these lemmas, we are now in a position to study the bounds on the concurrence of superpositions. We first consider the case that we have superpositions of two states.

III. SUPERPOSITIONS OF TWO STATES

To begin with, let us consider the more general case where the two component states in the superposition (1) are arbitrary, not necessarily orthogonal, and therefore the superposition is not normalized. If we define $|\Gamma'\rangle = |\Gamma\rangle/\|\Gamma\|$ as the normalized version of $|\Gamma\rangle$, then we can obtain bounds for the concurrence of the normalized version of the superposition. To do so, we plug Eq. (1) into Eq. (3), and by putting the result into the definition of concurrence, given in Eq. (2), we get the following relation for the concurrence of $|\Gamma'\rangle$

$$\begin{aligned}\|\Gamma\|^2 C(\Gamma') \; = \; \{\sum_\alpha\sum_\beta \mid (\alpha^*)^2 C_{\alpha\beta}(\psi) + (\beta^*)^2 C_{\alpha\beta}(\phi) \\ + \; 2(\alpha^*\beta^*)C_{\alpha\beta}(\psi,\phi) \mid^2\}^{1/2}, \quad (14)\end{aligned}$$

where $C_{\alpha\beta}(\psi,\phi)$ are defined in Eq. (5). This relation is general, irrespective of the orthogonality properties of $|\psi\rangle$ and $|\phi\rangle$. In the following subsections we will restrict our attention to some special cases.

A. Biorthogonal states

First we consider the more restrictive case where the two pure states $|\psi\rangle$ and $|\phi\rangle$ are biorthogonal states, that is, they satisfy both conditions given in Eqs. (9) and (10). In this case we can find an exact expression for the concurrence of superposition.

Theorem 1: Biorthogonal states. When $|\psi\rangle$ and $|\phi\rangle$ are biorthogonal states, then the concurrence of the superposition $|\Gamma\rangle = \alpha|\psi\rangle + \beta|\phi\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$, is given by

$$C(\Gamma) = \sqrt{|\alpha|^4 C^2(\psi) + |\beta|^4 C^2(\phi) + 4|\alpha\beta|^2}. \quad (15)$$

Proof. In this case, according to the lemma 1, the inner product of the concurrence vectors $\mathbf{C}(\psi)$, $\mathbf{C}(\phi)$ and $\mathbf{C}(\psi,\phi)$ are zero, therefore Eq. (14) reduces to

$$C(\Gamma) = \sqrt{|\alpha|^4 C^2(\psi) + |\beta|^4 C^2(\phi) + 4|\alpha\beta|^2 C^2(\psi,\phi)}. \quad (16)$$

But recall that $C(\psi,\phi) = 1$, if $|\psi\rangle$ and $|\phi\rangle$ are biorthogonal states (lemma 3). Using this we get Eq. (15).

Remark 1. If $|\psi\rangle$ and $|\phi\rangle$ are two biorthogonal states of a two-qubit system, then it requires that $C(\psi) = C(\phi) = 0$, therefore Eq. (15) reduces to $C(\Gamma) = 2|\alpha\beta|$.

B. One-sided orthogonal states

Now we consider the case of $|\psi\rangle$ and $|\phi\rangle$ being one-sided orthogonal but not necessarily biorthogonal, that is, they satisfy condition (9). In this case we can find the lower and upper bounds for the concurrence of superposition.

Theorem 2: One-sided orthogonal states. When $|\psi\rangle$ and $|\phi\rangle$ are one-sided orthogonal states, then the concurrence of the superposition $|\Gamma\rangle = \alpha|\psi\rangle + \beta|\phi\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$, satisfy

$$C(\Gamma) \le \sqrt{|\alpha|^4 C^2(\psi) + |\beta|^4 C^2(\phi) + 4|\alpha\beta|^2}, \quad (17)$$

and

$$C(\Gamma) \ge \sqrt{|\alpha|^4 C^2(\psi) + |\beta|^4 C^2(\phi)}. \quad (18)$$

Proof. In this case, again, according to lemma 1 the inner product of the concurrence vectors $\mathbf{C}(\psi)$, $\mathbf{C}(\phi)$ and $\mathbf{C}(\psi,\phi)$ are zero, therefore Eq. (16) is satisfied. But from lemma 3 we have $0 \le C^2(\psi,\phi) \le 1$, where completes the proof.

Remark 2. If $|\psi\rangle$ and $|\phi\rangle$ are two one-sided orthogonal states of a two-qubit system, then they are necessarily biorthogonal, so again $C(\psi) = C(\phi) = 0$, and we find $C(\Gamma) = 2|\alpha\beta|$.

C. Orthogonal states

Now we assume that the two component states of the superposition (1) are orthogonal, i.e. $\langle\psi|\phi\rangle = 0$, but not necessarily one-sided orthogonal. In this case we find the following bounds for the concurrence of superposition.

Theorem 3: Orthogonal states. Let $|\psi\rangle$ and $|\phi\rangle$ be two orthogonal states of a general bipartite system. Then the concurrence of the superposition $|\Gamma\rangle = \alpha|\psi\rangle + \beta|\phi\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$, satisfies

$$C(\Gamma) \leq |\alpha|^2 C(\psi) + |\beta|^2 C(\phi) + 2|\alpha\beta|, \tag{19}$$

and

$$C(\Gamma) \geq \left||\alpha|^2 C(\psi) - |\beta|^2 C(\phi)\right| - 2|\alpha\beta|. \tag{20}$$

proof: Successive application of the Minkowski inequality [12]

$$\left[\sum_{i=1}^{n} |x_i + y_i|^p\right]^{1/p} \leq \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n} |y_i|^p\right]^{1/p}, \tag{21}$$

with $p > 1$, to Eq. (14), together with the definition of the concurrence given in Eq. (2), lead to

$$C(\Gamma) \leq |\alpha|^2 C(\psi) + |\beta|^2 C(\phi) + 2|\alpha\beta| C(\psi, \phi). \tag{22}$$

Now using the fact that $C(\psi, \phi) \leq 1$ (lemma 3), directly leads to the upper bound (19).

Next in order to proof the lower bound (20), we make use of the inverse Minkowski inequality [12]

$$\begin{aligned} \left[\sum_{i=1}^{n} |x_i + y_i|^p\right]^{1/p} &\geq \left|\left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p} - \left[\sum_{i=1}^{n} |y_i|^p\right]^{1/p}\right| \\ &\geq \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p} - \left[\sum_{i=1}^{n} |y_i|^p\right]^{1/p} \end{aligned} \tag{23}$$

and apply it to Eq. (14), two times. First, by using the second line of Eq. (23), we separate the first two terms of Eq. (14) from the third one, and second we use the first line of Eq. (23) and separate between the first two terms. We get therefore

$$C(\Gamma) \geq \left||\alpha|^2 C(\psi) - |\beta|^2 C(\phi)\right| - 2|\alpha\beta| C(\psi, \phi). \tag{24}$$

Again, by using the fact that $C(\psi, \phi) \leq 1$, for orthogonal states, we deduce the advertised inequality (20).

Remark 3. If $|\psi\rangle$ and $|\phi\rangle$ are two orthogonal states of a two-qubit system, then $C(\psi, \phi) \leq \sqrt{1-\delta^2}$, where $\delta = \max\{C(\psi), C(\phi)\}$ [5]. Applying this to Eqs. (22) and (24), we obtain tighter bounds for the two-qubit case [5].

It is worth to mention, however, that although the upper bound (19) is exactly the same as the corresponding upper bound given by Niset *et al.* [5], the lower bound (20) is tighter than that they have obtained. The difference is, indeed, between the third term of Eq. (20), in the sense that the authors of [5] obtained $2|\alpha\beta|(1+\delta)$, with $\delta = \min(|\frac{\alpha}{\beta}|C(\psi_1), |\frac{\beta}{\alpha}|C(\psi_2))$, instead of $2|\alpha\beta|$. It is interesting to note that they have suspected that if a more appropriate expression exists, it would probably have the correction factor δ equal to zero.

D. Arbitrary states

We consider the more general case where the two component states in the superposition (1) are not orthogonal and therefore the superposition is not normalized. In this case we find the following bounds for the concurrence of the normalized version $|\Gamma'\rangle$.

Theorem 4: Arbitrary states. Let $|\psi\rangle$ and $|\phi\rangle$ be two arbitrary states of a general bipartite system. The concurrence of the normalized state $|\Gamma'\rangle = |\Gamma\rangle / \parallel \Gamma \parallel$, with $|\Gamma\rangle = \alpha|\psi\rangle + \beta|\phi\rangle$ and $|\alpha|^2 + |\beta|^2 = 1$, satisfies

$$\|\Gamma\|^2 C(\Gamma') \leq \min \begin{cases} |\alpha|^2 C(\psi) + |\beta|^2 C(\phi) + 2|\alpha\beta|\sqrt{1 + |\langle\psi|\phi\rangle|^2}, \\ |\alpha|^2 C(\psi) + |\beta^2 + 2\alpha\beta\langle\phi|\psi\rangle| C(\phi) + 2|\alpha\beta|\sqrt{1 - |\langle\psi|\phi\rangle|^2}, \\ |\alpha^2 + 2\alpha\beta\langle\psi|\phi\rangle| C(\psi) + |\beta|^2 C(\phi) + 2|\alpha\beta|\sqrt{1 - |\langle\psi|\phi\rangle|^2}, \end{cases} \tag{25}$$

and

$$\|\Gamma\|^2 C(\Gamma') \geq \max \begin{cases} \left||\alpha|^2 C(\psi) - |\beta|^2 C(\phi)\right| - 2|\alpha\beta|\sqrt{1 + |\langle\psi|\phi\rangle|^2}, \\ \left||\alpha|^2 C(\psi) - |\beta^2 + 2\alpha\beta\langle\phi|\psi\rangle| C(\phi)\right| - 2|\alpha\beta|\sqrt{1 - |\langle\psi|\phi\rangle|^2}, \\ \left||\alpha^2 + 2\alpha\beta\langle\psi|\phi\rangle| C(\psi) - |\beta|^2 C(\phi)\right| - 2|\alpha\beta|\sqrt{1 - |\langle\psi|\phi\rangle|^2}. \end{cases} \tag{26}$$

Note that when the two states $|\psi\rangle$ and $|\phi\rangle$ are orthogonal, then Eqs. (25) and (26) reduce to Eqs. (19) and (20), respectively.

Proof: For the upper bound (25), we should proof that the left-hand side of Eq. (25) is less that each term in the right-hand side. For the first line, we proceed exactly the same as that we done in the proof of the upper bound of theorem 3, but here we use the fact that

$C^2(\psi,\phi) \le 1 + |\langle\psi|\phi\rangle|^2$. Now to reach the second line of Eq. (25), we write $|\phi\rangle = \langle\psi|\phi\rangle\,|\psi\rangle + \sqrt{1-|\langle\psi|\phi\rangle|^2}|\psi_\perp\rangle$, where $|\psi_\perp\rangle$ is a vector orthogonal to $|\psi\rangle$. Putting this into Eq. (5), we find $C_{\alpha\beta}(\psi,\phi) = \langle\psi|\phi\rangle^* C_{\alpha\beta}(\psi) + \sqrt{1-|\langle\psi|\phi\rangle|^2} C_{\alpha\beta}(\psi,\psi_\perp)$, where can be used in Eq. (14), getting

$$\begin{aligned}\|\Gamma\|^2 C(\Gamma') &= \{\sum_\alpha\sum_\beta |\left((\alpha^*)^2 + 2\alpha^*\beta^*\langle\psi|\phi\rangle^*\right) C_{\alpha\beta}(\psi) \\ &+ (\beta^*)^2 C_{\alpha\beta}(\phi) \\ &+ 2(\alpha^*\beta^*)\sqrt{1-|\langle\psi|\phi\rangle|^2} C_{\alpha\beta}(\psi,\psi_\perp)\,|^2\}^{1/2}. \end{aligned} \tag{27}$$

Next, by successive application of the Minkowski inequality (21), followed by the fact that $C(\psi,\psi_\perp) \le 1$, we reach to the second line of the inequality (25). Alternatively, if we first expand $|\psi\rangle$ in terms of $|\phi\rangle$ and $|\phi_\perp\rangle$, we get the third line of the inequality (25). This completes the proof for the upper bound (25).

In a similar way, but using the inverse Minkowski inequality instead of the Minkowski inequality, one can obtain the lower bound (26).

Remark 4. If $|\psi\rangle$ and $|\phi\rangle$ are two arbitrary states of a two-qubit system, then $C(\psi,\phi) \le \sqrt{1-\delta^2}$, where $\delta = \max\{|C(\psi) - |\langle\psi|\phi\rangle||, |C(\phi) - |\langle\psi|\phi\rangle||\}$ [5]. Using this in the process of the above proof, one can obtain tighter bounds for the two-qubit case [5].

It is worth to mention that the bounds obtained in Eqs. (25) and (26) are, in general, tighter than the corresponding bounds given by Niset *et al.* [5]. In fact, the authors of [5] reached to the first line of relation (25) for the upper bound which is, in general, weaker than our upper bound. On the other hand, for the lower bound, they have obtained a term such as the first line of Eq. (26), but with $2|\alpha\beta|\sqrt{1+|\langle\psi|\phi\rangle|^2+\delta}$, $\delta = \min(|\frac{\alpha}{\beta}|C(\psi_1), |\frac{\beta}{\alpha}|C(\psi_2))$, instead of $2|\alpha\beta|\sqrt{1+|\langle\psi|\phi\rangle|^2}$.

In order to illustrate the bounds more intuitionally, let us consider a simple example. Let $|\psi\rangle$ and $|\phi\rangle$ be states of a $\mathbb{C}^3\otimes\mathbb{C}^3$ space, defined by

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|11\rangle + \frac{1}{2}|22\rangle, \tag{28}$$

$$|\phi\rangle = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{2}|11\rangle + \frac{1}{2}|22\rangle. \tag{29}$$

Evidently, these states are not orthogonal, $\langle\psi|\phi\rangle = 1/2$, and that they have equal concurrence, $C(\psi) = C(\phi) = \sqrt{5}/2$. In Fig. 1, we illustrate the upper and lower bounds of the superposition (1), with $\alpha = x$ and $\beta = -\sqrt{1-x^2}$. As it is evident from the figure, our bounds give stronger constraints than those derived in Ref. [5].

IV. SUPERPOSITIONS OF MORE THAN TWO STATES

In this section we consider a state of an arbitrary bipartite system constructed as a superposition of m pure states

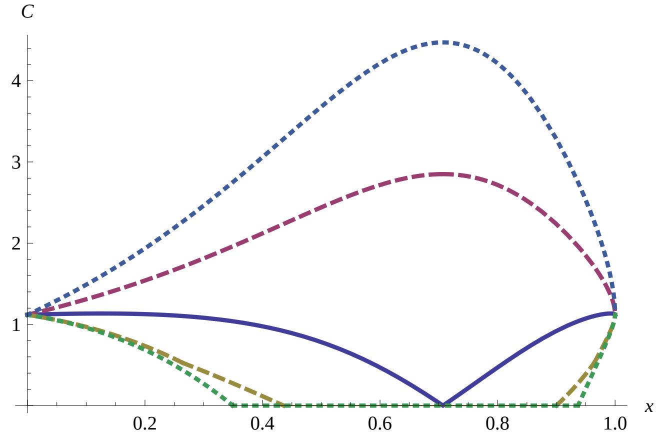

FIG. 1: (Color online) Concurrence of the normalized version of the superposition (1), with $|\psi\rangle$ and $|\phi\rangle$ as defined in the text. The solid line is the exact value of $C(\Gamma')$, while the two dashed lines correspond to the upper and lower bounds derived from Eqs. (25) and (26), and the two dotted lines correspond to the upper and lower bounds given in Ref. [5].

$$|\Gamma\rangle = \sum_{i=1}^m \gamma_i|\psi_i\rangle. \tag{30}$$

We first concern our attention to the more general case where the states $\{|\psi_i\rangle\}$ in the superposition (30) are not orthogonal and therefore the superposition is not normalized. We define $|\Gamma'\rangle = |\Gamma\rangle/\|\Gamma\|$ as the normalized version of $|\Gamma\rangle$, then we shall obtain bounds on the concurrence of the normalized version of the superposition.

To do so, we plug Eq. (30) into Eq. (3), and by putting the result into the definition of the concurrence given in Eq. (2), we get the following relation for the concurrence of $|\Gamma'\rangle$

$$\begin{aligned}\|\Gamma\|^2 C(\Gamma') &= \{\sum_\alpha\sum_\beta |\sum_{i=1}^m (\gamma_i^*)^2 C_{\alpha\beta}(\psi_i) \\ &+ 2\sum_{i<j}^m (\gamma_i^*\gamma_j^*) C_{\alpha\beta}(\psi_i,\psi_j)\,|^2\}^{1/2}. \end{aligned} \tag{31}$$

In the following subsections we will consider, separately, some special cases.

A. Biorthogonal states

First we consider the more restrictive case that the set of pure states $\{|\psi_i\rangle\}$ are biorthogonal states, that is, they satisfy conditions (9) and (10). In this case we will find an exact expression for the concurrence of a superposition.

Theorem 5: Biorthogonal states. When $\{|\psi_i\rangle\}$ are biorthogonal states, then the concurrence of the superposition $|\Gamma\rangle = \sum_{i=1}^m \gamma_i|\psi_i\rangle$, with $\sum_{i=1}^m |\gamma_i|^2 = 1$, is given by

$$C(\Gamma)=\sqrt{\sum_{i=1}^{m}|\gamma_i|^4C^2(\psi_i)+4\sum_{i<j}^{m}|\gamma_i\gamma_j|^2}. \qquad (32)$$

Proof. In this case, according to the lemma 2, the inner product of two different concurrence vectors are zero, i.e. $\mathbf{C}(\psi_i,\psi_j)\cdot\mathbf{C}(\psi_{i'},\psi_{j'})=0$ unless $\mathbf{C}(\psi_i,\psi_j)=\mathbf{C}(\psi_{i'},\psi_{j'})$. Therefore we have

$$C(\Gamma)=\sqrt{\sum_{i=1}^{m}|\gamma_i|^4C^2(\psi_i)+4\sum_{i<j}^{m}|\gamma_i\gamma_j|^2C^2(\psi_i,\psi_j)}. \qquad (33)$$

But recall that $C(\psi_i,\psi_j)=1$ if $|\psi_i\rangle$ and $|\psi_j\rangle$ are biorthogonal states. Using this we get Eq. (32).

B. One-sided orthogonal states

Now we consider the case that the set of pure states $\{|\psi_i\rangle\}$ are one-sided orthogonal states, that is, they satisfy the condition (9). In this case we can find the lower and upper bounds for the concurrence of superposition.

Theorem 6: One-sided orthogonal states. When $\{|\psi_i\rangle\}$ are one-sided orthogonal states, then the concurrence of the superposition $|\Gamma\rangle=\sum_{i=1}^m\gamma_i|\psi_i\rangle$, with $\sum_{i=1}^m|\gamma_i|^2=1$, satisfies

$$C(\Gamma)\le\sqrt{\sum_{i=1}^{m}|\gamma_i|^4C^2(\psi_i)+4\sum_{i<j}^{m}|\gamma_i\gamma_j|^2}, \qquad (34)$$

and

$$C(\Gamma)\ge\sqrt{\sum_{i=1}^{m}|\gamma_i|^4C^2(\psi_i)}. \qquad (35)$$

Proof. In this case again we have that $\mathbf{C}(\psi_i,\psi_j)\cdot\mathbf{C}(\psi_{i'},\psi_{j'})=0$ unless $\mathbf{C}(\psi_i,\psi_j)=\mathbf{C}(\psi_{i'},\psi_{j'})$, therefore Eq. (33) is satisfied. But from lemma 3 we have $0\le C^2(\psi_i,\psi_j)\le1$, where completes the proof.

C. Orthogonal states

In this subsection we assume that the set $\{|\psi_i\rangle\}$ are orthogonal. We find the following bounds for the concurrence of superposition.

Theorem 7: Orthogonal states. Let the set $\{|\psi_i\rangle\}$ be orthogonal states of a general bipartite system. Then the concurrence of the superposition $|\Gamma\rangle=\sum_{i=1}^m\gamma_i|\psi_i\rangle$, with $\sum_{i=1}^m|\gamma_i|^2=1$, satisfies

$$C(\Gamma)\le\sum_{i=1}^{m}|\gamma_i|^2C(\psi_i)+2\sum_{i<j}^{m}|\gamma_i\gamma_j|, \qquad (36)$$

and

$$C(\Gamma)\ge2\Delta-\sum_{i=1}^{m}|\gamma_i|^2C(\psi_i)-2\sum_{i<j}^{m}|\gamma_i\gamma_j|, \qquad (37)$$

where Δ is defined by

$$\Delta=\max\{|\gamma_i|^2C(\psi_i)\},\qquad i=1,\cdots,m. \qquad (38)$$

Proof. Applying the Minkowski inequality (21), repeatedly, to Eq. (31), followed by the fact that in this case we have $C(\psi_i,\psi_j)\le1$, we get the upper bound (36).

Next in order to proof the lower bound (37), we apply the second line of the inverse Minkowski inequality (23) to Eq. (31) and separate between the first m terms and the rest of the right-hand side. We get therefore

$$\begin{aligned}C(\Gamma)\ &\ge\ \sqrt{\sum_\alpha\sum_\beta\mid\sum_{i=1}^{m}(\gamma_i^*)^2C_{\alpha\beta}(\psi_i)\mid^2}\\ &-\ 2\sqrt{\sum_\alpha\sum_\beta\mid\sum_{i<j}^{m}(\gamma_i^*\gamma_j^*)C_{\alpha\beta}(\psi_i,\psi_j)\mid^2}.\end{aligned} \qquad (39)$$

Now by successive application of the inverse Minkowski inequality and the Minkowski inequality to the first and the second line of the above equation, respectively, we get

$$C(\Gamma)\ge2\Delta-\sum_{i=1}^{m}|\gamma_i|^2C(\psi_i)-2\sum_{i<j}^{m}|\gamma_i\gamma_j|C(\psi_i,\psi_j). \qquad (40)$$

Again making use of the fact that $C(\psi_i,\psi_j)\le1$, for orthogonal states, we get the lower bound (37).

D. Arbitrary states

In this subsection we generalize the previous theorems to include the general situation of the superposed states being arbitrary. We state the following theorem.

Theorem 8: Arbitrary states. Let $\{|\psi_i\rangle\}_{i=1}^m$ be a set of arbitrary normalized, but not necessarily orthogonal, pure states of a general bipartite system. The concurrence of the normalized state $|\Gamma'\rangle=|\Gamma\rangle/\parallel\Gamma\parallel$, with $|\Gamma\rangle=\sum_{i=1}^m\gamma_i|\psi_i\rangle$, $\sum_{i=1}^m|\gamma_i|^2=1$, satisfies

$$\|\Gamma\|^2 C(\Gamma') \leq \min \begin{cases} \sum_{i=1}^m |\gamma_i|^2 C(\psi_i) + 2\sum_{i<j}^m |\gamma_i\gamma_j| \sqrt{1+|\langle\psi_i|\psi_j\rangle|^2}\,, \\ \\ \min_{\mathbf{P}}\{\sum_{k\in\{k<l\}_-} |\gamma_k^2 + 2\sum_{l\in\{k<l\}_-} \gamma_k\gamma_l\langle\psi_k|\psi_l\rangle| C(\psi_k) \\ + \sum_{s\in\{r<s\}_+} |\gamma_s^2 + 2\sum_{r\in\{r<s\}_+} \gamma_r\gamma_s\langle\psi_s|\psi_r\rangle| C(\psi_s) \\ + \sum_{\bar{i}}' |\gamma_{\bar{i}}|^2 C(\psi_{\bar{i}}) + 2\sum_{i<j}^m |\gamma_i\gamma_j| \sqrt{1-|\langle\psi_i|\psi_j\rangle|^2}\}, \end{cases} \tag{41}$$

and

$$\|\Gamma\|^2 C(\Gamma') \geq \max \begin{cases} 2\Delta - \sum_{i=1}^m |\gamma_i|^2 C(\psi_i) - 2\sum_{i<j}^m |\gamma_i\gamma_j| \sqrt{1+|\langle\psi_i|\psi_j\rangle|^2}, \\ \\ \max_{\mathbf{P}}\{2\Delta' - \sum_{k\in\{k<l\}_-} |\gamma_k^2 + 2\sum_{l\in\{k<l\}_-} \gamma_k\gamma_l\langle\psi_k|\psi_l\rangle| C(\psi_k) \\ - \sum_{s\in\{r<s\}_+} |\gamma_s^2 + 2\sum_{r\in\{r<s\}_+} \gamma_r\gamma_s\langle\psi_s|\psi_r\rangle| C(\psi_s) \\ - \sum_{\bar{i}}' |\gamma_{\bar{i}}|^2 C(\psi_{\bar{i}}) - 2\sum_{i<j}^m |\gamma_i\gamma_j| \sqrt{1-|\langle\psi_i|\psi_j\rangle|^2}\}, \end{cases} \tag{42}$$

where $\min_{\mathbf{P}}$ ($\max_{\mathbf{P}}$) denotes minimum (maximum) over all possible partitions of the set of pairs $\{i<j\}$, with $1 \leq i < j \leq m$, into distinct subsets $\{k<l\}_-$ and $\{r<s\}_+$. The number of such possible partitions is $2^{m(m-1)/2}$. Also Δ is defined in Eq. (38), and Δ' is given by

$$\Delta' = \max \begin{cases} |\gamma_k^2 + 2\sum_{l\in\{k<l\}_-} \gamma_k\gamma_l\langle\psi_k|\psi_l\rangle| C(\psi_k), \\ |\gamma_s^2 + 2\sum_{r\in\{r<s\}_+} \gamma_r\gamma_s\langle\psi_s|\psi_r\rangle| C(\psi_s), \\ |\gamma_{\bar{i}}|^2 C(\psi_{\bar{i}}), \end{cases} \tag{43}$$

where $k \in \{k<l\}_-$, $s \in \{r<s\}_+$ and $\bar{i} \neq \{k,s \mid k \in \{k<l\}_-,\ s\in\{r<s\}_+\}$.

Proof: For the first inequality, we should proof that the left-hand side is less than each term in the right-hand side. To do so, we apply the Minkowski inequality (21) several times to Eq. (31), followed by the fact that $C^2(\psi_i,\psi_j) \leq 1+|\langle\psi_i|\psi_j\rangle|^2$. This proofs the first line of Eq. (41). Now to reach the second line of Eq. (41), we first partition the pairs $\{i<j\}$, with $1 \leq i < j \leq m$, into two distinct subsets of pairs $\{k<l\}_-$ and $\{r<s\}_+$. For the first subset we expand $|\psi_l\rangle$ in terms of $|\psi_k\rangle$ and $|\psi_{k\perp}\rangle$, i.e. $|\psi_l\rangle = \langle\psi_k|\psi_l\rangle\,|\psi_k\rangle + \sqrt{1-|\langle\psi_k|\psi_l\rangle|^2}|\psi_{k\perp}\rangle$, and for the second one we expand $|\psi_r\rangle$ in terms of $|\psi_s\rangle$ and $|\psi_{s\perp}\rangle$, i.e. $|\psi_r\rangle = \langle\psi_s|\psi_r\rangle\,|\psi_s\rangle + \sqrt{1-|\langle\psi_s|\psi_r\rangle|^2}|\psi_{s\perp}\rangle$. Putting these into Eq. (5), we find $C_{\alpha\beta}(\psi_k,\psi_l) = \langle\psi_k|\psi_l\rangle^* C_{\alpha\beta}(\psi_k) + \sqrt{1-|\langle\psi_k|\psi_l\rangle|^2}C_{\alpha\beta}(\psi_k,\psi_{k\perp})$ and $C_{\alpha\beta}(\psi_r,\psi_s) = \langle\psi_s|\psi_r\rangle^* C_{\alpha\beta}(\psi_s) + \sqrt{1-|\langle\psi_s|\psi_r\rangle|^2}C_{\alpha\beta}(\psi_s,\psi_{s\perp})$, respectively. Using these in Eq. (31) and after some calculations we find

$$\begin{aligned} \|\Gamma\|^2 C(\Gamma') = \{\sum_{\alpha\beta} | \sum_{\bar{i}}{}' [(\gamma_{\bar{i}}^*)^2] C_{\alpha\beta}(\psi_{\bar{i}}) \\ + \sum_{k\in\{k<l\}_-} [(\gamma_k^*)^2 + 2\sum_{l\in\{k<l\}_-} \gamma_k^*\gamma_l^*\langle\psi_k|\psi_l\rangle^*] C_{\alpha\beta}(\psi_k) \\ + \sum_{s\in\{r<s\}_+} [(\gamma_s^*)^2 + 2\sum_{r\in\{r<s\}_+} \gamma_r^*\gamma_s^*\langle\psi_s|\psi_r\rangle^*] C_{\alpha\beta}(\psi_s) \\ + 2\sum_{i<j} (\gamma_i^*\gamma_j^*) \sqrt{1-|\langle\psi_i|\psi_j\rangle|^2} C_{\alpha\beta}(\psi_i,\psi_{i\perp})|^2\}^{1/2}, \end{aligned} \tag{44}$$

where $\sum_{\bar{i}}'$ denotes sum over all possible values $1 \leq \bar{i} \leq m$ such that $\bar{i} \neq \{k,s \mid k\in\{k<l\}_-,\ s\in\{r<s\}_+\}$; For example, for $m=3$ a possible partition for pairs $1 \leq i < j \leq 3$ is $\{12\}_-$ and $\{13,23\}_+$, therefore $\bar{i}=2$, but for the partition $\{13\}_-$ and $\{12,23,\}_+$, $\bar{i}$ does not take any value. Now by successive application of the Minkowski inequality, followed by the fact that $C(\psi_i,\psi_{i\perp}) \leq 1$, we reach to the second line of the inequality (41). This completes the proof for the first inequality.

Next in order to proof the lower bound (42), we should proof that the left-hand side is greater than each line in the right-hand side. For the first line, we proceed just as that we done in the proof of the lower bound of theorem 7, but using the fact that, here, $C(\psi_i,\psi_j) \leq \sqrt{1+|\langle\psi_i\psi_j|\rangle|^2}$. On the other hand to proof the second line of Eq. (42), we apply the inverse Minkowski inequality to Eq. (44) and separate the first three terms from the last one, i.e.

$$\|\Gamma\|^2 C(\Gamma') = \{\sum_{\alpha\beta} |\sum_{\bar{i}}{}' (\gamma_{\bar{i}}^*)^2 C_{\alpha\beta}(\psi_{\bar{i}}) \tag{45}$$
$$+ \sum_{k\in\{k<l\}_-} (\gamma_k^*)^2 + 2 \sum_{l\in\{k<l\}_-} \gamma_k^*\gamma_l^* \langle\psi_k|\psi_l\rangle^* C_{\alpha\beta}(\psi_k)$$
$$+ \sum_{s\in\{r<s\}_+} (\gamma_s^*)^2 + 2 \sum_{r\in\{r<s\}_+} \gamma_r^*\gamma_s^* \langle\psi_s|\psi_r\rangle^* C_{\alpha\beta}(\psi_s)|^2\}^{1/2}$$
$$- \{\sum_{\alpha\beta} |2\sum_{i<j}(\gamma_i^*\gamma_j^*)\sqrt{1-|\langle\psi_i|\psi_j\rangle|^2} C_{\alpha\beta}(\psi_i,\psi_{i\perp})|^2\}^{1/2}.$$

Now successive application of the inverse Minkowski inequality and the Minkowski inequality to the first and the second terms of the above equation, respectively, followed by the fact that $C(\psi_i, \psi_i \perp) \leq 1$, leads to the lower bound (42).

V. CONCLUSION

We have used the notion of concurrence vector and have investigated upper and lower bounds on the concurrence of the state as a function of the concurrence of the superposed states. We have shown that the amount of entanglement quantified by the concurrence vector is exactly the same as that quantified by I-concurrence, so that we could compare our results to those given in [5]. We have shown that when the two superposed states are biorthogonal states, the concurrence of the superposition can be written, simply, as a function of the concurrence of the superposed states. For other situations such as one-sided orthogonal states, orthogonal states, and arbitrary states, we have derived simple relations for the upper and lower bounds of the superpositions. It is shown that our bounds are, in general, tighter than the bounds obtained in Ref. [5]. An extension of the results to the case with more than two states in the superpositions is also given. It follows from our results that the vectorial representation of the generalized concurrence is more suitable in investigation of the concurrence of superpositions.

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