Detecting Full N-Particle Entanglement in Arbitrarily High-Dimensional Systems with Bell-Type Inequality

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We derive a set of Bell-type inequalities for arbitrarily high-dimensional systems, based on the assumption of partial separability in the hybrid local-nonlocal hidden variable model. Partially entangled states would not violate the inequalities, and thus upon violation, these Bell-type inequalities are sufficient conditions to detect the full N-particle entanglement and validity of the hybrid local-nonlocal hidden variable description.

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Entanglement is one of the most fundamental features of quantum mechanics, and it lies at the heart of recent quantum information theory. As a result, many remarkable achievements, such as quantum teleportation [1] and the higher levels of security in cryptography [2] have been attained owing to the quantum entanglement.

Given N-particle quantum systems, the correlations among them have been the subject of several recent studies [3–8]. This is also motivated by the question whether the correlations in recent experiments on 3- or 4-particle systems are due to the full N-particle entanglement and not just combinations of quantum entanglement of smaller number of particles [9, 10]. For $N = 2$, the entanglement type of the bipartite system is humdrum, i.e., it is either entangled or separable. However, the situation is dramatically changed when $N \geq 3$, besides the totally separable states, there are partially entangled states and fully N-particle entangled states. Consider all possible decompositions of a N-particle state as a mixture of pure states $\rho^N = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$, if for any decomposition there is at least one $|\Psi_i\rangle$ showing N-particle entanglement, then we shall say that ρ^N exhibits full N-particle entanglement; if the state is not separable or not fully N-particle entangled, then we call that ρ^N is partially entangled.

The conventional "N-particle Bell inequalities" are designed to deny the local hidden variable (LHV) models [11–13]. For N-particle quantum systems, the partially entangled states and the fully N-particle entangled states may violate the same Bell inequality, namely, the conventional Bell inequalities do not distinguish at all the partially entangled states and the fully N-particle entangled states. Actually, a particle can decay into several particles, this gives rise to a natural question: Are the resultant decaying systems in a fully entangled state or just a partially entangled state? In 1987, Svetlichny triggered the problem and proposed a Bell-type inequality to distinguish full three-qubit entanglement from partially two-qubit ones [3]. The Svetlichny inequality is essentially different from the conventional Bell inequality, because the former is designed for a hybrid local-nonlocal hidden variable (HLNHV) model and the latter is for a LHV model. As the name implies, HLNHV models utilize the fusion of local and nonlocal descriptions based on the assumption of partial separability. Fifteen years later, Seevinck *et al.* and Collins *et al.* independently generalized the Svetlichny inequality from three-qubit case to arbitrarily N-qubit case [4, 5]. Upon violation, these N-qubit Bell-type inequalities are sufficient conditions for detecting full N-qubit entanglement.

In this Letter, we generalize the Svetlichny inequalities from the qubit case to N arbitrarily d-dimensional systems (N-qudit). These Bell-type inequalities are derived based on the assumption of partial separability, or more generally speaking, on the so-called HLNHV models, thus the quantum mechanical violations of these inequalities provide experimentally accessible conditions to detect the full N-qudit entanglement and rule out the HLNHV models.

Consider an experimental situation involving N particles in which two measurements $i_n = 1, 2$, $(n = 1, \cdots, N)$ can be performed on each particle. Each of the measurements has d possible outcomes: $x_{i_n} = 0, 1, \cdots, d-1$, $(i_n = 1, 2)$. We now follow Svetlichny's splendid ideas [3, 4] and make the following assumption of partial separability: The N-qudit system is composed of many subsystems, which might be correlated in any way (e.g. entangled) but are uncorrelated with respect to each other. Since we can always take any two subsystems jointly as a single one but still uncorrelated with respect to the others, we only need to focus our attention on the case that the composed system consists of only two uncorrelated subsystems involving $m < N$ and $N - m < N$ qudits, respectively. For simplicity, we also assume that the first subsystem is formed by the first m qudits and the other by the remaining qudits. Denote the probability of observing the results x_{i_n} by $P(x_{i_1}, x_{i_2}, \cdots, x_{i_N})$, then the partial separability assumption can be expressed as $P(x_{i_1}, x_{i_2}, \cdots, x_{i_N}) = \int_\Gamma d\lambda \rho(\lambda) P_1(x_{i_1}, x_{i_2}, \cdots, x_{i_m}|\lambda) \times P_2(x_{i_{m+1}}, \cdots, x_{i_N}|\lambda)$, where $P_1(x_{i_1}, x_{i_2}, \cdots, x_{i_m}|\lambda)$ and $P_2(x_{i_{m+1}}, \cdots, x_{i_N}|\lambda)$ are probabilities conditioned to the hidden variable λ; Γ is the total λ space and $\rho(\lambda)$ is a statistical distribution of λ, which satisfies $\rho(\lambda) \geq 0$ and $\int_\Gamma d\lambda \rho(\lambda) = 1$. Other decompositions can be described with a different value of m and different choices of the composing qudits. A HLNHV model can then be well defined based on the assumption of partial separability and the formula of the factorizable probability, readers who are interested in it may refer to Refs. [5, 14].

For convenience, we introduce two functions: $g_1(x+s_t)=\frac{S-M(x+s_t,d)}{S}, g_2(x+s_t)=\frac{S-M(-x-s_t,d)}{S}$. Here $S=(d-1)/2$ is the spin value of the particle; s_t means a shift of the argument x; $M[x,d]=(x, \mathtt{mod}\ d)$ and $0\leq M(x,d)\leq d-1$. Then the N-qudit Bell-type inequality reads

$$I^N=-\Big(\sum_{i_1,i_2,\cdots,i_N=1}^{2} Q_{i_1 i_2\cdots i_N}\Big)\leq 2^{N-1}, \qquad (1)$$

with

$$Q_{i_1 i_2\cdots i_N}\equiv \sum_{x_{i_1},\cdots,x_{i_N}=0}^{d-1} f^{i_1 i_2\cdots i_N} P(x_{i_1},x_{i_2},\cdots,x_{i_N}). \quad (2)$$

Let $\mathcal{I}\equiv i_1 i_2\cdots i_N$, and $t(\mathcal{I})$ denotes the times that the index "2" appears in the string $\mathcal{I}$, then in above inequality $f^{\mathcal{I}}=1-\frac{M[x_{i_1}+\cdots+x_{i_N}+s_t,d]}{S}$ ($\equiv g_1$ for simplicity hereafter) if $t(\mathcal{I})$ is even and $f^{\mathcal{I}}=1-\frac{M[-x_{i_1}-\cdots-x_{i_N}-s_t,d]}{S}$ ($\equiv g_2$ hereafter) if $t(\mathcal{I})$ is odd, and $s_t\equiv s_{t(\mathcal{I})}=3\times(1-[\frac{t(\mathcal{I})}{2}])$, where $[\frac{t}{2}]$ means the integer part of $\frac{t}{2}$. One easily sees that the above inequality is symmetric under permutations of the N particles. Essentially, the inequality (1) is a kind of probabilistic Bell-type inequality after one substitutes (2) into (1). We have expressed it in a form of (1) because of two reasons: (i) making the inequality succinct and (ii) $Q_{\mathcal{I}}$ may be regarded as the generalized correlation functions of N-qudit in comparison to the typical form of correlation functions of qubits. As for the coefficient $f^{\mathcal{I}}$, its possible values are equal to $S_z/S\in\{-1,-1+1/S,\cdots,1\}$, where S_z is expectation value of the z-component of the spin operators. Especially $f^{\mathcal{I}}$ has only two possible values ±1 when $S=1/2$, in turn $Q_{\mathcal{I}}$ reduces to the typical form of correlation functions of qubits. In the following, we shall prove the upper bound of the inequality is 2^{N-1}.

Three qudits.— Our inequality for three qudits takes $f^{111}=f^{122}=f^{221}=f^{212}=g_1$ and $f^{222}=f^{112}=f^{121}=f^{211}=g_2$; the shift $s_{t(111)}=s_{t(112)}=s_{t(121)}=s_{t(211)}=3$ and $s_{t(122)}=s_{t(212)}=s_{t(221)}=s_{t(222)}=0$. We assume that the first subsystem is formed by the first two qudits and the second subsystem by the third qudit and the two subsystems are uncorrelated. Hence, in a HLNHV model, due to our two-setting scenario, four kinds of outcomes for the first two qudits $x_{i_1}+x_{i_2}$ and two kinds of outcomes for the third qudit x_{i_3} are independent of each other, so we simply denote $x_{i_1}+x_{i_2}$ by a single variable ξ_{ij} and x_{i_3} by ζ_k, which run from 0 to $d-1$. Moreover, any nondeterministic local variable model can be made deterministic by adding additional variables [15], we only need to consider the *deterministic* versions [16] of the HLNHV model in which for each value of λ the outcomes of the measurements are completely determined, namely, the probabilities of obtaining each of the possible measurements is either 0 or 1. More specifically, for each value of λ, we have predetermined values for the outcomes of ξ_{ij} and ζ_k, $(i,j,k=1,2)$.

Another preliminary knowledge is some properties of our two-qudit inequality

$$I^2=-Q_{11}-Q_{12}-Q_{21}-Q_{22}\leq 2 \qquad (3)$$

where $f^{11}=f^{22}=g_1, f^{12}=f^{21}=g_2$, $s_{t(11)}=s_{t(12)}=s_{t(21)}=3$, and $s_{t(22)}=0$. Specifically, $I^2=-g_1(r_{11}+3)-g_2(r_{12}+3)-g_2(r_{21}+3)-g_1(r_{22})$. Here $r_{ij}\equiv\alpha_i+\beta_j$, α_i being the outcome of the first qudit for the i-th measurement and β_j being that of the second qudit for the j-th measurement. The inequality (3) is an equivalent form of the well-known Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality [17] for two-qudit, which is usually written as $I_{\rm CGLMP}=Q_{11}+Q_{12}+Q_{21}-Q_{22}\leq 2$, where $f^{11}=g_2, f^{22}=f^{12}=f^{21}=g_1$, $s_{t(11)}=s_{t(12)}=s_{t(21)}=s_{t(22)}=0$ in our notation. We also specifically rewrite it as $I_{\rm CGLMP}=g_2(r'_{11})+g_1(r'_{12})+g_1(r'_{21})-g_1(r'_{22})$, and $r'_{ij}\equiv\alpha'_i+\beta'_j$. By using the identity $g_1(x)=-g_2(x+1)$, one arrives at $I^2=g_2(r_{11}+4)+g_1(r_{12}+2)+g_1(r_{21}+2)-g_1(r_{22})$. After setting $\alpha'_1=\alpha_1+2$, $\alpha'_2=\alpha_2$, $\beta'_1=\beta_1+2$, $\beta'_2=\beta_2$, one immediately finds that I^2 is equivalent to the CGLMP inequality. Hereafter we simply call (3) as the CGLMP inequality.

We start to prove $I^3\leq 4$. Define $\mathbb{I}_1\equiv-Q_{111}-Q_{112}-Q_{121}-Q_{122}=-g_1(\xi_{11}+\zeta_1+3)-g_2(\xi_{11}+\zeta_2+3)-g_2(\xi_{12}+\zeta_1+3)-g_1(\xi_{12}+\zeta_2)$. Because $I^2=-g_1(\alpha_1+\beta_1+3)-g_2(\alpha_1+\beta_2+3)-g_2(\alpha_2+\beta_1+3)-g_1(\alpha_2+\beta_2)$, if one sets $\xi_{11}=\alpha_1$, $\xi_{12}=\alpha_2$, $\zeta_1=\beta_1$, $\zeta_2=\beta_2$, then one easily finds that $\mathbb{I}_1$ equivalent to I^2, thus $\mathbb{I}_1\leq 2$. Similarly, define $\mathbb{I}_2\equiv-Q_{211}-Q_{212}-Q_{221}-Q_{222}=-g_2(\xi_{21}+\zeta_1+3)-g_1(\xi_{21}+\zeta_2)-g_1(\xi_{22}+\zeta_1)-g_2(\xi_{22}+\zeta_2)$, and set $\xi_{22}=\alpha_1+3$, $\xi_{21}=\alpha_2$, $\zeta_1=\beta_1$, $\zeta_2=\beta_2$, then $\mathbb{I}_2$ is equivalent to I^2, so $\mathbb{I}_2\leq 2$. Thus $I^3=\mathbb{I}_1+\mathbb{I}_2\leq 4$. This ends the proof.

Arbitrary N qudits.— Based on the CGLMP inequality (3), now we can prove the N-qudit Bell-type inequality (1) as what we have done in the three-qudit case. For further convenience and without losing the generalization, we denote the correlation function $-Q_I$ by (k), here $k=t(\mathcal{I})$. For instance, inequality (3) can be denoted by $(0)+(1)+(1)+(2)\leq 2$. An observant reader may notice that $(0)+(1)+(1)+(2)$ possesses the structure $g_1+g_2+g_2+g_1$. This is essential in our proof.

Firstly we will show that a n-qudit Bell-type inequality can always be rearranged into a grouping so that the index string $\mathcal{I}$ appears in a manner that in every element $[(p)+(q)]$ of this kind of grouping, the times that index "2" appears satisfy $t(q)=t(p)+1$, that is, $I^n=\sum_{k=0}^{n-1}T(k)\times[(k)+(k+1)]$, here $T(k)$ indicates the times that the element $[(k)+(k+1)]$ appears, then we obtain iterative equations: $T(k)=C_n^k-T(k-1)=\sum_{i=0}^{k}(-1)^{k-i}C_n^i$, which yields $\mathcal{W}=\sum_{k=0}^{n-1}T(k)=2^{n-1}$. Since there are two terms in every $T(k)$, so $2\mathcal{W}=2^n$ is exactly the number of terms in a n-particle inequality for two settings. Therefore such a rearrangement always exists in our inequality. Secondly, the set of index strings of the whole system can be generated from that of subsystems by connection. Considering a N-qudit system consisting of two subsystems of m qudits and $N-m$ qudits, respectively. The connection of two index strings $i_1 i_2 \cdots i_m$ and $i_{m+1} \cdots i_N$ makes $i_1 i_2 \cdots i_m i_{m+1} \cdots i_N$ indicating the kind of measurement for each qudit in the whole system. According to our two-setting scenario, in subsystem m we have a set of 2^m index strings $i_1 i_2 \cdots i_m$ and in subsystem $N-m$ we have a set of 2^{N-m} index strings $i_{m+1} \cdots i_N$; by connection of index strings of two subsystems we have totally $2^m \cdot 2^{N-m} = 2^N$ index strings $i_1 i_2 \cdots i_m i_{m+1} \cdots i_N$. In this sense, we say that the set of index strings for the whole system can be generated from that of subsystems. Thus connection of two index strings $i_1 i_2 \cdots i_m$, $i'_1 i'_2 \cdots i'_m$ for subsystem m and two index strings $i_{m+1} \cdots i_N$, $i'_{m+1} \cdots i'_N$ for subsystem $N-m$ makes four index strings for the whole system N, namely, $i_1 \cdots i_m i_{m+1} \cdots i_N$, $i_1 \cdots i_m i'_{m+1} \cdots i'_N$, $i'_1 \cdots i'_m i_{m+1} \cdots i_N$, and $i'_1 \cdots i'_m i'_{m+1} \cdots i'_N$. Since index strings can appear in the manner of $[(k)+(k+1)]$ series, we have a group of four correlation functions associated with two index strings for an element $[(p)+(p+1)]$ in subsystem m and two index strings for an element $[(q)+(q+1)]$ in subsystem $N-m$, i.e., $[(p)+(p+1)] \otimes [(q)+(q+1)] = [(p+q)+(p+q+1)+(p+q+1)+(p+q+2)] \equiv [(k)+(k+1)+(k+1)+(k+2)]$, by using the property $t(i_1 \cdots i_m; i_{m+1} \cdots i_N) = t(i_1 \cdots i_m) + t(i_{m+1} \cdots i_N)$. They should be in the form of either $g_1+g_2+g_2+g_1$ or $g_2+g_1+g_1+g_2$.

Case a: k is even, we have

$$\begin{aligned}&(k)+(k+1)+(k+1)+(k+2)\\&= g_1(\xi_{i_1\cdots i_m} + \zeta_{i_{m+1}\cdots i_N} + 3 - 3[k/2])\\&+ g_2(\xi_{i_1\cdots i_m} + \zeta_{i'_{m+1}\cdots i'_N} + 3 - 3[(k+1)/2])\\&+ g_2(\xi_{i'_1\cdots i'_m} + \zeta_{i_{m+1}\cdots i_N} + 3 - 3[(k+1)/2])\\&+ g_1(\xi_{i'_1\cdots i'_m} + \zeta_{i'_{m+1}\cdots i'_N} + 3 - 3[(k+2)/2]),\end{aligned} \quad (4)$$

by definition $r_{i_1 i_2 \cdots i_N} = \xi_{i_1\cdots i_m} + \zeta_{i_{m+1}\cdots i_N}$. If we set $\xi_{i_1\cdots i_m} = \alpha_1 + 3a$, $\xi_{i'_1\cdots i'_m} = \alpha_2 + 3a$, $\zeta_{i_{m+1}\cdots i_N} = \beta_1$, $\zeta_{i'_{m+1}\cdots i'_N} = \beta_2$, where $t(i'_1 \cdots i'_m) = t(i_1 \cdots i_m) + 1$, $t(i'_{m+1} \cdots i'_N) = t(i_{m+1} \cdots i_N) + 1$ and $a = k/2$, then this group of correlation functions is equivalent to the CGLMP inequality (3) and thus its upper bound is 2.

Case b: k is odd, we have $(k)+(k+1)+(k+1)+(k+2) = g_2+g_1+g_1+g_2$. We make the rearrangement as:

$$\begin{aligned}&(k+1)+(k+2)+(k)+(k+1)\\&= g_1(\xi_{i'_1\cdots i'_m} + \zeta_{i_{m+1}\cdots i_N} + 3 - 3[(k+1)/2])\\&+ g_2(\xi_{i'_1\cdots i'_m} + \zeta_{i'_{m+1}\cdots i'_N} + 3 - 3[(k+2)/2])\\&+ g_2(\xi_{i_1\cdots i_m} + \zeta_{i_{m+1}\cdots i_N} + 3 - 3[k/2])\\&+ g_1(\xi_{i_1\cdots i_m} + \zeta_{i'_{m+1}\cdots i'_N} + 3 - 3[(k+1)/2]).\end{aligned} \quad (5)$$

If we set $\xi_{i_1\cdots i_m} = \alpha_2 + 3(b-1)$, $\xi_{i'_1\cdots i'_m} = \alpha_1 + 3b$, $\zeta_{i_{m+1}\cdots i_N} = \beta_1$, $\zeta_{i'_{m+1}\cdots i'_N} = \beta_2$, where $t(i'_1 \cdots i'_m) = t(i_1 \cdots i_m) + 1$, $t(i'_{m+1} \cdots i'_N) = t(i_{m+1} \cdots i_N) + 1$, $b = (k+1)/2$, then this group of correlation functions is equivalent to inequality (3) and thus its upper bound is 2.

Based on above analysis, such a group of correlation functions $[(k)+(k+1)+(k+1)+(k+2)]$ is always less than 2. For I^N, the total number of such groups is $2^{m-1} \cdot 2^{N-m-1} = 2^{N-2}$, therefore the upper bound of I^N is 2^{N-1}. This ends the proof.

We now turn to study quantum violations of inequality (1) for Greenberger-Horne-Zeilinger (GHZ) states

$$|\psi\rangle^N_{\rm GHZ} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |jj\cdots j\rangle, \quad (6)$$

which are fully entangled states of N-qudit. Quantum mechanical joint probability is calculated by

$$\begin{aligned}&P^{QM}(x_{i_1}, x_{i_2}, \cdots, x_{i_N}) = \mathtt{Tr}[\rho^N (U_{i_1} \otimes U_{i_2} \otimes \cdots \otimes U_{i_N})\\&\times (\Pi_{i_1} \otimes \Pi_{i_2} \otimes \cdots \otimes \Pi_{i_N})(U^\dagger_{i_1} \otimes U^\dagger_{i_2} \otimes \cdots \otimes U^\dagger_{i_N})],\end{aligned} \quad (7)$$

where $\Pi_{i_n} = |x_{i_n}\rangle\langle x_{i_n}|$ and U_{i_n}, $(n = 1, 2, ..., N)$, are projectors and unitary transformation operators for the corresponding qudits, respectively. As for the unitary transformations U_{i_n}, it is sufficient to consider the unbiased symmetric multi-port beamsplitters [18] when we study the GHZ states. The matrix elements of an unbiased symmetric multi-port beamsplitter are given by $U_{kl}(\vec{\phi}) = \omega^{kl} \exp(i\phi_l)/\sqrt{d}$, where $\omega = \exp(2i\pi/d)$ and $\vec{\phi} = (\phi_0, \phi_1, \cdots, \phi_{d-1})$, are the settings of the appropriate phase angles. Explicitly, for the first qudit, the phase angles are $\vec{\phi}_{i_1=1} = (\phi_{10}, \phi_{11}, \cdots, \phi_{1,d-1})$ and $\vec{\phi}_{i_1=2} = (\phi_{20}, \phi_{21}, \cdots, \phi_{2,d-1})$ due to the two-setting scenario. Numerical calculations show that, when maximal violation of the Bell-type inequality for the GHZ states occurs, the optimal angles can be $\vec{\phi}_{i_n=1} = \vec{\phi}_{i_1=1}$, $\vec{\phi}_{i_n=2} = \vec{\phi}_{i_1=2}$, $(n = 1, 2, \cdots, N)$, $\phi_{1k} = k\phi_{11}$, $\phi_{2k} = k\phi_{21}$, $\phi_{11} = m_1 \frac{\pi}{2d}$, $\phi_{21} = m_2 \frac{\pi}{2d}$, and $m_1 = 15/N$, $m_2 = m_1 - 6$. For $N = 2$, the result correctly recovers that of the CGLMP inequality [17], namely $[I^2]^{\max} = 4d \sum_{k=0}^{[d/2]-1} (1 - \frac{2k}{d-1})(q_k - q_{-(k+1)})$, where $q_c = 1/\{2d^3 \sin^2[\pi(c+1/4)/d]\}$. The maximal violations increase with dimension d, for examples, $[I^2_{d=2}]^{\max} = 2\sqrt{2} \simeq 2.82843$ and $[I^2_{d=3}]^{\max} = (12 + 8\sqrt{3})/9 \simeq 2.87293$. For arbitrary N-qudit, the maximal violation is

$$[I^N]^{\max} = 2^{N-2} \times [I^2]^{\max}. \quad (8)$$

One can show how sensitive the Bell-type inequality is by considering the factor $\mathcal{R}$ defined by the maximal violation of the inequality over the upper bound for true N-body entanglement [8], i.e.,

$$\mathcal{R} = [I^N]^{\max}/2^{N-1}. \quad (9)$$

It is easy to have $\mathcal{R} = [I^2]^{\max}/2$ for any N-qudit system, which means the inequality (1) has the same strong violation as the CGLMP inequality. For $d = 2$, the Bell-type inequality (1) is an equivalent version of the Svetlichny inequality for N-qubit, accordingly $\mathcal{R} = \sqrt{2}$, which recovers the result in Ref. [4, 5] (see Eq. (14) in [5]).

So far we have discussed the violations of the pure GHZ states. If white noise is added, the pure state turns to a mixed state as

$$\rho^N(V) = V\rho_{\rm GHZ} + (1-V)\rho_{\rm noise}, \qquad (10)$$

where $\rho_{\rm noise} = 1/d^N$, 1 is the unit operator, V is the so-called visibility, and $0 \leq V \leq 1$. One easily shows that such a mixed state violates the inequality (1) if and only if $V > V_{\rm cr}$, where $V_{\rm cr} = 1/\mathcal{R}$ is the critical value of visibility. The LHV description of the state (10) is not allowed if $V > V_{\rm cr}$. For $d = 2$ and 3, the critical values $V_{\rm cr}$ are 0.70711 and 0.69615, respectively. Moreover, for any partially entangled states $\rho^N = \rho^m \otimes \rho^{N-m}$, the quantum joint probability are factorizable, i.e., $P^{QM}(x_{i_1}, x_{i_2}, \cdots, x_{i_N}) = P_1^{QM}(x_{i_1}, x_{i_2,\cdots,x_{i_m}}) \times P_2^{QM}(x_{i_{m+1}}, x_{i_{m+2}}, \cdots, x_{i_N})$. Consequently, our inequality holds for any partially entangled states or any convex mixture of them, any violation is a sufficient condition to confirm full N-particle entanglement and rule out the HLNHV description.

Summary.— Based on the assumption of partial separability we have derived a set of Bell-type inequalities for arbitrarily high-dimensional systems. Partially entangled states would not violate the inequalities, thus upon violation, the Bell-type inequalities are sufficient conditions to detect the full N-qudit entanglement and rule out the HLNHV description. It is observed that the Bell-type inequalities for multi-qudit ($d \geq 3$) violate the hybrid local-nonlocal realism more strongly than the Svetlichny ones for qubits, and the quantum violations increase with d. Furthermore, how to generalize the Bell-type inequality to the multi-setting one remains a significant topic to be investigated further.

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[1] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[2] N. Gisin, G. Ribordy, W. Tittel, H. Zbinden, Rev. Mod. Phys. **74**, 145 (2002).
[3] G. Svetlichny, Phys. Rev. D **35**, 3066 (1987).
[4] M. Seevinck and G. Svetlichny, Phys. Rev. Lett. **89**, 060401 (2002).
[5] D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, Phys. Rev. Lett. **88**, 170405 (2002).
[6] S. Ghose, N. Sinclair, S. Debnath, P. Rungta, and R. Stock, Phys. Rev. Lett. **102**, 250404 (2009).
[7] O. Gühne and M. Seevinck, New J. Phys. **12**, 053002 (2010).
[8] W. Laskowski and M. Żukowski, Phys. Rev. A **72**, 062112 (2005).
[9] A. Rauschenbeutel, G. Nogues, S. Osnaghi, P. Bertet, M. Brune, J. Raimond, and S. Haroche, Science **288**, 2024 (2000).
[10] J.-W. Pan, D. Bouwmeester, M. Daniell, H. Weinfurter, and A. Zeilinger, Nature (London) 403, 515 (2000).
[11] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990); M. Ardehali, Phys. Rev. A **46**, 5375 (1992); A. V. Belinskii and D. N. Klyshko, Phys. Usp. **36**, 653 (1993).
[12] R. F. Werner and M. M. Wolf, Phys. Rev. A **64**, 032112 (2001).
[13] M. Żukowski and Č. Brukner, Phys. Rev. Lett. **88**, 210401 (2002).
[14] P. Mitchell, S. Popescu, and D. Roberts, Phys. Rev. A **70**, 060101(R) (2004).
[15] I. Percival, Phys. Lett. A **244**, 495 (1998).
[16] A. Fine, Phys. Rev. Lett. **48**, 291 (1982).
[17] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. **88**, 040404 (2002).
[18] M. Żukowski, A. Zeilinger, and M. A. Horne, Phys. Rev. A **55**, 2564 (1997).