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Green's Function expansion of scalar and vector fields in the presence of a medium

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Abstract

Based on a canonical approach and functional-integration techniques, a series expansion of Green's function of a scalar field, in the presence of a medium, is obtained. A series expansion for Lifshitz-energy, in finite-temperature, in terms of the susceptibility of the medium is derived and the whole formalism is generalized to the case of electromagnetic field in the presence of some dielectrics. A covariant formulation of the problem is presented.

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I. INTRODUCTION

Quantum field theory is the quantum mechanics of continuous systems and fully developed in quantum electrodynamics which is the most successful theory in physics [1]. Quantum field theory trough path-integrals bridge to statistical mechanics and its applications include many branches of physics like, particle physics, condensed-matter physics, atomic physics, astrophysics and even economics [2]. Usually we are interested in a quantum field which has to be considered in the presence of a matter field described by some bosonic fields. For example, in quantum optics there are situations where the electromagnetic field quantization should be achieved in the presence of a general magnetodielectric medium [3–5] or in calculating the effect of matter fields on Casimir forces [6, 7]. In these cases the matter field should be included directly into the process of quantization. Unfortunately there are very few problems where the interested physical quantities, like for example, the Casimir force, can be determined analytically and so finding an effective approximation method is necessary. A fundamental quantity in a quantum field theory is the propagator or the Green's function [8] from which many physical quantities may be extracted. Here, using path-integrals, and based on a microscopic approach, we begin from a Lagrangian and obtain an expansion for the two-point correlation function i.e., the Green's function in terms of the susceptibility function of the medium for both scalar and electromagnetic fields in the presence of an arbitrary linear magnetodielectric medium. As an example of applications of these expansions for the case of a real scalar field we have introduced an expansion for the free energy or Lifshitz energy in the presence of some arbitrary dielectrics [9, 10]. Also we have considered the covariant formulation of the electromagnetic field in the presence of a linear magnetodielectric [11] which may have applications in quantum optics or dynamical Casimir effects [12].

II. SCALAR FIELD

Let us start the section with a simple but efficient field theory which have a wide range of applications in many branches of physics, i.e., the Lagrangian of a real Klein-Gordon field in $3+1$-dimensional space-time ($x = (\mathbf{x}, x^0) \in \mathbb{R}^{3+1}$), with the following Lagrangian density

$$\mathcal{L}_s = \frac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x) - \frac{1}{2}m^2\varphi^2(x), \tag{1}$$

and let the medium be modeled by a continuum of harmonic oscillators which is usually called the Hopfield model of a reservoir [13]

$$\mathcal{L}_m = \frac{1}{2}\int_0^\infty d\omega\,\left(\dot{Y}^2_\omega(x) - \omega^2 Y^2_\omega(x)\right), \tag{2}$$

the interaction between the scalar field and its medium is assumed to be linear and described by

$$\mathcal{L}_{int} = \int_0^\infty d\omega\, f(\omega,\mathbf{x})\dot{Y}_\omega(x)\varphi(x). \tag{3}$$

Having the total Lagrangian we can quantize the system using path-integral techniques. An important quantity in any field theory is the generating functional from which n-point correlation functions can be obtained from successive functional derivatives. Here our purpose is to find two-point correlation functions or Green's functions in terms of the susceptibility of the medium. For this purpose let us first find the free generating functional which can be written as

$$\begin{aligned} W_0[J,\{J_\omega\}] &= \int D\varphi\, e^{\frac{\imath}{\hbar}\int d^4x\,\{-\frac{1}{2}\varphi[\Box+m^2]\varphi+J\varphi\}}\int\prod_\omega DY_\omega\, e^{\frac{\imath}{\hbar}\int d^4x\,\int_0^\infty d\omega\{-\frac{1}{2}Y_\omega[\partial_t^2+\omega^2]Y_\omega+J_\omega Y_\omega\}} \\ &= \int D\varphi e^{-\frac{1}{2}\langle\varphi|\hat{A}|\varphi\rangle+\langle J|\varphi\rangle}\int\prod_\omega DY_\omega e^{-\frac{1}{2}\int_0^\infty\{\langle Y_\omega|\hat{B}_\omega|Y_\omega\rangle+\langle J_\omega|Y_\omega\rangle\}} \end{aligned} \tag{4}$$

where we have defined

$$\begin{aligned} \hat{A} &= \frac{\imath}{\hbar}(\Box+m^2), & \hat{B}_\omega &= \frac{\imath}{\hbar}(\partial_t^2+\omega^2), \\ \rho(x) &= \frac{\imath}{\hbar}J(x), & \rho_\omega(x) &= \frac{\imath}{\hbar}J_\omega(x). \end{aligned} \tag{5}$$

Now using the following formula

$$\int D\varphi(x)e^{-\frac{1}{2}\langle\varphi|\hat{A}|\varphi\rangle+\langle\rho|\varphi\rangle} = (det\hat{A})^{-\frac{1}{2}}e^{\frac{1}{2}\langle\rho|\hat{A}^{-1}|\rho\rangle} \tag{6}$$

Eq. (4) can be rewritten as

$$W_0[J,\{J_\omega\}] = Ne^{\frac{1}{2}\langle\rho|\hat{A}^{-1}|\rho\rangle}e^{\frac{1}{2}\int_0^\infty d\omega\,\langle\rho_\omega|\hat{B}_\omega^{-1}|\rho_\omega\rangle} \tag{7}$$

where $N = (det\hat{A})^{-\frac{1}{2}}\prod_\omega(det\hat{B}_\omega)^{-\frac{1}{2}}$ is a renormalization factor. Also from the following definitions

$$\begin{aligned} \frac{\imath}{\hbar}(\Box+m^2)\,G^0(x,x') &= \delta^4(x-x') \\ \frac{\imath}{\hbar}(\partial_t^2+\omega^2)\,G^0_\omega(x,x') &= \delta^4(x-x') \end{aligned} \tag{8}$$

we will find

$$\hat{A}^{-1} = G^0(x,x') = \imath\hbar\int \frac{d^4k}{(2\pi)^4}\frac{e^{\imath k\cdot(x-x')}}{k^2-m^2}$$

$$\hat{B}_\omega^{-1} = G^0_\omega(x,x') = \imath\hbar\delta^3(\mathbf{x}-\mathbf{x}')\int \frac{d\,k^0}{2\pi}\frac{e^{\imath k^0(x^0-x'^0)}}{(k^0)^2-\omega^2} \tag{9}$$

with the following Fourier transforms

$$\tilde{G}^0(k) = \frac{\imath\hbar}{k^2-m^2},$$

$$\tilde{G}^0_\omega(k^0) = \frac{\imath\hbar}{(k^0)^2-\omega^2}. \tag{10}$$

respectively. The free generating functional can now be written as

$$W_0[J,\{J_\omega\}] = Ne^{-\frac{1}{2\hbar^2}\int d^4x\int d^4x' J(x)G^0(x-x')J(x')}e^{-\frac{1}{2\hbar^2}\int d^4x\int d^4x'\int_0^\infty J_\omega(x)G^0_\omega(x-x')J_\omega(x')}, \tag{11}$$

and the interacting generating functional can be obtained from the free generating functional using the following formula [8]

$$\begin{aligned} W[J,\{J_\omega\}] &= e^{\frac{\imath}{\hbar}\int d^4x\int_0^\infty d\omega\, f(\omega,x)\left(\frac{\hbar}{\imath}\frac{\delta}{\delta J(x)}\right)\frac{\partial}{\partial x^0}\left(\frac{\hbar}{\imath}\frac{\delta}{\delta J_\omega(x)}\right)}W_0[J,\{J_\omega\}] \\ &= N\, e^{-\imath\hbar\int_0^\infty d\omega\int d^4x\, f(\omega,x)\frac{\delta}{\delta J(x)}\frac{\partial}{\partial x^0}\frac{\delta}{\delta J_\omega(x)}} \\ &\times\; e^{-\frac{1}{2\hbar^2}\int d^4x\int d^4x' J(x)G^0(x-x')J(x')}e^{-\frac{1}{2\hbar^2}\int d^4x\int d^4x'\int_0^\infty d\omega\, J_\omega(x)G^0_\omega(x-x')J_\omega(x')} \end{aligned} \tag{12}$$

Having the generating functional, the two-point function, i.e. the Green's function can be obtained as

$$G(x,x') = (\frac{\hbar}{\imath})^2\frac{\delta^2}{\delta J(x)\delta J(x')}W[J,\{J_\omega\}]\bigg|_{j,\{j_\omega\}=0}. \tag{13}$$

Now let us assume that the coupling function between the Klein-Gordon field and its medium is weak-one can also assume that the susceptibility of the medium is not far from vacuum-such that it can be considered as an expansion parameter which can be used to find a series solution for the Green's function or the correlation function. Using Eq.(12) and after some straightforward calculations, we find the following expansion for the Green's function in frequency domain

$$\begin{aligned} G(\mathbf{x}-\mathbf{x}',\omega) &= G^0(\mathbf{x}-\mathbf{x}',\omega) + \int_\Omega d^3\mathbf{z_1}G^0(\mathbf{x}-\mathbf{z_1},\omega)[\omega^2\tilde{\chi}(\omega,\mathbf{z_1})]G^0(\mathbf{z_1}-\mathbf{x}',\omega) + \\ &\int_\Omega\int_\Omega d^3\mathbf{z_1}d^3\mathbf{z_2}\,G^0(\mathbf{x}-\mathbf{z_1},\omega)[\omega^2\tilde{\chi}(\omega,\mathbf{z_1})]G^0(\mathbf{z_1}-\mathbf{z_2},\omega)[\omega^2\tilde{\chi}(\omega,\mathbf{z_2})]G^0(\mathbf{z_2}-\mathbf{x}',\omega)+\cdots \end{aligned} \tag{14}$$

and it can be easily shown that it satisfies the Green's function equation which we will find in the next section. Note that since $\tilde{\chi}(\omega,\mathbf{x})=0$, for $\mathbf{x}\notin\Omega$, so we can rewrite the expansion (14) in a more compact or matrix form as fallows

$$\begin{aligned} G(\omega) &= G^0(\omega)+G^0(\omega)\,[\omega^2\,\tilde{\chi}(\omega)]\,G^0(\omega)+G^0(\omega)\,[\omega^2\,\tilde{\chi}(\omega)]G^0(\omega)[\omega^2\,\tilde{\chi}(\omega)]\,G^0(\omega)+\cdots \\ &= G^0(\omega)\,[\mathbb{I}-\omega^2\tilde{\chi}(\omega)\,G^0(\omega)]^{-1}. \end{aligned} \tag{15}$$

A. Equations of motion

In this section we find the equations of motion for the fields and in particular we obtain a Langevin type equation for the scalar field

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}=\frac{\partial\mathcal{L}}{\partial\varphi}\Longrightarrow(\Box+m^2)\,\varphi=\int_0^\infty d\omega\,f(\omega,\mathbf{x})\dot{Y}_\omega(x), \tag{16}$$

$$\partial_\mu(\frac{\delta\mathcal{L}}{\delta_\mu Y_\omega})=\frac{\delta\mathcal{L}}{\delta Y_\omega}\Longrightarrow\ddot{Y_\omega}+\omega^2 Y_\omega=-f(\omega,\mathbf{x})\,\dot{\varphi}(x). \tag{17}$$

By solving Eq.(17) and inserting it into Eq.(13), we find a Langevin equation for the Klein-Gordon field

$$(\Box+m^2)\,\varphi(x)+\frac{\partial}{\partial t}\int_{-\infty}^{t}dt'\,\chi(t-t',\mathbf{x})\frac{\partial}{\partial t'}\,\varphi(\mathbf{x},t')=\xi(x), \tag{18}$$

where $\chi(\tau,\mathbf{x})$ is the susceptibility function or the memory of the medium with the following Fourier transform

$$\tilde{\chi}(\omega,\mathbf{x})=\int_0^\infty d\omega'\,\frac{f^2(\omega',\mathbf{x})}{\omega'^2-\omega^2+\imath 0^+}. \tag{19}$$

The source field $\xi(x)$ is defined by

$$\xi(x)=\int_0^\infty d\omega\,f(\omega,\mathbf{x})\dot{Y}_\omega^N(x) \tag{20}$$

where

$$Y_\omega^N(\mathbf{x},t)=\cos(\omega t)Y_\omega(\mathbf{x},0)+\frac{\sin(\omega t)}{\omega}\dot{Y}_\omega(\mathbf{x},0). \tag{21}$$

From Eqs.(20,21), we see that the source field depends on initial values of the reservoir fields so it can be considered as a noise field. The Green's function of Eq.(18) satisfies

$$(\Box+m^2)\,G(\mathbf{x}-\mathbf{x}',t-t')+\frac{\partial}{\partial t}\int_{-\infty}^{t}dt''\,\chi(t-t'',\mathbf{x})\frac{\partial}{\partial t''}G(\mathbf{x}-\mathbf{x}',t''-t')=\delta(\mathbf{x}-\mathbf{x}',t-t'). \tag{22}$$

In a homogeneous medium, where the memory function is position independent, Eq.(22) can be solved easily in reciprocal space

$$\begin{aligned}\tilde{G}(\mathbf{k},\omega) &= \frac{1}{\mathbf{k}^2-\omega^2+m^2-\omega^2\tilde{\chi}(\omega)} = \frac{1}{\mathbf{k}^2-\omega^2+m^2-\int d\omega' \frac{\omega^2 f^2(\omega')}{\omega'^2-\omega^2+\imath 0^+}} \\ &= \frac{1}{\mathbf{k}^2-\omega^2\epsilon(\omega)+m^2}\end{aligned} \tag{23}$$

where $\epsilon(\omega) = 1+\tilde{\chi}(\omega)$ can be considered as the dielectric function corresponding to the medium. From Eq.(23) it is clear that the Green's function in the presence of a homogeneous medium can be obtained from the Green's function of the free space simply by substituting ω^2 with $\omega^2\epsilon(\omega)$. Eq.(18) in frequency-space can be written as

$$\left(-\nabla^2-\omega^2\epsilon(\omega,\mathbf{x})+m^2\right)G(\mathbf{x}-\mathbf{x}',\omega) = \delta(\mathbf{x}-\mathbf{x}') \tag{24}$$

In some simple geometries the dielectric function $\epsilon(\omega,\mathbf{x})$ depends on $\mathbf{x}$ as follows

$$\epsilon(\omega,\mathbf{x}) = \begin{cases}\epsilon(\omega) & \text{if } \mathbf{x}\in\Omega \\ 1 & \text{if } \mathbf{x}\notin\Omega\end{cases} \tag{25}$$

where Ω is a region or the union of regions where the space is filled with a homogeneous but frequency dependent medium with the dielectric function $\epsilon(\omega)$. In this case the Green's function can be found in some regular geometries [6] but for an arbitrary dielectric function it is quite complicated and in this case we find a series solution in terms of free Green's function and susceptibility of the medium. Note that in some geometries, electromagnetic field can be considered as two massless Klein-Gordon fields, and the scalar formalism can help for example in obtaining Lifshitz energies or Casimir forces in such geometries [7].

B. Partition function

Having the expansion (14) let us find the partition function in the presence of some dielectrics defined by the dielectric function $\epsilon(\omega,\mathbf{x})$ which as a special case may be given by (25). The partition function of a real scalar field in the presence of a medium according to the modified Green's function given by (24) can be written as

$$\Xi = \int D\varphi\, e^{\frac{\imath}{\hbar}S} = \int D\varphi\, e^{\frac{\imath}{\hbar}\int d^4x\,\mathcal{L}} \tag{26}$$

where $\mathcal{L}$ is given by (1). The partition function in frequency domain can be written as [14]

$$\Xi = \int D\varphi \, e^{-\frac{\imath}{2\hbar}\int \frac{d\omega}{2\pi} \int d^3\mathbf{x}\, \tilde{\varphi}(\mathbf{x},-\omega)[-\omega^2\epsilon(\omega,\mathbf{x})-\nabla^2+m^2]\tilde{\varphi}(\mathbf{x},\omega)}. \tag{27}$$

If we make a Wick rotation $\omega = \imath\nu$ in frequency domain the action will be Euclidean and the free energy can be determined from $E = -\frac{\hbar}{\tau}\ln\Xi$, where τ is the duration of interaction which is taken to be sufficiently large. Using standard path-integral techniques we will find the free energy in finite temperature T as

$$E = k_B T \sum_{l=0}^{\infty}{}' \ln \det[\hat{K}(\nu_l;\mathbf{x},\mathbf{x}'] \tag{28}$$

where $\nu_l = 2\pi l k_b T/\hbar$ is the Matsubara frequency, k_B is the Boltzman constant and the prime over the summation means that the term corresponding to $l = 0$, should be given a half weight. The kernel $\hat{K}(\nu_l;\mathbf{r},\mathbf{r}') = [\nu_l^2\epsilon(\imath\nu_l,\mathbf{r}) - \nabla^2]\delta^3(\mathbf{r}-\mathbf{r}')$. Using the identity, $\ln\det[\hat{K}] = tr\,\ln[\hat{K}]$ and the fact that $\hat{K}(\nu_l;\mathbf{r},\mathbf{r}') = G^{-1}(\imath\nu_l;\mathbf{r},\mathbf{r}')$ we find

$$E = -k_B T \sum_{l=0}^{\infty}{}' \, tr\,\ln[G(\imath\nu_l;\mathbf{r},\mathbf{r}'] \tag{29}$$

now using the expansion (14), we find the following expansion for free energy in terms of the susceptibility

$$E = k_B T \sum_{l=0}^{\infty}{}' \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int d^3\mathbf{r}_1 \cdots d^3\mathbf{r}_n \, G^0(\imath\nu_l;\mathbf{r}_1-\mathbf{r}_2)\cdots G^0(\imath\nu_l;\mathbf{r}_n-\mathbf{r}_1)\,\chi(\imath\nu_l,\mathbf{r}_1)\cdots\chi(\imath\nu_l,\mathbf{r}_n) \tag{30}$$

where $G^0(\mathbf{r}-\mathbf{r}';\imath\nu_l)$ is given by

$$G^0(\imath\nu_l;\mathbf{r}-\mathbf{r}') = \frac{1}{4\pi}\frac{e^{-\sqrt{m^2+\nu_l^2}\,|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{31}$$

which corresponds to a Yukawa potential with the modified mass $\sqrt{m^2+\nu_l^2}$.

III. ELECTROMAGNETIC FIELD

In this section we use the Coulomb gauge i.e. $\nabla\cdot\mathbf{A} = 0$, $A^0 = 0$ and find a similar expansion for the Green's function of the electromagnetic field in the presence of some arbitrary regions of matter which as an example can have applications in calculating the Casimir forces. For this purpose let us take the total Lagrangian density as follows [15]

$$\mathcal{L} = \frac{1}{2}\epsilon_0(\mathbf{E}^2 - \frac{1}{\mu_0}\mathbf{B}^2) + \frac{1}{2}\int_0^{\infty} d\omega(\dot{\mathbf{Y}}_\omega{}^2(x) - \omega^2\mathbf{Y}_\omega{}^2(x)) + \int d\omega f(\omega,\mathbf{x})\mathbf{A}\cdot\dot{\mathbf{Y}}_\omega \tag{32}$$

The interacting generating functional can be written in terms of the vector potential and the medium fields as

$$\begin{aligned} W &= \int D[\mathbf{A}]\prod_{\omega} D[\mathbf{Y}_\omega]\exp\frac{\imath}{\hbar}\int d^4x\Big\{-\frac{1}{2}A_i\hat{K}_{ij}A_j-\int_0^\infty\frac{1}{2}Y_{\omega,i}(\partial_t^2+\omega^2)\,\delta_{ij}\,Y_{\omega,j} \\ &+ \int_0^\infty d\omega f(\omega,\mathbf{x})A_i\dot{Y}_{\omega,i}+J_iA_i+\int_0^\infty d\omega\,J_{\omega,i}Y_{\omega,i}\Big\} \end{aligned} \tag{33}$$

where summation over repeated indices is assumed and the kernel $\hat{K}_{ij}$ is defined by

$$\hat{K}_{ij}=\left[\epsilon_0\partial_0^2-\frac{1}{\mu_0}\nabla^2\right]\delta_{ij}+\frac{1}{\mu_0}\partial_i\partial_j \tag{34}$$

Now from the equation

$$G_{ij}(x,x')=(\frac{\hbar}{\imath})^2\frac{\delta^2}{\delta J_i(x)\delta J_j(x')}W[J,\{J_\omega\}]\Big|_{j,\{j_\omega\}=0}. \tag{35}$$

and similar calculations we will find the following expansion for the Green's function in frequency domain

$$\begin{aligned} G_{ij}(\mathbf{x}-\mathbf{x}',\omega) &= G^0_{ij}(\mathbf{x}-\mathbf{x}',\omega)+\int_\Omega d^3\,\mathbf{z_1}G^0_{il}(\mathbf{x}-\mathbf{z_1},\omega)[\omega^2\tilde{\chi}(\omega,\mathbf{z_1})]G^0_{lj}(\mathbf{z_1}-\mathbf{x}',\omega)+ \\ &\int_\Omega\int_\Omega d^3\mathbf{z_1}d^3\mathbf{z_2}\,G^0_{il}(\mathbf{x}-\mathbf{z_1},\omega)[\omega^2\tilde{\chi}(\omega,\mathbf{z_1})]G^0_{lm}(\mathbf{z_1}-\mathbf{z_2},\omega)[\omega^2\tilde{\chi}(\omega,\mathbf{z_2})]G^0_{mj}(\mathbf{z_2}-\mathbf{x}',\omega)+\cdots \end{aligned} \tag{36}$$

which in matrix form can be written as

$$\begin{aligned} \mathbb{G}(\omega) &= \mathbb{G}^0(\omega)+\mathbb{G}^0(\omega)[\omega^2\,\tilde{\chi}(\omega)]\mathbb{G}^0(\omega)+\mathbb{G}^0(\omega)[\omega^2\,\tilde{\chi}(\omega)]\mathbb{G}^0(\omega)[\omega^2\,\tilde{\chi}(\omega)]\mathbb{G}^0(\omega)+\cdots \\ &= \mathbb{G}^0(\omega)\left[\mathbb{I}-\omega^2\,\tilde{\chi}(\omega)\,\mathbb{G}^0(\omega)\right]^{-1}. \end{aligned} \tag{37}$$

A. Equations of motion

From Lagrangian density (2) we find the following equations

$$\hat{K}_{ij}A_j=\int_0^\infty d\omega\,f(\omega,\mathbf{x})\dot{Y}_{\omega,i} \tag{38}$$

$$\ddot{Y}_{\omega,i}+\omega^2Y_{\omega,i}=-f(\omega,\mathbf{x})\dot{A}_i \tag{39}$$

Solving Eq.(39) and inserting the solution into Eq.(38) we find

$$\hat{K}_{ij}A_j+\frac{\partial}{\partial t}\int_0^\infty d\omega\,f^2(\omega,\mathbf{x})\int dt'\,G_\omega(t-t')\frac{\partial}{\partial t'}A_i(t')=\int_0^\infty d\omega\,f(\omega,\mathbf{x})\dot{Y}^N_{\omega,i} \tag{40}$$

where

$$G_\omega(t-t') = \int \frac{d\omega'}{2\pi} \frac{e^{\imath\omega'(t-t')}}{\omega^2-\omega'^2} \tag{41}$$

and $Y^N_{\omega,i}$ is the homogeneous solution of Eq.(39) which depends on the initial values of the medium fields and can be considered as a noise or fluctuating field which does not affect the Green's function. Using Eqs.(20) and (41) we can rewrite Eq.(28) as

$$\hat{K}_{ij}A_j + \frac{\partial}{\partial t}\int_0^\infty \frac{d\omega'}{2\pi}\,\tilde{\chi}(\omega',\mathbf{x})\int dt'\, e^{\imath\omega'(t-t')}\frac{\partial}{\partial t'}A_i(t') = \int_0^\infty d\omega\, f(\omega,\mathbf{x})\dot{Y}^N_{\omega,i} \tag{42}$$

which in frequency-domain can be written as

$$\left[(-\epsilon_0\omega^2 - \frac{1}{\mu_0}\nabla^2)\,\delta_{ij} + \frac{1}{\mu_0}\partial_i\partial_j\right]\tilde{A}_j(\mathbf{x},\omega) - \omega^2\tilde{\chi}(\omega,\mathbf{x})\tilde{A}_i(\omega,\mathbf{x}) = \int_0^\infty d\omega\,\imath\omega\, f(\omega,\mathbf{x})\tilde{Y}^N_{\omega,i}(\omega,\mathbf{x}) \tag{43}$$

The Green's function of Eq.(43) satisfies

$$\left[(-\epsilon_0(1+\tilde{\chi}(\omega,\mathbf{x})\,\omega^2\,\delta_{ij}) - \frac{1}{\mu_0}\nabla^2)\,\delta_{ij} + \frac{1}{\mu_0}\partial_i\partial_j\right]\, G_{jk}(\mathbf{x},\mathbf{x}',\omega) = \frac{1}{\mu_0}\delta^3(\mathbf{x}-\mathbf{x}')\delta_{ij} \tag{44}$$

which using the definitions $\epsilon(\omega,\mathbf{x}) = \epsilon_0[1+\tilde{\chi}(\omega,\mathbf{x}]$ and $c^{-2} = \epsilon_0\mu_0$ can be written as

$$\left[-\frac{\omega^2}{c^2}\epsilon(\omega,\mathbf{x})\delta_{ij} - \nabla^2\delta_{ij} + \partial_i\partial_j\right]\, G_{jk}(\mathbf{x},\mathbf{x}',\omega) = \delta^3(\mathbf{x}-\mathbf{x}')\delta_{ik}, \tag{45}$$

and it can be easily shown that the Green's function (36) satisfies Eq.(45).

A similar approach can be followed to find the partition function in terms of the susceptibility of the medium as follows

$$\begin{aligned} E \;=\; & k_BT\sum_{l=0}^{\infty\;\prime}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int d^3\mathbf{x}_1\cdots\int d^3\mathbf{x}_n\, G^0_{i_1i_2}(\mathbf{x}_1-\mathbf{x}_2;\imath\nu_l)\cdots G^0_{i_ni_1}(\mathbf{x}_n-\mathbf{x}_1;\imath\nu_l) \\ \times\; & \chi(\imath\nu_l,\mathbf{x}_1)\cdots\chi(\imath\nu_l,\mathbf{x}_n), \end{aligned} \tag{46}$$

where the free Green's function $G^0_{ij}(\mathbf{x}-\mathbf{x}';\imath\nu_l)$ satisfies Eq.(45) with $\epsilon(\omega,\mathbf{x}) = 1$ and $\omega = \imath\nu_l$. By defining $\mathbf{r} = \mathbf{x}-\mathbf{x}'$, we find

$$G^0_{ij}(\mathbf{r};\imath\nu_l) = \frac{\nu_l^2}{c^2}\frac{e^{-\frac{\nu_l r}{c}}}{4\pi r}\left[\delta_{ij}(1+\frac{c}{\nu_l r}+\frac{c^2}{\nu_l^2 r|^2}) - \frac{r_ir_j}{r^2}(1+\frac{3c}{\nu_l r}+\frac{3c^2}{\nu_l^2r^2})\right] + \frac{1}{3}\delta_{ij}\delta^3(\mathbf{r}). \tag{47}$$

In zero temperature, the summation over the positive integer l is replaced by an integral according to the rule $\hbar\int_0^\infty \frac{d\nu}{2\pi}f(\imath\nu) \;\rightarrow\; k_BT\sum_{l=0}^{\infty\;\prime} f(\imath\nu_l)$. For a nice discussion of Casimir-Lifshitz interaction between dielectrics of arbitrary geometry see [10].

B. Covariant formulation

In reference [11] electromagnetic field quantization in a moving medium has been investigated by considering the medium to be modeled by a continuum of tensor fields. But the medium can also be modeled by a continuum of scalar fields, i.e., Klein-Gordon fields which we will follow here. So we consider the electromagnetic field interacting with a moving medium, a situation which can have applications in dynamic Casimir effects. For this purpose we consider the following Lorentz invariant Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\int_0^\infty d\omega\, [\partial_\mu Y_\omega \partial^\mu Y_\omega - \omega^2 Y_\omega^2] + \int_0^\infty d\omega\, f^{\mu\nu}(\omega, \mathbf{x})\, Y_\omega \partial_\mu A_\nu. \tag{48}$$

where $f^{\mu\nu}(\omega, \mathbf{x})$ is an antisymmetric coupling tensor which couples the electromagnetic field to its medium and is related to the susceptibility of the medium trough Eq.(52). From Euler-Lagrange equations we find

$$\Box\, A^\nu = -\partial_\mu K^{\mu\nu}(x), \tag{49}$$

$$(\Box + \omega^2)\, Y_\omega(x) = f^{\mu\nu}(\omega, x)\partial_\mu A_\nu(x) \tag{50}$$

where $\Box = \partial_t^2 - \nabla^2$, and the antisymmetric tensor $K^{\mu\nu}(x) = \int_0^\infty d\omega\, f^{\mu\nu}(\omega, x) Y_\omega$ can be considered as the polarization tensor of the medium. By solving Eq.(50) and inserting it into Eq.(49) we find

$$\Box\, A^\mu(x) - \int d^4x'\, \partial_\nu\, \partial'_\alpha\, \chi^{\nu\mu\alpha\beta}(x, x') A_\beta(x') = -\partial_\eta K^{N,\eta\mu} \tag{51}$$

where $K^{N,\mu\nu} = \int_0^\infty d\omega\, f^{\mu\nu}(\omega, x) Y_\omega^N$, and Y_ω^N is the homogeneous solution of Eq.(50) which can be considered as a noise field. The susceptibility tensor $\chi^{\nu\mu\alpha\beta}(x, x')$ is defined by

$$\chi^{\nu\mu\alpha\beta}(x, x') = \int_0^\infty d\omega\, f^{\mu\nu}(\omega, x)\, G_\omega^0(x - x') f^{\alpha\beta}(\omega, x') \tag{52}$$

where $G_\omega^0(x - x')$ is given by Eq.(9). The Green's function in this case satisfies the following equation

$$\Box\, G_{\mu\nu}(x - x') - \int d^4x''\, g_{\mu\delta}\, \partial_\gamma\, \partial''_\alpha\, \chi^{\gamma\delta\alpha\beta}(x, x'')\, G_{\beta\nu}(x'' - x) = g_{\mu\nu}\, \delta^4(x - x') \tag{53}$$

and for the Green's function we find the following expansion in terms of the susceptibility tensor

$$G_{\mu\nu}(x, x') = G_{\mu\nu}^0(x - x') + \int d^4z_1\, d^4z_2\, G_{\mu\nu_1}^0(x - z_1)\, \Gamma^{\nu_1\nu_2}(z_1, z_2)\, G_{\nu_2\nu}^0(z_2 - x') +$$
$$\int d^4z_1 \cdots d^4z_4\, G_{\mu\nu_1}^0(x - z_1)\, \Gamma^{\nu_1\nu_2}(z_1, z_2)\, G_{\nu_2\nu_3}^0(z_2 - z_3)\, \Gamma^{\nu_3\nu_4}(z_3, z_4)\, G_{\nu_4\nu}^0(z_4 - x') + \cdots$$

where for simplicity we have defined $\Gamma^{\nu_1\nu_2}(z_1, z_2) = \partial_{\mu_1}\partial_{\mu_2}\chi^{\mu_1\nu_1\mu_2\nu_2}(z_1, z_2)$. So given the susceptibility tensor of the medium, one can find the Green's function perturbatively in terms of the susceptibility. Having the Green's function-at least perturbatively- we can investigate for example the dynamical energy configurations, which is closely related to the problems of dynamical Casimir effect, which is under consideration.

IV. CONCLUSION

Based on a Lagrangian approach, scalar and vector field theory in the presence of a medium, modeled by a continuum of Klein-Gordon fields, considered and a series expansion for the Green's function of the theory in terms of the susceptibility of the medium obtained. From the partition function of the scalar field, an expression for the free energy in terms of the susceptibility of the medium obtained and the formalism generalized to the case of electromagnetic field in the presence of some dielectrics. Also, the covariant form of the electromagnetic field in the presence of moving media investigated and an expression for the Green's function in terms of the susceptibility tensor obtained which can have applications in dynamical Casimir effects.

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