Phase space spinor amplitudes for spin-$\frac{1}{2}$ systems

P. Watson*and A.J. Bracken†
Centre for Mathematical Physics
Department of Mathematics
University of Queensland
Brisbane 4072
Queensland
Australia

Abstract

The concept of phase space amplitudes for systems with continuous degrees of freedom is generalized to finite-dimensional spin systems. Complex amplitudes are obtained on both a sphere and a finite lattice, in each case enabling a more fundamental description of pure spin states than that previously given by Wigner functions on either the sphere or lattice. In each case the Wigner function can be expressed as the star product of the amplitude and its conjugate, so providing a generalized Born interpretation of amplitudes that emphasizes their more fundamental status. The case of spin-$\frac{1}{2}$ is treated in detail, and it is shown that the phase space amplitudes transform correctly as spinors under under rotations, on both the sphere and the lattice.

* *Email:* pw.cmp@optusnet.com.au
† *Email:* a.bracken@uq.edu.au

1 Introduction

Weyl quantization [1] is a one to one map between functions on a phase space Γ and operators acting on a Hilbert space $\mathcal{H}$. Over the years the Weyl-Wigner (WW) transform $\mathcal{W}$, which is the inverse of Weyl's quantization map, has been used to develop the phase space formulation of quantum mechanics that has successfully been applied to systems with continuous degrees of freedom [2,3].

There have been several attempts to generalize the Weyl correspondence to include finite spin degrees of freedom within the WW formalism. The treatments of deGroot and Suttorp [4] and O'Connell and Wigner [9] combined the continuous phase space picture with the finite spin degrees of freedom into a single WW transform. Buot [5] formulated a discrete WW transform on a periodic lattice phase space; Chumakov *et al* [6] defined a WW type quasi-probability function as a linear combination of spherical harmonics on the phase space of the sphere $\mathcal{S}^2$; and Berezin [7,8] introduced a general method of quantization by representing functions on $\mathcal{S}^2$ in terms of covariant Q-symbols and contravariant P-symbols.

The lattice phase space approach was further generalized to an array of N orthogonal states to obtain a discrete Wigner function. For example Hannay and Berry [10] obtained a discrete Wigner function that resembled a δ function on each of the $(2N)^2$ points of a $2N \times 2N$ lattice phase space indexed by the set of integers modulo $2N$. Wootters [11] generated a general class of discrete Wigner functions (applicable to spin systems) by constructing an $N \times N$ lattice phase space indexed by the set of integers modulo N, where N is a prime number, or a Cartesian product of lattices in the case of composite N. Gibbons *et al* [12] found a class of discrete Wigner functions on an $N \times N$ lattice phase space of a finite field composed of N^k elements where N is prime and $k \in \mathbb{Z}^+$. Other approaches that dealt with spin systems but from a different perspective are those of Chandeler *et al* [13] who, motivated by the statistical characteristic function, generated three dimensional quasi-probability density functions for spin-$\frac{1}{2}$ systems; and Leonhardt [14], who applied precession tomography to spin systems to derive an expression for a discrete Wigner function.

Phase space amplitudes have been defined for continuous degrees of freedom by [15–21]. They provide representations of quantum state vectors rather than the density operator, and as such are more fundamental objects than Wigner functions [21]. In this paper we extend the concept of phase space amplitudes to finite-dimensional spin systems. We begin with a brief review of phase space amplitudes and then modify the main results to define phase space spin amplitudes. Expressions for spin amplitudes and their corresponding Wigner functions are obtained on both the sphere $\mathcal{S}^2$ and a lattice of dimension $(2j+1)^2$. Our primary focus will be on spin-$\frac{1}{2}$ amplitudes and explicit expressions for these will be given in terms of spherical harmonics and functions defined on each of the $(2j+1)^2 = 4$ lattice points. For convenience we consider all variables to be dimensionless and set Planck's constant $\hbar$ to unity.

2 Phase space amplitudes

In the phase space formulation of quantum mechanics, state vectors for a system with continuous degrees of freedom (p, q) only are represented by complex phase space amplitudes [15–21]. These are constructed from a pure state vector $|\psi\rangle$ by introducing a fixed vector $|\varphi\rangle$ of unit length, to form the outer (dyadic) product

$$\widehat{\Psi} \equiv |\psi\rangle\langle\varphi|. \tag{1}$$

For each choice of $|\varphi\rangle$, a set of distinct amplitudes for variable $|\psi\rangle$ are found as the images $\mathcal{W}(\widehat{\Psi})$ under the WW transform,

$$\mathcal{W}(\widehat{\Psi})(p,q)/\sqrt{2\pi} \longleftrightarrow \Psi(p,q) = \mathrm{tr}\left(\widehat{\Psi}\widehat{\Delta}(p,q)\right)/\sqrt{2\pi}. \tag{2}$$

Here $\widehat{\Delta}(p,q)$ is the Weyl-Wigner-Stratonovich (WWS) operator kernel [22].

A phase space amplitude contains all essential information about a quantum state, and this representation of state vectors in phase space has many important properties. For instance, phase space amplitudes can be used to calculate expectation values according to

$$\langle\widehat{A}\rangle = \mathrm{tr}(\widehat{A}\widehat{\Psi}\widehat{\Psi}^{\dagger}) \longrightarrow \int \overline{\Psi(p,q)}\,[(A\star\Psi)(p,q)]\,\mathrm{d}\Gamma, \qquad (\mathrm{d}\Gamma = \mathrm{d}p\mathrm{d}q), \tag{3}$$

and transition probabilities

$$\langle\psi_1|\psi_2\rangle = \int \overline{\Psi_1(p,q)}\Psi_2(p,q)\,\mathrm{d}\Gamma, \tag{4}$$

where in particular

$$\langle\psi|\psi\rangle = \int \overline{\Psi(p,q)}\Psi(p,q)\,\mathrm{d}\Gamma = 1, \tag{5}$$

and therefore the quantity $|\Psi|^2$ can be regarded as a quasi-probability distribution over Γ. In (3), the non-commutative star-product of phase space functions $A(p,q)$ and $B(p,q)$ appears [24], given by

$$\begin{aligned}(A\star B)(p,q) \;&=\; \left(\frac{1}{\pi}\right)^2 \int \mathrm{tr}\left(\widehat{\Delta}(p,q)\widehat{\Delta}(p',q')\widehat{\Delta}(p'',q'')\right) A(p',q')B(p'',q'') \\ &\qquad \times\, \mathrm{d}p'\mathrm{d}q'\mathrm{d}p''\mathrm{d}q''.\end{aligned} \tag{6}$$

The mapping from $|\psi\rangle$ to $\Psi(p,q)$ is linear so that the amplitudes preserve the superposition and interference properties of state vectors. In contrast, the Wigner function $W(p,q)$, which is defined for a pure state $|\psi\rangle$ as the image $\mathcal{W}(|\psi\rangle\langle\psi|)$ and commonly expressed as [23]

$$\mathcal{W}(|\psi\rangle\langle\psi|)(p,q)/2\pi \longleftrightarrow W(p,q) = \frac{1}{2\pi}\int e^{ipx}\overline{\psi(q+x/2)}\psi(q-x/2)\,\mathrm{d}x, \qquad (7)$$

does not have these properties.

In analogy to the way that configuration space wave functions determine a probability density, phase space amplitudes generalize the Born interpretation by determining the Wigner function with the help of the star product

$$W(p,q) = (\Psi \star \overline{\Psi})(p,q), \qquad (8)$$

which emphasizes the more fundamental status of these amplitudes.

3 Spin amplitudes on the sphere $\mathcal{S}^2$

The inverse of Weyl's quantization map assigns to each spin observable $\widehat{J}_k,\ k = 1,2,3,$ represented by $(2j+1)\times(2j+1)$ hermitian matrices with the fixed spin quantum number $j \in 0, \frac{1}{2}, 1\cdots$, a real valued function on the phase space of the sphere $\mathcal{S}^2$. However, pure spin states described by vectors $|j,m\rangle$ in the Hilbert space $\mathcal{H}_j = \mathbb{C}^{2j+1}$ which are eigenvectors of $\widehat{J}_3$ and $\widehat{J}^2$,

$$\widehat{J}_3|j,m\rangle = m|j,m\rangle,\ \widehat{J}^2|j,m\rangle = j(j+1)|j,m\rangle, \qquad m = j, j-1, \cdots -j, \qquad (9)$$

do not have images defined on $\mathcal{S}^2$.

Within the framework of phase space amplitudes this problem is circumvented by first observing that spin states are vectors $|\psi\rangle$ with $2j+1$ complex components $\psi_m = \langle j,m|\psi\rangle$ in $\mathcal{H}_j$ and second, in a manner similar to the continuous case, by introducing a fixed spin state $|\varphi\rangle$ which allows one to extend the amplitude operator of (1) to

$$\widehat{\Psi} \equiv |\psi\rangle\langle\varphi| \longleftrightarrow (\psi_m\overline{\varphi}_{m'}) = \begin{pmatrix} \psi_j\overline{\varphi}_j & \psi_j\overline{\varphi}_{j-1} & \cdots \\ \psi_{j-1}\overline{\varphi}_j & \ddots & \\ \vdots & & \psi_{-j}\overline{\varphi}_{-j} \end{pmatrix}. \qquad (10)$$

Normalization of $|\varphi\rangle$ requires that

$$\sum_{m=-j}^{j} |\varphi_m|^2 = 1. \qquad (11)$$

A basic feature of Weyl quantization is the existence of an operator valued function or WWS kernel, the determination of which is essential for the definition not only of the Wigner functions on $\mathcal{S}^2$, as in the literature [6], but also of the spin amplitudes in which we are interested here. A WWS kernel based on the axiomatic postulates of Stratonovich [22] and applicable to spin systems is known [25–27] and given by

$$\begin{aligned} \widehat{\Delta}^j(\theta,\phi) &= \sum_{m',m''=-j}^{j} Z^j_{m'm''} |j,m''\rangle\langle j,m'| && \text{(12a)} \\ Z^j_{m'm''} &= \sqrt{\frac{4\pi}{2j+1}} \sum_{l=0}^{2j} \sum_{m=-l}^{l} \epsilon^j_l (-1)^{j-m'} C^{j\ \ \ \ l\ \ \ j}_{m''\ -m'\ m}\, Y_{l,m}(\theta,\phi), && \text{(12b)} \end{aligned}$$

where $C^{j\ \ \ \ l\ \ \ j}_{m''\ -m'\ m}$ are the standard Clebsch-Gordan coefficients, $Y_{l,m}$ are the spherical harmonics and ϵ^j_l are constants such that $\epsilon^j_0 = 1 \text{ and } \epsilon^j_l = \pm 1$.

It is now a straightforward matter using (10) and (12), after making a suitable choice for the arbitrary fixed spin vector $|\varphi\rangle$, to define the spin amplitudes on $\mathcal{S}^2$ as

$$\Psi(\theta,\phi) = \mathrm{tr}\left(\widehat{\Psi}\widehat{\Delta}^j(\theta,\phi)\right). \tag{13}$$

Whatever choice of fixed spin vector is made, the corresponding Wigner function on $\mathcal{S}^2$, which is defined as

$$W(\theta,\phi) = \mathrm{tr}(|\psi\rangle\langle\psi|\widehat{\Delta}^j(\theta,\phi)), \tag{14}$$

is then expressed in terms of spin amplitudes by the star product

$$W(\theta,\phi) = \Psi(\theta,\phi) \star \overline{\Psi(\theta,\phi)}. \tag{15}$$

There are various methods available to evaluate star products for spin systems [28, 29] but for our purposes it will be appropriate to use the integral form by suitably modifying (6) to get

$$\begin{aligned} W(\theta,\phi) &= \left(\frac{2j+1}{4\pi}\right)^2 \int \mathrm{tr}\left(\widehat{\Delta}^j(\theta,\phi)\widehat{\Delta}^j(\theta',\phi')\widehat{\Delta}^j(\theta'',\phi'')\right) \\ &\qquad \times \Psi(\theta',\phi')\overline{\Psi(\theta'',\phi'')}\, \mathrm{d}\Omega'\mathrm{d}\Omega'', \end{aligned} \tag{16}$$

where $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta\mathrm{d}\phi$ is the invariant measure on $\mathcal{S}^2$.

All functions defined on a sphere can be expanded in terms of a complete set of spherical harmonics, hence we can write the spin amplitudes (13) in the form

$$\Psi(\theta,\phi) = \sum_{l=0}^{2j}\sum_{m=-l}^{l} a_{lm} Y_{l,m}(\theta,\phi), \tag{17}$$

where, from the orthogonality property of the spherical harmonics, the coefficients are given by

$$a_{lm} = \int_0^{2\pi} \int_0^{\pi} \Psi(\theta, \phi) \overline{Y_{l,m}(\theta, \phi)} \sin\theta \; \mathrm{d}\theta \mathrm{d}\phi. \tag{18}$$

Formula (17) suggests that spin amplitudes will transform as tensors under rotations, but we will see that, to the contrary, these amplitudes transform as spinors when $2j$ is odd-integral, in particular, when $j = 1/2$.

A double application of (17) leads to

$$\Psi(\theta, \phi)\overline{\Psi(\theta, \phi)} = \sum_{l_1, m_1} \sum_{l_2, m_2} a_{l_1 m_1} \overline{a}_{l_2 m_2} Y_{l_1, m_1}(\theta, \phi) \overline{Y_{l_2, m_2}(\theta, \phi)}. \tag{19}$$

The product of two spherical harmonics, with the same arguments can be written as a linear combination of single spherical harmonics in terms of the $3j$-symbols, hence (19) can be re-expressed as

$$\begin{aligned} \Psi(\theta, \phi)\overline{\Psi(\theta, \phi)} \;\; &= \;\; \sum_{l_1, m_1} \sum_{l_2, m_2} \sum_{l, m} \sqrt{\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}} a_{l_1 m_1} \overline{a}_{l_2 m_2} \\ &\quad \times \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix} \overline{Y_{l,m}(\theta, \phi)}. \end{aligned} \tag{20}$$

The Wigner function has a similar expansion, so that for any choice of fixed vector we can write

$$W(\theta, \phi) = \sum_{l=0}^{2j} \sum_{m=-l}^{l} b_{lm} Y_{l,m}(\theta, \phi), \tag{21}$$

with coefficients

$$b_{lm} = \int_0^{2\pi} \int_0^{\pi} W(\theta, \phi) \overline{Y_{l,m}(\theta, \phi)} \sin\theta \; \mathrm{d}\theta \mathrm{d}\phi. \tag{22}$$

The symplectic group of transformations that arises in the case of continuous degrees of freedom, acting on functions of variables (p, q) in a 2f-dimensional phase space, is replaced by the group $SU(2)$ of rotations acting on functions on the phase space of the sphere $\mathcal{S}^2$. Rotated spin amplitudes are generated from

$$\Psi_R(\theta, \phi) = (\mathcal{W}(\widehat{R}) \star \Psi)(\theta, \phi), \tag{23}$$

where the rotation operator is given, for spin-$\frac{1}{2}$, by

$$\widehat{R}(\gamma, \alpha, \beta) = e^{-i\gamma\widehat{\sigma}_z/2} e^{-i\alpha\widehat{\sigma}_y/2} e^{-i\beta\widehat{\sigma}_z/2}. \tag{24}$$

We will demonstrate below that although the rotated spin amplitudes are expressed as linear combinations of spherical harmonics they nevertheless transform as spinors.

For the case of spin-$\frac{1}{2}$ systems ($j=\frac{1}{2}$), setting the constants $\epsilon_l^{1/2}=+1$, the WWS kernel (12) reduces to

$$\widehat{\Delta}^{1/2}(\theta,\phi)=(\widehat{I}+\sqrt{3}\,\mathbf{n}\cdot\widehat{\sigma})/2, \tag{25}$$

where $\widehat{I}$ is a 2×2 unit matrix, $\widehat{\sigma}$ are the Pauli operators

$$\widehat{\sigma}_x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \widehat{\sigma}_y=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \widehat{\sigma}_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{26}$$

and $\mathbf{n}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$ is an arbitrary unit vector which is parameterized by the spherical polar coordinates $0\leq\theta\leq\pi$, $0\leq\phi<2\pi$.

The spin-$\frac{1}{2}$ amplitudes then follow by inserting (10) and (25) into (13) to get

$$\Psi(\theta,\phi)=\sqrt{\pi}\sum_{l=0}^{1}\sum_{m=-l}^{l}a_{lm}Y_{l,m}(\theta,\phi), \tag{27}$$

where the coefficients are given by

$$\begin{aligned}
a_{00} &= \psi_{1/2}\overline{\varphi}_{1/2}+\psi_{-1/2}\overline{\varphi}_{-1/2},\\
a_{1-1} &= \sqrt{2}\psi_{-1/2}\overline{\varphi}_{1/2},\\
a_{10} &= \psi_{1/2}\overline{\varphi}_{1/2}-\psi_{-1/2}\overline{\varphi}_{-1/2},\\
a_{11} &= -\sqrt{2}\psi_{1/2}\overline{\varphi}_{-1/2}.
\end{aligned} \tag{28}$$

Using (20), we find for the product

$$\Psi(\theta,\phi)\overline{\Psi(\theta,\phi)}=\sqrt{\pi}\sum_{l=0}^{1}\sum_{m=-l}^{l}b_{lm}Y_{l,m}(\theta,\phi), \tag{29}$$

with the associated coefficients

$$\begin{aligned}
b_{00} &= 1,\\
b_{1-1} &= \frac{1}{\sqrt{2}}(\psi_{-1/2}\overline{\psi}_{1/2}+\varphi_{-1/2}\overline{\varphi}_{1/2}),\\
b_{10} &= |\psi_{1/2}|^2|\overline{\varphi}_{1/2}|^2-|\psi_{-1/2}|^2|\overline{\varphi}_{-1/2}|^2,\\
b_{11} &= -\frac{1}{\sqrt{2}}(\psi_{1/2}\overline{\psi}_{-1/2}+\varphi_{1/2}\overline{\varphi}_{-1/2}).
\end{aligned} \tag{30}$$

The components of the fixed state $|\varphi\rangle$ can be chosen appropriately in (10) and (16) to simplify calculations, for example $\varphi_{1/2}=1,\varphi_{-1/2}=0$.

The corresponding Wigner functions are obtained by evaluating (16) with (25) and (27), for any choice of fixed spin vector, to get

$$W(\theta,\phi)=\sqrt{\pi}\sum_{l=0}^{1}\sum_{m=-l}^{l}c_{lm}Y_{l,m}(\theta,\phi), \tag{31}$$

and its coefficients

$$\begin{aligned} c_{00} &= 1, \\ c_{1-1} &= \sqrt{2}\psi_{-1/2}\overline{\psi}_{1/2}, \\ c_{10} &= |\psi_{1/2}|^2-|\psi_{-1/2}|^2, \\ c_{11} &= -\sqrt{2}\psi_{1/2}\overline{\psi}_{-1/2}. \end{aligned} \tag{32}$$

It is now possible to immediately write down the Wigner functions for any arbitrary spin vector by simply inserting appropriately chosen spin components.

It has been established that spin amplitudes are functions on $\mathcal{S}^2$, and it only remains to be shown that they do indeed transform as spinors under rotations. To do this consider a rotation through an angle α about the y-axis given by

$$\widehat{R}=\cos\tfrac{1}{2}\alpha\widehat{I}-i\sin\tfrac{1}{2}\alpha\widehat{\sigma}_y. \tag{33}$$

The image $\mathcal{W}(\widehat{R})$ of this rotation is given by the mapping

$$\mathcal{W}(\cos\tfrac{1}{2}\alpha\widehat{I}-i\sin\tfrac{1}{2}\alpha\widehat{\sigma}_y)(\theta,\phi)\longrightarrow\cos\tfrac{1}{2}\alpha-i\sqrt{3}\sin\tfrac{1}{2}\alpha\sin\theta\sin\phi. \tag{34}$$

Inserting this into (23), and evaluating the star product, leads to an expression of the form

$$\Psi_R(\theta,\phi)=\sqrt{\pi}\sum_{l=0}^{1}\sum_{m=-l}^{l}a_{lm}Y_{l,m}(\theta,\phi), \tag{35}$$

for the rotated spin amplitudes, with corresponding coefficients given by

$$\begin{aligned} a_{00} &= (\psi_{1/2}\overline{\varphi}_{1/2}+\psi_{-1/2}\overline{\varphi}_{-1/2})\cos\tfrac{1}{2}\alpha-(\psi_{-1/2}\overline{\varphi}_{1/2}+\psi_{1/2}\overline{\varphi}_{-1/2})\sin\tfrac{1}{2}\alpha, \\ a_{10} &= (\psi_{1/2}\overline{\varphi}_{1/2}-\psi_{-1/2}\overline{\varphi}_{-1/2})\cos\tfrac{1}{2}\alpha-(\psi_{-1/2}\overline{\varphi}_{1/2}+\psi_{1/2}\overline{\varphi}_{-1/2})\sin\tfrac{1}{2}\alpha, \\ a_{1-1} &= \sqrt{2}\psi_{-1/2}\overline{\varphi}_{1/2}\cos\tfrac{1}{2}\alpha+\sqrt{2}\psi_{1/2}\overline{\varphi}_{1/2}\sin\tfrac{1}{2}\alpha, \\ a_{11} &= -\sqrt{2}\psi_{1/2}\overline{\varphi}_{-1/2}\cos\tfrac{1}{2}\alpha+\sqrt{2}\psi_{-1/2}\overline{\varphi}_{-1/2}\sin\tfrac{1}{2}\alpha. \end{aligned} \tag{36}$$

Setting $\alpha=2\pi$ in the above set of coefficients produces a change of sign in $\Psi(\theta,\phi)$. Therefore, even though the spin amplitudes are functions on $\mathcal{S}^2$, they nevertheless transform as spinors.

4 Spin amplitudes on the lattice

Consider a quantum system characterized by two incompatible physical observables A and B each with an associated finite number of (non-degenerate) eigenvalues $\alpha = \{\alpha_i : i = -j, \cdots, j\}$ and $\beta = \{\beta_i : i = -j, \cdots, j\}$ respectively. From these we construct a lattice consisting of $(2j+1)^2$ points (α, β) in the plane. A generalized WW correspondence rule between observables $\widehat{X}_j$ and their corresponding symbols $X(\alpha, \beta)$ on each of the lattice points can be defined by

$$\widehat{X}_j = \frac{1}{2j+1} \sum_{\alpha\beta} X(\alpha,\beta) \widehat{\Delta}(\alpha,\beta), \tag{37a}$$

$$X(\alpha,\beta) = \mathrm{tr}\left(\widehat{X}_j \widehat{\Delta}(\alpha,\beta)\right), \tag{37b}$$

where $\widehat{\Delta}(\alpha, \beta)$ is a $(2j+1) \times (2j+1)$ lattice kernel matrix defined at the lattice points (α, β) and required to satisfy the following properties

$$\widehat{\Delta}(\alpha,\beta) = \widehat{\Delta}^{\dagger}(\alpha,\beta) \tag{38a}$$

$$\mathrm{tr}(\widehat{\Delta}(\alpha,\beta)) = 1 \tag{38b}$$

$$\mathrm{tr}(\widehat{\Delta}(\alpha,\beta)\widehat{\Delta}(\alpha',\beta')) = (2j+1)\ \delta_{\alpha\alpha'}\delta_{\beta\beta'} \tag{38c}$$

$$\sum_{\alpha\beta} \widehat{\Delta}(\alpha,\beta) = (2j+1)\ \widehat{1}. \tag{38d}$$

One possible form for the kernel $\widehat{\Delta}(\alpha, \beta)$ that guarantees the above properties are satisfied is [5, 11]

$$\widehat{\Delta}(\alpha,\beta) = \sum_{\alpha'} e^{-\frac{2\pi i}{2j+1}\beta\alpha'} |\alpha + \alpha'/2\rangle\langle\alpha - \alpha'/2|, \qquad \alpha' = -j, \cdots, j, \tag{39}$$

where the $|\cdot\rangle\langle\cdot|$ are $(2j+1) \times (2j+1)$ matrices, whose elements are determined from the properties (38). Alternatively, the matrix elements of $\widehat{\Delta}(\alpha, \beta)$ with respect to the basis $|\beta\rangle$ are found from (39) and given by the expression

$$\langle\beta'|\widehat{\Delta}(\alpha,\beta)|\beta''\rangle = e^{\frac{2\pi i}{2j+1}\alpha(\beta''-\beta')} \sum_{\alpha'} e^{-\frac{2\pi i}{2j+1}\alpha'[\beta+(\beta'+\beta'')/2]}. \tag{40}$$

A double application of (37a) in conjunction with (37b), leads to the star product

$$(X \star Y)(\alpha,\beta) = \frac{1}{(2j+1)^2} \sum_{\alpha'\alpha''\beta'\beta''} X(\alpha',\beta')Y(\alpha'',\beta'')\mathrm{tr}\left(\widehat{\Delta}(\alpha,\beta)\widehat{\Delta}(\alpha',\beta')\widehat{\Delta}(\alpha'',\beta'')\right), \tag{41}$$

of two symbols defined on each of the lattice points.

The image of the amplitude operator $\widehat{\Psi}$ of (10) under the action of (37b) gives the amplitudes on the lattice as

$$\Psi(\alpha,\beta) = \mathrm{tr}\left(\widehat{\Psi}\widehat{\Delta}(\alpha,\beta)\right), \tag{42}$$

with the corresponding Wigner functions, obtained by evaluating the star product (41) with (42), given by

$$W(\alpha,\beta) = \frac{1}{2j+1}(\Psi \star \overline{\Psi})(\alpha,\beta). \tag{43}$$

The lattice associated with spin $j = \frac{1}{2}$ consists of an array of four points designated by $\{(0,0),(0,1),(1,0),(1,1)\}$. Defined on the lattice is a spin amplitude Ψ and an associated Wigner function. The matrix elements for the lattice kernel are in this case obtained directly from (40) and found to be

$$\begin{aligned}
\widehat{\Delta}_{00} &= \begin{pmatrix} 1 & \frac{1}{2}(1-i) \\ \frac{1}{2}(1+i) & 0 \end{pmatrix},\ \widehat{\Delta}_{10} = \begin{pmatrix} 1 & -\frac{1}{2}(1-i) \\ -\frac{1}{2}(1+i) & 0 \end{pmatrix}, \\
\widehat{\Delta}_{01} &= \begin{pmatrix} 0 & \frac{1}{2}(1+i) \\ \frac{1}{2}(1-i) & 1 \end{pmatrix},\ \widehat{\Delta}_{11} = \begin{pmatrix} 0 & -\frac{1}{2}(1+i) \\ -\frac{1}{2}(1-i) & 1 \end{pmatrix}.
\end{aligned} \tag{44}$$

The symplectic Fourier transform of the kernel $\widehat{\Delta}(\alpha,\beta)$, is

$$\widehat{D}(\alpha,\beta) = \frac{1}{2j+1}\sum_{\alpha'\beta'} e^{\frac{2\pi i}{2j+1}(\alpha\beta'-\alpha'\beta)}\widehat{\Delta}(\alpha',\beta'), \tag{45}$$

and in particular

$$\widehat{D}(\alpha,\beta) = \tfrac{1}{2}\left(\widehat{\Delta}(0,0) + (-1)^{\alpha}\widehat{\Delta}(0,1) + (-1)^{\beta}\widehat{\Delta}(1,0) + (-1)^{\alpha+\beta}\widehat{\Delta}(1,1)\right), \tag{46}$$

which leads to the Pauli operators

$$\widehat{D}(0,0) = \widehat{I},\ \widehat{D}(0,1) = \widehat{\sigma}_x,\ \widehat{D}(1,1) = \widehat{\sigma}_y,\ \widehat{D}(1,0) = \widehat{\sigma}_z. \tag{47}$$

Combining (10) and (44) to evaluate (42) we have for the spin amplitudes

$$\begin{aligned}
\Psi(0,0) &= \overline{\varphi}_{1/2}(\psi_{1/2} + \psi_{-1/2}) + \tfrac{1}{2}(1+i)(\psi_{1/2}\overline{\varphi}_{-1/2} - \psi_{-1/2}\overline{\varphi}_{1/2}), \\
\Psi(0,1) &= \psi_{-1/2}(\overline{\varphi}_{-1/2} + \overline{\varphi}_{1/2}) + \tfrac{1}{2}(1-i)(\psi_{1/2}\overline{\varphi}_{-1/2} - \psi_{-1/2}\overline{\varphi}_{1/2}), \\
\Psi(1,0) &= \overline{\varphi}_{1/2}(\psi_{1/2} - \psi_{-1/2}) - \tfrac{1}{2}(1+i)(\psi_{1/2}\overline{\varphi}_{-1/2} - \psi_{-1/2}\overline{\varphi}_{1/2}), \\
\Psi(1,1) &= \psi_{-1/2}(\overline{\varphi}_{-1/2} - \overline{\varphi}_{1/2}) - \tfrac{1}{2}(1-i)(\psi_{1/2}\overline{\varphi}_{-1/2} - \psi_{-1/2}\overline{\varphi}_{1/2}).
\end{aligned} \tag{48}$$

These lead immediately to

$$\begin{aligned}
\Psi(0,0)\overline{\Psi(0,0)} &= |(\psi_{1/2}+\psi_{-1/2})\overline{\varphi}_{1/2}+i(\overline{\varphi}_{1/2}+\overline{\varphi}_{-1/2})\psi_{1/2}|^2/2,\\
\Psi(0,1)\overline{\Psi(0,1)} &= |(\varphi_{1/2}+\varphi_{-1/2})\overline{\psi}_{-1/2}+i(\overline{\psi}_{1/2}+\overline{\psi}_{-1/2})\varphi_{1/2}|^2/2,\\
\Psi(0,1)\overline{\Psi(0,1)} &= |(\psi_{1/2}-\psi_{-1/2})\overline{\varphi}_{1/2}+i(\overline{\varphi}_{1/2}-\overline{\varphi}_{-1/2})\psi_{1/2}|^2/2,\\
\Psi(1,1)\overline{\Psi(1,1)} &= |(\varphi_{1/2}-\varphi_{-1/2})\overline{\psi}_{-1/2}+i(\overline{\psi}_{1/2}-\overline{\psi}_{-1/2})\varphi_{1/2}|^2/2, \qquad (49)
\end{aligned}$$

Using the amplitudes (48) in (43) it is found that, whatever choice is made for the fixed spin state, the Wigner functions are

$$\begin{aligned}
W(0,0) &= [\overline{\psi}_{1/2}(\psi_{1/2}+\psi_{-1/2})+\tfrac{1}{2}(1+i)(\psi_{1/2}\overline{\psi}_{-1/2}-\psi_{-1/2}\overline{\psi}_{1/2})]/2,\\
W(0,1) &= [\psi_{-1/2}(\overline{\psi}_{-1/2}+\overline{\psi}_{1/2})+\tfrac{1}{2}(1-i)(\psi_{1/2}\overline{\psi}_{-1/2}-\psi_{-1/2}\overline{\psi}_{1/2})]/2,\\
W(1,0) &= [\overline{\psi}_{1/2}(\psi_{1/2}-\psi_{-1/2})-\tfrac{1}{2}(1+i)(\psi_{1/2}\overline{\psi}_{-1/2}-\psi_{-1/2}\overline{\psi}_{1/2})]/2,\\
W(1,1) &= [\psi_{-1/2}(\overline{\psi}_{-1/2}-\overline{\psi}_{1/2})-\tfrac{1}{2}(1-i)(\psi_{1/2}\overline{\psi}_{-1/2}-\psi_{-1/2}\overline{\psi}_{1/2})]/2. \qquad (50)
\end{aligned}$$

It is now a straightforward matter to obtain the Wigner functions for any arbitrary pointing spin vector by inserting into (50) the appropriate spin components. Note that even on a lattice the Wigner functions can take negative values and therefore do not allow for the familiar probabilistic interpretation as can be seen for example by setting $\psi_{1/2}=(1-i)/\sqrt{2}, \psi_{-1/2}=\sqrt{2}(1+i)$.

The images $\mathcal{W}(\widehat{R})$ of the rotation given by (34) on each of the four lattice points are

$$R(0,0)=R(1,1)=e^{-i\alpha/2}, \;\; R(0,1)=R(1,0)=e^{i\alpha/2}. \qquad (51)$$

The spin amplitudes on each of the lattice points are then rotated according to

$$\Psi_R(\alpha,\beta)=(R\star\Psi)(\alpha,\beta), \qquad (52)$$

and using the star product (41) yields

$$\begin{aligned}
\Psi_R(0,0) &= \overline{\varphi}_{1/2}(\psi_{1/2}+\psi_{-1/2})\cos\tfrac{1}{2}\alpha+\overline{\varphi}_{1/2}(\psi_{1/2}-\psi_{-1/2})\sin\tfrac{1}{2}\alpha+\tfrac{1}{2}(1+i)\\
&\quad\times((\psi_{1/2}\overline{\varphi}_{-1/2}-\psi_{-1/2}\overline{\varphi}_{1/2})\cos\tfrac{1}{2}\alpha-(\psi_{1/2}\overline{\varphi}_{1/2}+\psi_{-1/2}\overline{\varphi}_{-1/2})\sin\tfrac{1}{2}\alpha),\\
\Psi_R(0,1) &= \psi_{-1/2}(\overline{\varphi}_{-1/2}+\overline{\varphi}_{1/2})\cos\tfrac{1}{2}\alpha+\psi_{1/2}(\overline{\varphi}_{1/2}+\overline{\varphi}_{-1/2})\sin\tfrac{1}{2}\alpha+\tfrac{1}{2}(1-i)\\
&\quad\times((\psi_{1/2}\overline{\varphi}_{-1/2}-\psi_{-1/2}\overline{\varphi}_{1/2})\cos\tfrac{1}{2}\alpha-(\psi_{1/2}\overline{\varphi}_{1/2}+\psi_{-1/2}\overline{\varphi}_{-1/2})\sin\tfrac{1}{2}\alpha),\\
\Psi_R(1,0) &= \overline{\varphi}_{1/2}(\psi_{1/2}-\psi_{-1/2})\cos\tfrac{1}{2}\alpha-\overline{\varphi}_{1/2}(\psi_{1/2}+\psi_{-1/2})\sin\tfrac{1}{2}\alpha-\tfrac{1}{2}(1+i)\\
&\quad\times((\psi_{1/2}\overline{\varphi}_{-1/2}-\psi_{-1/2}\overline{\varphi}_{1/2})\cos\tfrac{1}{2}\alpha-(\psi_{1/2}\overline{\varphi}_{1/2}+\psi_{-1/2}\overline{\varphi}_{-1/2})\sin\tfrac{1}{2}\alpha),\\
\Psi_R(1,1) &= \psi_{-1/2}(\overline{\varphi}_{-1/2}-\overline{\varphi}_{1/2})\cos\tfrac{1}{2}\alpha+\psi_{1/2}(\overline{\varphi}_{-1/2}-\overline{\varphi}_{1/2})\sin\tfrac{1}{2}\alpha-\tfrac{1}{2}(1-i)\\
&\quad\times((\psi_{1/2}\overline{\varphi}_{-1/2}-\psi_{-1/2}\overline{\varphi}_{1/2})\cos\tfrac{1}{2}\alpha-(\psi_{1/2}\overline{\varphi}_{1/2}+\psi_{-1/2}\overline{\varphi}_{-1/2})\sin\tfrac{1}{2}\alpha).
\end{aligned}$$

(53)

Setting $\alpha = 2\pi$ in the above set of amplitudes results in a change of sign in $\Psi(\alpha, \beta)$ and thus shows that the spin amplitudes defined on the lattice transforms as a spinor under rotations.

5 Concluding remarks

We have extended the concept of phase space amplitudes to finite spin systems and introduced the notion of a spin amplitude to represent pure spin states in phase space. Spin amplitudes and their associated Wigner functions have been expressed as expansions in terms of the spherical harmonic functions on the sphere. A generalized Weyl correspondence has been adapted to define spin amplitudes and Wigner functions on a lattice.

As an example the case of spin-$\frac{1}{2}$ has been described in detail revealing some important features of the approach. The liberty to choose any normalized fixed spin reference state $|\varphi\rangle$ introduces a degree of arbitrariness into the definition of the spin amplitudes, analogous to the freedom to choose a "window state" in the continuous case [21]. It is an important problem to optimise this choice for a given quantum spin system. We emphasize however, as can be seen from (8) and (43), that the associated Wigner functions are independent of this choice, just as in the continuous case.

The spin amplitudes are completely described in terms of a combination of system and fixed state spinor components and these amplitudes transform as spinors under rotations on both the sphere and lattice thus further supporting their fundamental status.

It is our view that the representation of spin states in phase space holds the promise of new physical insights and a novel perspective in areas like quantum computing, quantum optics and information theory. We hope to return to some of these aspects in the future.

References

[1] H. Weyl, The Theory of Groups and Quantum Mechanics (Dover Publications, New York, 1950).

[2] M. Hillery, R. F. O'Connell. M. O. Scully and, E. P. Wigner, Phys. Rep. **106**, 121 (1984).

[3] H. W. Lee, Phys. Rep. **259**, 147 (1994).

[4] S. R. deGroot and L. G. Suttorp, Foundations of Electrodynamics (North Holland, Amsterdam, 1972).

[5] F. A. Buot, Phys. Rev. **B10**, 3700 (1974).

[6] S. M. Chumakov, A. B. Klimov and, K. B. Wolf, Phys. Rev. **A61**, 034101 (2000).

[7] F. A. Berezin, J. Math. USSR-Izv. **8**, 1109 (1974).

[8] F. A. Berezin, Comm. Math. Phys. **40**, 153 (1975).

[9] R. F. O'Connell and E. P. Wigner, Phys. Rev. A **30**, 2613 (1984).

[10] J. H. Hannay and M. V. Berry, Physica, **1D**, 267 (1980).

[11] W. K. Wootters, Ann. of Phys. **176**, 1 (1987).

[12] K. S. Gibbons, M. J. Hoffman and, W. K. Wootters, Phys. Rev. **A70**, 062101 (2004).

[13] C. Chandler, L. Cohen, C. Lee, M. Scully and, K. Wodkiewicz, Found. of Phys. **22**, 867 (1992).

[14] U. Leonhardt, Phys. Rev. **A53**, 2998 (1996).

[15] G. Torres-Vega and J. H. Frederick, J. Chem. Phys. **98**, 3103-3120 (1993).

[16] J. E. Harriman, J. Chem. Phys. **100**, 3651-3661 (1994).

[17] J. J. Wlodarz, J. Chem. Phys. **100**, 7476-7480 (1994).

[18] M. D. Oliveira, M. C. B. Fernandes, F. C. Khanna, A. E. Santana and, J. D. M. Vianna, Ann. Phys., NY **312**, 492-510 (2004).

[19] M. A. de Gosson, J. Phys. A: Math. Gen. **38**, 9263-9287 (2005).

[20] T. B. Smith, J. Phys. A: Math. Gen. **39**, 1469-1484 (2006).

[21] A. J. Bracken and P. Watson, J. Phys. A: Math. Gen. **43**, 395304 (2010).

[22] R. L. Stratonovich, JETP, 891, (1957).

[23] E. Wigner, Phys. Rev. **40**, 749 (1932).

[24] J. E. Moyal, Proc. Camb. Phil. Soc., **45**, 99 (1949).

[25] J. C. Varilly and J. M. Gracia-Bondia, Ann. of Phys. **190**, 107 (1989).

[26] J. P. Amiet and M. B. Cibils, J. Phys. A:Math. Gen. **24**, 1515 (1991).

[27] S. Heiss and S. Weigert, Phys. Rev. **A63**, 012105 (2000).

[28] A. B. Klimov and P. Espinoza, J. Phys. A:Math. Gen. **31**, 8435 (2002).

[29] D. Zueco and I. Calvo, J. Phys. A:Math. Gen. **40**, 4635 (2007).