Optimal Joint Scheduling and Resource Allocation in OFDMA Downlink Systems with Imperfect Channel-State Information

Rohit Aggarwal*, Mohamad Assaad†, C. Emre Koksal*, and Philip Schniter*
*Dept. of ECE, The Ohio State University, Columbus, OH 43210.
Email: {aggarwar,koksal,schniter}@ece.osu.edu
†Supélec, Gif sur Yvette, France. Email: mohamad.assaad@supelec.fr

Abstract

In this paper, we address the problem of joint scheduling and resource allocation for the downlink of an orthogonal frequency division multiple access (OFDMA) based wireless network. Since perfect current knowledge of channel state information (CSI) for all users may be difficult to maintain at the base-station, especially when the number of users and/or subchannels is large, we consider resource allocation under imperfect CSI, where the channel state is described by a generic probability distribution. In particular, we model the resource allocation problem as the maximization of an expected sum utility over user allocations, powers, and code rates, subject to an instantaneous sum-power constraint. First, we consider the "continuous" case where multiple users and/or code rates can time-share a single OFDMA subchannel and time slot. This yields a non-convex optimization problem that we convert into a convex optimization problem and solve optimally using a dual optimization approach. Second, we consider the "discrete" case where only a single user and code rate is allowed per OFDMA subchannel per time slot. For the mixed-integer optimization problem that arises, we discuss the connections it has with the continuous case and show that it can solved optimally in some situations. For the other situations, we present a bound on the optimality gap. For both cases, we provide algorithmic implementations of the obtained solution. Finally, we study, numerically, the performance of the proposed algorithms under varying degrees of CSI uncertainty and OFDMA system configurations.

I. Introduction

In the downlink of a wireless orthogonal frequency division multiple access (OFDMA) system, the base station (BS) delivers data to a pool of users whose channels vary in both time and frequency. Since bandwidth and power resources are limited, the BS would like to allocate them most efficiently, e.g., by pairing users with strong subchannels and distributing power in the most effective manner. At the same time, the BS may need to maintain per-user quality-of-service (QoS) constraints, such as a minimum reliable rate for each user. Overall, the BS faces a resource allocation problem where the goal is to maximize an efficiency-related quantity (e.g., the sum of log-throughput) under particular (e.g., power) constraints. Although the optimal allocation of resources is clearly a function of the instantaneous channel state of all users at all subchannels, it is difficult in practice to maintain perfect instantaneous channel state information (CSI), and so resource allocation must be accomplished under imperfect CSI.

In this paper, we consider the problem of simultaneous user scheduling, power allocation, and rate optimization in an OFDMA downlink system when only a generic channel-state distribution is available at the BS. Here, the use of a generic channel-state distribution allows us considerable flexibility in the modeling of channel uncertainty. In particular, we consider the problem of maximizing expected sum-utility subject to a constraint on sum-power under two scenarios. In the first scenario, we allow multiple users and/or code rates to time-share any given subchannel and time slot. This formulation results in a non-convex optimization problem. We show that it can be converted into a convex optimization problem and solved optimally using a dual optimization approach. For this, we propose an algorithm that converges exponentially fast to the optimal solution. In the second scenario, we allow at most one user-MCS combination to be allocated on any subchannel. This formulation results in a mixed-integer optimization problem. We discuss connections between the two scenarios and attack the second problem using the solution obtained in the first. For some cases, we show that the obtained solution has an optimality gap (i.e., difference between the obtained and optimal performance) of zero, while for the other cases, we bound the optimality gap. In addition, we propose an algorithmic implementation of the proposed solution. Finally, we study, numerically, the performance of the proposed algorithms under different OFDMA system configurations.

We now discuss related work. The problem of OFDMA downlink scheduling and resource allocation under *perfect* CSI has been addressed in a number of publications (e.g., [1]–[6]). In [1], a subchannel, bit-rate, and power allocation algorithm was developed to minimize power consumption while maintaining a data rate requirement. The authors of [2] proposed a low-complexity power-adaptation algorithm to maximize sum-rate. They found that sum-rate is maximized when each subchannel is assigned to the user with the single best channel gain for that subchannel, and when the transmit power is distributed over subchannels using a water-filling policy. In [5], a weighted-sum ergodic-capacity maximization problem was formulated to exploit time, frequency, and multi-user diversity while enforcing different notions of fairness. Non-convex optimization problems of weighted sum-rate maximization and weighted sum-power minimization were solved using a Lagrange dual decomposition method in [6].

The above works assume the availability of perfect transmitter CSI that would be very difficult to maintain in practice. We claim that a more practical formulation of the OFDMA downlink resource allocation problem would assume imperfect CSI. The effect of imperfect CSI has been widely studied for single-user OFDM (e.g., [7]–[9]). In [7], channel prediction was used to mitigate the effect of outdated CSI on the performance of adaptive OFDM systems. The effect of OFDM channel estimation error, as well as that of outdated CSI, were studied for the variable bit-rate case in [8]. In [9], an optimal power loading algorithm for rate maximization was derived based on average and outage capacity criteria, and it was concluded that the outage rate of the system may be greatly reduced due to CSI error. Resource allocation strategies under imperfect CSI for a multi-user (e.g., OFDMA) downlink system has been studied in [10]–[12]. In [10], the authors considered the problem of ergodic weighted sum-rate maximization for user scheduling and resource allocation, and studied the impact of channel estimation error on OFDMA performance, where channel estimation error resulted pilot-aided MMSE channel estimation. In contrast, we consider a general utility maximization framework wherein no restrictive assumptions are made on the generation of imperfect CSI. In [12], a margin adaptive resource allocation framework was studied to cope with feedback delay and outdated CSI. In particular, the problem of total transmit power minimization, subject to strict constraints on conditional expected user capacities, was investigated. In contrast, we focus on maximizing a more general concave goodput-based utility subject to a sum-power constraint.

The rest of the paper is organized as follows. In Section II, we outline our system model and frame the optimization problems that we intend to solve. In Section III, we consider the "continuous" problem, where each subchannel can be shared by multiple users and rates, and present an optimal solution. In Section IV, we consider the "discrete" problem, where each subchannel can support at most one combination of user and rate per time slot. There we show that, under certain conditions, the continuous and discrete problems become equivalent, allowing us to apply our approach to the continuous problem. When these conditions do not hold, we propose a practical algorithm that approximately solves the discrete problem and bound its performance. In Section V, we compare the performance of the proposed algorithms to reference algorithms under various settings. Finally, in Section VI, we conclude.

II. System Model

We consider a downlink OFDMA system with N subchannels and K active users ($N, K \in \mathbb{Z}^+$). During every channel use, a symbol (of any signaling scheme) is transmitted using a particular OFDMA subchannel. It propagates through a fading channel on the way to its intended mobile recipient. The OFDMA subchannels are assumed to be non-interfering with gains that are time-invariant over each symbol duration. Furthermore, the subchannels associated with a particular user are assumed to be statistically independent of those associated with other users. Thus, the successful reception of a transmitted symbol depends on the corresponding subchannel's SNR γ, power p, and modulation and coding scheme (MCS) m. Here, we assume that MCS $m \in \{1, \ldots, M\}$ corresponds to a transmission rate of r_m bits per symbol and a symbol error probability of $\epsilon_m(p\gamma)$. The symbol error probability is a function of the received SNR $p\gamma$ because we treat the subchannel SNR γ as an exogenous parameter.

Given a symbol error rate of $\epsilon_m(p\gamma)$, the goodput $g = \big(1 - \epsilon_m(p\gamma)\big) r_m$ quantifies the expected number of bits per symbol that can be transmitted without error. In the sequel, we will focus on maximizing goodput-based utilities of the form $U(g)$, where $U(\cdot)$ is any twice-differentiable strictly-increasing concave function. To make the problem tractable, we will assume symbol error probabilities of the form[1] $\epsilon_m(p\gamma) = a_m e^{-b_m p\gamma}$, where a_m and b_m are known constants (see, e.g., [10]).

Before we can precisely state our scheduling and resource allocation (SRA) problem, we need to introduce some additional notations. First, to indicate how subchannels are partitioned among users and rates in each time-slot, we introduce the proportionality indicator $I_{n,k,m}$, where $I_{n,k,m} = 1$ means that subchannel n is fully dedicated to user k at MCS m, and $I_{n,k,m} = 0$ means that subchannel n is totally unavailable to user k at MCS m. The subchannel resource constraint is then expressed as $\sum_{k,m} I_{n,k,m} \leq 1$ for all n. In the sequel, we consider two flavors of the SRA problem, a "continuous" one where each subchannel can be shared among multiple users and/or rates per time slot (i.e., $I_{n,k,m} \in [0, 1]$), and a "discrete" one where each subchannel can be allocated to at most one user/rate combination per time slot (i.e., $I_{n,k,m} \in \{0, 1\}$). Next, we introduce $p_{n,k,m} \geq 0$ as the power that would be expended on subchannel n if it was fully allocated to the user/rate combination (k, m). With this definition, the total expended power becomes $\sum_{n,k,m} I_{n,k,m} p_{n,k,m}$. Finally, we introduce $\gamma_{n,k}$ as the n^{th} subchannel's SNR for user k. Although we assume that the BS does not know the SNR realizations $\{\gamma_{n,k}\}$, we assume that it does know the (marginal) distribution of each $\gamma_{n,k}$.

Our objective is to maximize the expected sum utility $\mathrm{E}\big\{ \sum_{n,k,m} I_{n,k,m} U_{n,k,m}(g_{n,k,m}) \big\}$, where $g_{n,k,m}$ denotes the goodput that is contributed from subchannel n by user k with MCS m if that subchannel was

[1] While models of the form $a_m e^{-b_m p\gamma}$ are typically used to describe *bit* error probability [10], we can adapt this model to *symbol* error probability using a generalization of the goodput metric. For example, under bit error probability $a_m e^{-b_m p\gamma}$ and d independent bits per symbol, we would generalize the goodput expression to $g = (1 - a_m e^{-b_m p\gamma})^d r_m$. While all results in this paper can be easily extended to this generalized goodput expression, we consider only the case $d = 1$, for simplicity.

fully allocated to that user/MCS combination. Here, the expectation is taken over the subchannel-SNRs $\{\gamma_{n,k}\}$, which in turn affect the goodputs $\{g_{n,k,m}\}$. The utility function $U_{n,k,m}(\cdot)$ is used to transform goodput into a quality-of-service (QoS) or fairness[2] metric, e.g., maximin fairness or proportional fairness [13]. We assume $U_{n,k,m}(\cdot)$ to be any generic real-valued function that is twice differentiable, strictly-increasing, and concave with $U_{n,k,m}(0) < \infty$. Therefore, $U'_{n,k,m}(\cdot) > 0$ and $U''_{n,k,m}(\cdot) \leq 0$, where $'$ denotes the derivative. Incorporating a sum-power constraint to our objective, our SRA problem then becomes

$$\text{SRA} \triangleq \max_{\substack{\{p_{n,k,m} \geq 0\} \\ \{I_{n,k,m}\}}} \mathrm{E}\left\{\sum_{n=1}^{N}\sum_{k=1}^{K}\sum_{m=1}^{M} I_{n,k,m} U_{n,k,m}\big((1 - a_m e^{-b_m p_{n,k,m}\gamma_{n,k}}) r_m\big)\right\} \tag{1}$$

$$\text{s.t.} \quad \sum_{k,m} I_{n,k,m} \leq 1 \ \forall n \quad \text{and} \quad \sum_{n,k,m} I_{n,k,m} p_{n,k,m} \leq P_{\mathsf{con}}$$

In Section III, we solve the SRA problem for the continuous case $I_{n,k,m} \in [0,1]$, and in Section IV we solve it for the discrete case $I_{n,k,m} \in \{0,1\}$.

III. Optimal Scheduling and Resource Allocation with subchannel sharing

In this section, we address the SRA problem in the case where $I_{n,k,m} \in [0,1] \ \forall (n,k,m)$. Recall that this problem arises when sharing of any subchannel by multiple users and/or multiple MCS combinations is allowed. We refer to this problem as the continuous scheduling and resource allocation (CSRA) problem. Defining $\boldsymbol{I}$ as the $N \times K \times M$ matrix with $(n,k,m)^{\text{th}}$ element as $I_{n,k,m}$ and the domain of $\boldsymbol{I}$ as

$$\mathcal{I}_{\mathsf{CSRA}} := \left\{\boldsymbol{I} : \boldsymbol{I} \in [0,1]^{N \times K \times M}, \ \textstyle\sum_{k,m} I_{n,k,m} \leq 1 \ \forall n\right\},$$

the CSRA problem can be stated as

$$\text{CSRA} := \min_{\substack{\{p_{n,k,m} \geq 0\} \\ \boldsymbol{I} \in \mathcal{I}_{\mathsf{CSRA}}}} -\sum_{n,k,m} I_{n,k,m} \, \mathrm{E}\left\{U_{n,k,m}\big((1 - a_m e^{-b_m p_{n,k,m}\gamma_{n,k}}) r_m\big)\right\} \ \text{s.t.} \sum_{n,k,m} I_{n,k,m} \, p_{n,k,m} \leq P_{\mathsf{con}}. \tag{2}$$

This problem has a non-convex constraint set, making it a non-convex optimization problem. In order to convert it into a convex optimization problem, we write the "actual" power allocated to user k at MCS m on subchannel n as $x_{n,k,m} = I_{n,k,m} \, p_{n,k,m}$. Then, the problem becomes

$$\text{CSRA} = \min_{\substack{\{x_{n,k,m} \geq 0\} \\ \boldsymbol{I} \in \mathcal{I}_{\mathsf{CSRA}}}} \sum_{n,k,m} I_{n,k,m} \, F_{n,k,m}(I_{n,k,m}, x_{n,k,m}) \quad \text{s.t.} \sum_{n,k,m} x_{n,k,m} \leq P_{\mathsf{con}}, \tag{3}$$

[2] In our formulation, if $U_{n,k,m}(\cdot)$ was chosen to incentivize "fairness" constraints, then the fairness would be imposed jointly over users, subchannels, and rates. If one instead desired to impose fairness over users only, then an optimization problem of the form $\sum_k \mathrm{E}\left\{U_k\left(\sum_{n,m} I_{n,k,m} p_{n,k,m}\right)\right\}$ may be more appropriate. However, this latter problem may require a different optimization approach than the one taken in this paper.

where $F_{n,k,m}(\cdot,\cdot)$ is given by

$$F_{n,k,m}(I_{n,k,m}, x_{n,k,m}) = \begin{cases} -\,\mathrm{E}\left\{U_{n,k,m}\Big((1-a_m e^{-b_m x_{n,k,m}\gamma_{n,k}/I_{n,k,m}})r_m\Big)\right\} & \text{if } I_{n,k,m} \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The modified problem in (3) is a convex optimization problem with a convex objective function (shown in Appendix A) and linear inequality constraint. Moreover, Slater's condition is satisfied at $I_{n,k,m} = \frac{1}{2KM}$ and $x_{n,k,m} = \frac{P_{\text{con}}}{N} I_{n,k,m},\ \forall n,k,m$. Hence, the solution of (3) is the same as that of its dual problem (i.e., zero duality gap) [14]. Let us denote the optimal $\boldsymbol{I}$ and $\boldsymbol{x}$ for (3) by $\boldsymbol{I}^*_{\text{CSRA}}$ and $\boldsymbol{x}^*_{\text{CSRA}}$, respectively, and let $\boldsymbol{p}^*_{\text{CSRA}}$ be the corresponding $\boldsymbol{p}$.

Writing the dual formulation, using μ as the dual variable, the Lagrangian of (3) is

$$L(\mu, \boldsymbol{I}, \boldsymbol{x}) = \sum_{n,k,m} I_{n,k,m}\, F_{n,k,m}(I_{n,k,m}, x_{n,k,m}) + \Big(\sum_{n,k,m} x_{n,k,m} - P_{\text{con}}\Big)\mu, \tag{5}$$

where we use $\boldsymbol{x}$ to denote the $N \times K \times M$ matrix $[x_{n,k,m}]$. The corresponding unconstrained dual problem, then, becomes

$$\begin{aligned} &\max_{\mu \geq 0} \min_{\substack{\boldsymbol{x} \succeq 0 \\ \boldsymbol{I} \in \mathcal{I}_{\text{CSRA}}}} L(\mu, \boldsymbol{I}, \boldsymbol{x}) \\ &= \max_{\mu \geq 0} \min_{\boldsymbol{I} \in \mathcal{I}_{\text{CSRA}}} L(\mu, \boldsymbol{I}, \boldsymbol{x}^*(\mu, \boldsymbol{I})) = \max_{\mu \geq 0} L(\mu, \boldsymbol{I}^*(\mu), \boldsymbol{x}^*(\mu, \boldsymbol{I}^*(\mu))) = L(\mu^*, \boldsymbol{I}^*(\mu^*), \boldsymbol{x}^*(\mu^*, \boldsymbol{I}^*(\mu^*))), \end{aligned} \tag{6}$$

where $\boldsymbol{x} \succeq 0$ means that $x_{n,k,m} \geq 0\ \forall n,k,m$, $\boldsymbol{x}^*(\mu, \boldsymbol{I})$ denotes the optimal $\boldsymbol{x}$ for a given μ and $\boldsymbol{I}$, $\boldsymbol{I}^*(\mu) \in \mathcal{I}_{\text{CSRA}}$ denotes the optimal $\boldsymbol{I}$ for a given μ, and μ^* denotes the optimal μ. In the next few subsections, we will optimize the Lagrangian according to (6) w.r.t. $\boldsymbol{x}$, $\boldsymbol{I}$, and μ in Section III-A, Section III-B, and Section III-C, respectively. This will be followed by the proposed iterative algorithm to solve CSRA problem in Section III-D. We will, then, end the discussion on CSRA problem with few insightful properties of the obtained optimal CSRA solution in Section III-E.

A. *Optimizing over total powers, $\boldsymbol{x}$, for a given μ and user-MCS allocation matrix $\boldsymbol{I}$*

The Lagrangian in (5) is a convex function of $\boldsymbol{x}$. Therefore, any local minima of the function is the global minima. Calculating the derivative of $L(\mu, \boldsymbol{I}, \boldsymbol{x})$ w.r.t. $x_{n,k,m}$, we get

$$\begin{aligned} &\frac{\partial L(\mu, \boldsymbol{I}, \boldsymbol{x})}{\partial x_{n,k,m}} \\ &= \begin{cases} \mu & \text{if } I_{n,k,m} = 0 \\ \mu - a_m b_m r_m\, \mathrm{E}\left\{U'_{n,k,m}\Big((1-a_m e^{-b_m x_{n,k,m}\gamma_{n,k}/I_{n,k,m}})r_m\Big)\gamma_{n,k} e^{-b_m x_{n,k,m}\gamma_{n,k}/I_{n,k,m}}\right\} & \text{otherwise.} \end{cases} \end{aligned} \tag{7}$$

Clearly, if $I_{n,k,m} = 0$, $L(\cdot,\cdot,\cdot)$ is an increasing[3] function of $x_{n,k,m}$ since $\mu \geq 0$. Therefore, $x^*_{n,k,m}(\mu, \boldsymbol{I}) = 0$. If however $I_{n,k,m} \neq 0$, then $\frac{\partial L(\mu,\boldsymbol{I},\boldsymbol{x})}{\partial x_{n,k,m}}$ is an increasing function of $x_{n,k,m}$ since $U'_{n,k,m}(\cdot)$ is a decreasing function of $x_{n,k,m}$. Thus, we have

$$\mu - a_m b_m r_m \,\mathrm{E}\left\{U'_{n,k,m}\big((1 - a_m e^{-b_m x_{n,k,m}\gamma_{n,k}/I_{n,k,m}})r_m\big)\gamma_{n,k} e^{-b_m x_{n,k,m}\gamma_{n,k}/I_{n,k,m}}\right\} = 0 \tag{8}$$

for some positive value of $x_{n,k,m}$ if and only if $0 \leq \mu \leq a_m b_m r_m U'_{n,k,m}\big((1-a_m)r_m\big)\,\mathrm{E}\{\gamma_{n,k}\}$. Therefore,

$$x^*_{n,k,m}(\mu, \boldsymbol{I}) = \begin{cases} \tilde{x}_{n,k,m}(\mu, \boldsymbol{I}) & \text{if } 0 \leq \mu \leq a_m b_m r_m U'_{n,k,m}\big((1-a_m)r_m\big)\,\mathrm{E}\{\gamma_{n,k}\} \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

where $\tilde{x}_{n,k,m}(\mu, \boldsymbol{I})$ satisfies

$$\mu \,=\, a_m b_m r_m \,\mathrm{E}\left\{U'_{n,k,m}\big((1 - a_m e^{-b_m \tilde{x}_{n,k,m}(\mu,\boldsymbol{I})\gamma_{n,k}/I_{n,k,m}})r_m\big)\gamma_{n,k} e^{-b_m \tilde{x}_{n,k,m}(\mu,\boldsymbol{I})\gamma_{n,k}/I_{n,k,m}}\right\}. \tag{10}$$

From (10), we observe that $\tilde{x}_{n,k,m}(\mu, \boldsymbol{I}) = \tilde{p}_{n,k,m}(\mu) I_{n,k,m}$ where $\tilde{p}_{n,k,m}(\mu)$ satisfies

$$\mu \,=\, a_m b_m r_m \,\mathrm{E}\left\{U'_{n,k,m}\big((1 - a_m e^{-b_m \tilde{p}_{n,k,m}(\mu)\gamma_{n,k}})r_m\big)\gamma_{n,k} e^{-b_m \tilde{p}_{n,k,m}(\mu)\gamma_{n,k}}\right\}. \tag{11}$$

Combining the above observations, we can write for any $\boldsymbol{I} \in \mathcal{I}_{\mathsf{CSRA}}$ and (n,k,m) that

$$x^*_{n,k,m}(\mu, \boldsymbol{I}) = I_{n,k,m}\, p^*_{n,k,m}(\mu), \tag{12}$$

where

$$p^*_{n,k,m}(\mu) = \begin{cases} \tilde{p}_{n,k,m}(\mu) & \text{if } 0 \leq \mu \leq a_m b_m r_m U'_{n,k,m}\big((1-a_m)r_m\big)\,\mathrm{E}\{\gamma_{n,k}\} \\ 0 & \text{otherwise,} \end{cases} \tag{13}$$

and $\tilde{p}_{n,k,m}(\mu)$ satisfies (11). Note that if such a $\tilde{p}_{n,k,m}(\mu)$ exists that satisfies (11), then it is unique. This is because, in (11), $U'_{n,k,m}(\cdot)$ is a continuous decreasing positive function and $e^{-b_m \tilde{p}_{n,k,m}(\mu)\gamma_{n,k}}$ is a strictly-decreasing continuous function of $\tilde{p}_{n,k,m}(\mu)$ which makes the right side of (11) a strictly-decreasing continuous function of $\tilde{p}_{n,k,m}(\mu)$. Therefore, in the domain of its existence, $\tilde{p}_{n,k,m}(\mu)$ is unique and decreases continuously with increase in μ. Consequently, $x^*_{n,k,m}(\mu, \boldsymbol{I})$ is a decreasing continuous function of μ. A sample plot showing the variation of $p^*_{n,k,m}(\mu)$ w.r.t. μ has been shown in Fig. 1.

[3]We use the terms “increasing” and “decreasing” interchangeably with “non-decreasing” and “non-increasing”, respectively. The terms “strictly-increasing” and “strictly-decreasing” are used when appropriate.

B. Optimizing over user-MCS allocation matrix $\boldsymbol{I}$ for a given μ

Substituting $\boldsymbol{x}^*(\mu, \boldsymbol{I})$ from (12) into (5), we get the Lagrangian as follows.

$$\begin{aligned} &L(\mu, \boldsymbol{I}, \boldsymbol{x}^*(\mu, \boldsymbol{I})) \\ &= -\mu P_{\mathsf{con}} + \sum_n \underbrace{\sum_{k,m} I_{n,k,m} \left[\overbrace{- \mathrm{E}\left\{ U_{n,k,m}\big((1 - a_m e^{-b_m p^*_{n,k,m}(\mu)\gamma_{n,k}}) r_m\big) \right\} + \mu p^*_{n,k,m}(\mu)}^{V_{n,k,m}(\mu, p^*_{n,k,m}(\mu))} \right]}_{L_n(\mu, \boldsymbol{I}_n)}, \end{aligned} \tag{14}$$

where $\boldsymbol{I}_n = \{I_{n,k,m}\ \forall (k,m)\}$. Since the above Lagrangian contains the sum of $L_n(\mu, \boldsymbol{I}_n)$ over n, minimizing $L_n(\mu, \boldsymbol{I}_n)$ for every n (over all possible $\boldsymbol{I}_n$) minimizes the Lagrangian. Now, $L_n(\mu, \boldsymbol{I}_n)$ is an linear function of $\{I_{n,k,m}\ \forall (k,m)\}$ that satisfies $\sum_{k,m} I_{n,k,m} \leq 1$. Therefore, $L_n(\mu, \boldsymbol{I}_n)$ is minimized by the $\boldsymbol{I}_n$ that gives maximum possible weight to the (k,m) combination with the most negative value of $V_{n,k,m}(\mu, p^*_{n,k,m}(\mu))$. To write this mathematically, let us define, for each μ and subchannel n, a set of participating user-MCS combinations that yield the same most-negative value of $V_{n,k,m}(\mu, p^*_{n,k,m}(\mu))$ over all (k,m) as follows:

$$S_n(\mu) \triangleq \left\{ (k,m) : (k,m) = \operatorname*{argmin}_{(k',m')} V_{n,k',m'}(\mu, p^*_{n,k',m'}(\mu)), \text{ and } V_{n,k,m}(\mu, p^*_{n,k,m}(\mu)) \leq 0 \right\}. \tag{15}$$

If $S_n(\mu)$ is a null or a singleton set, then the optimal allocation on subchannel n is given by

$$I^*_{n,k,m}(\mu) = \begin{cases} 1 & \text{if } (k,m) \in S_n(\mu) \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

However, if $|S_n(\mu)| > 1$ (cardinality greater than one), then multiple (k,m) combinations contribute equally towards the minimum value of $L_n(\mu, \boldsymbol{I})$ and thus the optimum can be reached by sharing subchannel n. In particular, let us suppose that $S_n(\mu) = \{\big(k_1(n), m_1(n)\big), \ldots, \big(k_{|S_n(\mu)|}(n), m_{|S_n(\mu)|}(n)\big)\}$. Then, the optimal allocation of subchannel n is given by

$$I^*_{n,k,m}(\mu) = \begin{cases} I_{n,k_i(n),m_i(n)} & \text{if } (k,m) = (k_i(n), m_i(n)) \text{ for some } i \in \{1, \ldots, |S_n(\mu)|\} \\ 0 & \text{otherwise,} \end{cases} \tag{17}$$

where the vector $(I_{n,k_1(n),m_1(n)}, \ldots, I_{n,k_{|S_n(\mu)|}(n),m_{|S_n(\mu)|}(n)})$ is any point in the unit-$(|S_n(\mu)|-1)$ simplex, i.e., it belongs to the space $[0,1]^{|S_n(\mu)|}$ and satisfies

$$\sum_{i=1}^{|S_n(\mu)|} I_{n,k_i(n),m_i(n)} = 1. \tag{18}$$

C. Optimizing over μ

In order to optimize over μ, we can calculate the Lagrangian optimized for a given value of μ as

$$L(\mu, \boldsymbol{I}^*(\mu), \boldsymbol{x}^*(\mu, \boldsymbol{I}^*(\mu))) \\ = -\mu P_{\mathsf{con}} + \sum_{n,k,m} I^*_{n,k,m}(\mu)\Big[-\mathrm{E}\Big\{U_{n,k,m}\big((1-a_m e^{-b_m p^*_{n,k,m}(\mu)\gamma_{n,k}})r_m\big)\Big\} + \mu p^*_{n,k,m}(\mu)\Big], \quad (19)$$

and then maximize it over all possible values of $\mu \geq 0$ to find μ^*. Notice from (16)-(18) that we have $\sum_{k,m} I^*_{n,k,m}(\mu^*) = 1$ for at least one n. Otherwise, $\boldsymbol{I}^*(\mu^*) = \mathbf{0}$ which, clearly, is not the optimal solution. Therefore, $\mu^* \geq \mu_{\mathsf{min}} > 0$, where

$$\mu_{\mathsf{min}} = \min_{n,k,m} a_m b_m r_m \,\mathrm{E}\left\{U'_{n,k,m}\big((1-a_m e^{-b_m P_{\mathsf{con}}\gamma_{n,k}})r_m\big)\gamma_{n,k}e^{-b_m P_{\mathsf{con}}\gamma_{n,k}}\right\} \quad (20)$$

is obtained by taking $\tilde{p}_{n,k,m}(\mu) \to P_{\mathsf{con}}$ for all (n,k,m) in the right side of (11). Since $p^*_{n,k,m}(\mu)$ is a decreasing continuous function of μ (seen in Section III-A), we have $\sum_{n,k,m} x^*_{n,k,m}(\mu, \boldsymbol{I}) > P_{\mathsf{con}}$ for all $\boldsymbol{I} \neq \mathbf{0}$ and $\mu < \mu_{\mathsf{min}}$. We can also obtain an upper bound $\mu^* \leq \mu_{\mathsf{max}}$, where

$$\mu_{\mathsf{max}} = \max_{n,k,m} a_m b_m r_m U'_{n,k,m}\big((1-a_m)r_m\big)\,\mathrm{E}\{\gamma_{n,k}\}, \quad (21)$$

is obtained by taking $\tilde{p}_{n,k,m}(\mu) \to 0$ in the right side of (11). Thus, for any $\mu > \mu_{\mathsf{max}}$, $x^*_{n,k,m}(\mu, \boldsymbol{I}) = 0 \;\forall n, k, m, \boldsymbol{I}$. Since, the primal objective in (3) is not maximized when zero power is allocated on all subchannels, we have $\mu^* \in [\mu_{\mathsf{min}}, \mu_{\mathsf{max}}] \subset (0, \infty)$.

At the optimal μ, i.e., μ^*, if we have $|S_n(\mu^*)| \leq 1 \;\forall n$, then the optimal CSRA allocation, $\boldsymbol{I}^*_{\mathsf{CSRA}}$, equals $\boldsymbol{I}^*(\mu^*)$ and can be calculated using (16). Moreover, the optimal power allocation $\boldsymbol{p}^*_{\mathsf{CSRA}}$ allocates

$$p^*_{n,k,m,\mathsf{CSRA}} = \begin{cases} p^*_{n,k,m}(\mu^*) & \text{if } I^*_{n,k,m}(\mu^*) \neq 0 \\ 0 & \text{otherwise.} \end{cases} \quad (22)$$

to every possible (n,k,m) combination. However, if for some n, we have $|S_n(\mu^*)| > 1$, then ambiguity arises due to multiple possibilities of $\boldsymbol{I}^*(\mu^*)$ obtained via (17). In order to find the optimal user-MCS allocation in such cases, we use the fact that the CSRA problem in (3) is a convex optimization problem whose optimal solution satisfies the sum-power constraint with equality, i.e.,

$$\sum_{n,k,m} x^*_{n,k,m}(\mu^*, \boldsymbol{I}^*(\mu^*)) = \sum_{n,k,m} I^*_{n,k,m}(\mu^*) p^*_{n,k,m}(\mu^*) = P_{\mathsf{con}}. \quad (23)$$

This is because $\mu^* \geq \mu_{\mathsf{min}} > 0$ (shown earlier) and the complementary slackness condition gives that $\mu^*\big(\sum_{n,k,m} x^*_{n,k,m}(\mu^*, \boldsymbol{I}^*(\mu^*)) - P_{\mathsf{con}}\big) = 0$. Now, the total power allocated to any subchannel n at μ^* is $\sum_{i=1}^{|S_n(\mu^*)|} I_{n,k_i(n),m_i(n)}\, p^*_{n,k_i(n),m_i(n)}(\mu^*)$ where $\{I_{n,k_i(n),m_i(n)}\}_{i=1}^{|S_n(\mu^*)|}$ satisfies (18). This quantity is dependent on the choice of values for $\{I_{n,k_i(n),m_i(n)}\}_{i=1}^{|S_n(\mu^*)|}$ and takes on any value between an upper and lower bound given by the following equation:

$$\min_i p^*_{n,k_i(n),m_i(n)}(\mu^*) \leq \sum_{i=1}^{|S_n(\mu^*)|} I_{n,k_i(n),m_i(n)}\, p^*_{n,k_i(n),m_i(n)}(\mu^*) \leq \max_i p^*_{n,k_i(n),m_i(n)}(\mu^*). \tag{24}$$

Note that the existence of at least one $\boldsymbol{I} = \boldsymbol{I}^*(\mu^*)$ satisfying

$$\sum_n \sum_i I_{n,k_i(n),m_i(n)}\, p^*_{n,k_i(n),m_i(n)}(\mu^*) = P_{\mathsf{con}} \tag{25}$$

is guaranteed by the optimality of the dual solution (of our convex CSRA problem over a closed constraint set). Therefore, we necessarily have $\sum_n \min_i p^*_{n,k_i(n),m_i(n)}(\mu^*) \leq P_{\mathsf{con}}$, and $\sum_n \max_i p^*_{n,k_i(n),m_i(n)}(\mu^*) \geq P_{\mathsf{con}}$. In addition, all choices of user-MCS allocations, $\boldsymbol{I}^*(\mu^*)$, given by (17) that satisfy the equality $\sum_{n,k,m} I^*_{n,k,m}(\mu^*)\, p^*_{n,k,m}(\mu^*) = P_{\mathsf{con}}$, are optimal for the CSRA problem.

In the case that the optimal solution $\boldsymbol{I}^*(\mu^*)$ is non-unique, i.e., $|S_n(\mu^*)| > 1$ for some n, then one instance of $\boldsymbol{I}^*(\mu^*)$ can be found as follows. For each subchannel n, define

$$(k_{\mathsf{max}}(n,\mu^*), m_{\mathsf{max}}(n,\mu^*)) := \operatorname*{argmax}_i p^*_{n,k_i(n),m_i(n)}(\mu^*), \tag{26}$$

$$(k_{\mathsf{min}}(n,\mu^*), m_{\mathsf{min}}(n,\mu^*)) := \operatorname*{argmin}_i p^*_{n,k_i(n),m_i(n)}(\mu^*), \tag{27}$$

and find the value of $\lambda \in [0,1]$ for which

$$\lambda\Big(\sum_n p_{n,k_{\mathsf{min}}(n,\mu^*),m_{\mathsf{min}}(n,\mu^*)}(\mu^*)\Big) + (1-\lambda)\Big(\sum_n p_{n,k_{\mathsf{max}}(n,\mu^*),m_{\mathsf{max}}(n,\mu^*)}(\mu^*)\Big) = P_{\mathsf{con}}, \tag{28}$$

i.e.,

$$\lambda = \frac{\sum_n p_{n,k_{\mathsf{max}}(n,\mu^*),m_{\mathsf{max}}(n,\mu^*)}(\mu^*) - P_{\mathsf{con}}}{\sum_n p_{n,k_{\mathsf{max}}(n,\mu^*),m_{\mathsf{max}}(n,\mu^*)}(\mu^*) - \sum_n p_{n,k_{\mathsf{min}}(n,\mu^*),m_{\mathsf{min}}(n,\mu^*)}(\mu^*)}. \tag{29}$$

Now, defining two specific allocations, $\boldsymbol{I}^{\mathsf{min}}(\mu^*)$ and $\boldsymbol{I}^{\mathsf{max}}(\mu^*)$, as

$$I^{\mathsf{min}}_{n,k,m}(\mu^*) = \begin{cases} 1 & (k,m) = (k_{\mathsf{min}}(n,\mu^*), m_{\mathsf{min}}(n,\mu^*)) \\ 0 & \text{otherwise}, \end{cases}, \quad I^{\mathsf{max}}_{n,k,m}(\mu^*) = \begin{cases} 1 & (k,m) = (k_{\mathsf{max}}(n,\mu^*), m_{\mathsf{max}}(n,\mu^*)) \\ 0 & \text{otherwise}, \end{cases} \tag{30}$$

respectively, the optimal user-MCS allocation is given by $\boldsymbol{I}^*_{\mathsf{CSRA}} = \lambda \boldsymbol{I}^{\mathsf{min}}(\mu^*) + (1-\lambda)\boldsymbol{I}^{\mathsf{max}}(\mu^*)$. The corresponding optimal power allocation is then given by (22). It can be seen that this solution satisfies the subchannel constraint as well as the sum power constraint with equality, i.e.,

$$\sum_{n,k,m} I^*_{n,k,m}(\mu^*) p^*_{n,k,m}(\mu^*) = \sum_{n,k,m} x^*_{n,k,m}(\mu^*, \boldsymbol{I}^*(\mu^*)) = P_{\mathsf{con}}.$$

Two interesting observations can be made from the above discussion. Firstly, for any choice of concave utility functions $U_{n,k,m}(\cdot)$, there exists an optimal scheduling and resource allocation strategy that allocates each subchannel to at most 2 user-MCS combinations. Therefore for allocating N subchannels, even if more than $2N$ user-MCS options are available, at most $2N$ such options will be used. Secondly, if $\boldsymbol{I}^{\min}(\mu^*) = \boldsymbol{I}^{\max}(\mu^*)$, then the optimal CSRA solution allocates power to at most one (k, m) combination for every subchannel, i.e., no subchannel is shared among any two or more user-MCS combinations. This observation will motivate the SRA problem's solution without subchannel sharing in Section IV.

D. Algorithmic implementation

In practical systems, it is not possible to search exhaustively over $\mu \in [\mu_{\mathsf{min}}, \mu_{\mathsf{max}}]$. Thus, we propose an algorithm that will reach solutions in close (and adjustable) proximity to the optimal. In this algorithm, we first narrow down the location of μ^* lies by using a bisection-search over $[\mu_{\mathsf{min}}, \mu_{\mathsf{max}}]$ for the optimum total power allocation, and then find a set of resource allocation decisions, $(\boldsymbol{I}, \boldsymbol{x})$, that achieve a total utility close to the optimal.

To proceed in this direction, with the aim of developing a framework to do bisection-search over μ, let us define the total optimal allocated power for a given value of μ as follows:

$$X^*_{\mathsf{tot}}(\mu) \triangleq \sum_{n,k,m} x^*_{n,k,m}(\mu, \boldsymbol{I}^*(\mu)), \tag{31}$$

where $\boldsymbol{I}^*(\mu)$ and $\boldsymbol{x}^*(\mu, \boldsymbol{I}^*(\mu))$ (defined in (6)) minimize the Lagrangian (defined in (5)) for a given μ. The following lemma relates the variation of $X^*_{\mathsf{tot}}(\mu)$ with respect to μ.

Lemma 1. *The total optimal power allocation, $X^*_{\mathsf{tot}}(\mu)$, is a monotonically decreasing function of μ.*

Proof: A proof sketch is given in Appendix B. For the full proof, see [15]. ■

A sample plot of $X^*_{\mathsf{tot}}(\mu)$ and $L(\mu, \boldsymbol{I}^*(\mu), \boldsymbol{x}^*(\mu, \boldsymbol{I}^*(\mu)))$ as a function of μ is shown in Figure 2. From the figure, three observations can be made. First, as μ increases, the optimal total allocated power decreases, as expected from Lemma 1. Second, as expected, the Lagrangian is maximized for that value of μ at which $X^*_{\mathsf{tot}}(\mu) = P_{\mathsf{con}}$. Third, the optimal total power allocation varies continuously in the region of μ where the optimal allocation, $\boldsymbol{I}^*(\mu)$, remains constant and takes a jump (negative) when $\boldsymbol{I}^*(\mu)$ changes. This happens for the following reason. We know that for any (n, k, m), $p^*_{n,k,m}(\mu)$ is a continuous function of μ. Thus, when the optimal allocation remains constant over a range of μ, the total power allocated, $\sum_{n,k,m} I^*_{n,k,m}(\mu) p^*_{n,k,m}(\mu)$ also varies continuously with μ. However, at the point of discontinuity (say $\tilde{\mu}$), multiple optimal allocations achieve the same optimal value of Lagrangian. In other words, $|S_n(\tilde{\mu})| > 1$ for some n. In that case, $X^*_{\mathsf{tot}}(\tilde{\mu})$ can take any value in the interval

$$\left[\sum_n p^*_{n,k_{\mathsf{min}}(n),m_{\mathsf{min}}(n)}(\tilde{\mu}),\ \ \sum_n p^*_{n,k_{\mathsf{max}}(n),m_{\mathsf{max}}(n)}(\tilde{\mu})\right]$$

while achieving the same minimum value of the Lagrangian at $\tilde{\mu}$. Applying Lemma 1, we have

$$X^*_{\text{tot}}(\tilde{\mu}-\Delta_1) \;\geq\; \sum_n p^*_{n,k_{\max}(n),m_{\max}(n)}(\tilde{\mu}) \;\geq\; X^*_{\text{tot}}(\tilde{\mu}) \;\geq\; \sum_n p^*_{n,k_{\min}(n),m_{\min}(n)}(\tilde{\mu}) \;\geq\; X^*_{\text{tot}}(\tilde{\mu}+\Delta_2)$$

for any $\Delta_1, \Delta_2 > 0$, causing a jump of $\left(\sum_n p^*_{n,k_{\min}(n),m_{\min}(n)}(\tilde{\mu}) - \sum_n p^*_{n,k_{\max}(n),m_{\max}(n)}(\tilde{\mu})\right)$ in the total optimal power allocation at $\tilde{\mu}$.

Lemma 1 allows us to do a bisection-search over μ since $X^*_{\text{tot}}(\mu)$ is a decreasing function of μ and the optimal μ is the one at which $X^*_{\text{tot}}(\mu) = P_{\text{con}}$. In particular, if $\mu^* \in [\underline{\mu}, \bar{\mu}]$ for some $\underline{\mu}$ and $\bar{\mu}$, then $\mu^* \in \left[\frac{\underline{\mu}+\bar{\mu}}{2}, \bar{\mu}\right]$ if $X^*_{\text{tot}}\left(\frac{\underline{\mu}+\bar{\mu}}{2}\right) > P_{\text{con}}$, otherwise $\mu^* \in \left[\underline{\mu}, \frac{\underline{\mu}+\bar{\mu}}{2}\right]$. Using this concept, we propose an algorithm in Table I that finds an interval $[\underline{\mu}, \bar{\mu}]$, such that $\mu^* \in [\underline{\mu}, \bar{\mu}]$ and $\bar{\mu} - \underline{\mu} \leq \kappa$ where κ (> 0) is a tuning-parameter, and allocates resources based on optimal resource allocations at $\underline{\mu}$ and $\bar{\mu}$.

The following lemma characterizes the relationship between the tuning parameter κ and the accuracy of the obtained solution.

Lemma 2. *Let $\mu^* \in [\underline{\mu}, \bar{\mu}]$ be the point where the proposed CSRA algorithm stops, and the total utility obtained by the proposed algorithm and the optimal CSRA solution be $\hat{U}_{\text{CSRA}}(\underline{\mu}, \bar{\mu})$ and U^*_{CSRA}, respectively. Then, $0 \leq U^*_{\text{CSRA}} - \hat{U}_{\text{CSRA}}(\underline{\mu}, \bar{\mu}) \leq (\bar{\mu} - \underline{\mu})P_{\text{con}}$.*

Proof: For proof, see Appendix C. ■

Since our algorithm stops when $\bar{\mu} - \underline{\mu} \leq \kappa$, from Lemma 2, the gap between the obtained utility and the optimal utility is bounded by $P_{\text{con}}\kappa$. Moreover, $\lim_{\underline{\mu}\to\bar{\mu}} \hat{U}_{\text{CSRA}}(\underline{\mu}, \bar{\mu}) = U^*_{\text{CSRA}}$.

The proposed algorithm requires at most $\left\lceil \log_2\left(\frac{\mu_{\max}-\mu_{\min}}{\kappa}\right)\right\rceil$ iterations of μ in order to find $\bar{\mu}$, and $\underline{\mu}$ such that $\bar{\mu} - \underline{\mu} \leq \kappa$ and $\mu^* \in [\underline{\mu}, \bar{\mu}]$. Therefore, measuring the complexity of the algorithm by the number of times (11) must be solved for a given (n, k, m, μ), the proposed algorithm takes at most

$$NKM \left\lceil \log_2\left(\frac{\mu_{\max}-\mu_{\min}}{\kappa}\right)\right\rceil \tag{32}$$

steps. We use this method of measuring complexity because it allows us to compare all algorithms in this paper easily. Since, for a given κ, the number of steps taken by the proposed bisection algorithm is proportional to $\log_2 \kappa$, the algorithm converges exponentially fast to the optimal solution.

E. Some properties of the CSRA solution

In this sub-section, we study few properties of the CSRA solution that give valuable insights into the optimal resource allocation strategy for any given value of Lagrange multiplier, μ. Let us fix a $\tilde{\mu} \in [\mu_{\min}, \mu_{\max}]$. Now, if $|S_n(\tilde{\mu})| \leq 1, \forall n$, then the optimal allocation at $\tilde{\mu}$, $\boldsymbol{I}^*(\tilde{\mu})$ is given by (16) which reveals that $\boldsymbol{I}^*(\tilde{\mu}) \in \{0,1\}^{N\times K\times M}$. In this case, the definition of $\mathcal{I}_{\mathsf{CSRA}}$ implies that every subchannel is allocated to at most one user-MCS combination. Note that this is precisely the constraint we impose in the later part of this paper. Let us now consider the case where $|S_n(\tilde{\mu})| > 1$ for some n is possible.

Lemma 3. *For any $\tilde{\mu} > 0$, there exists a $\delta > 0$ such that for all $\mu \in (\tilde{\mu}-\delta, \tilde{\mu}+\delta) \setminus \{\tilde{\mu}\}$, there exists an optimal allocation, $\boldsymbol{I}^*(\mu) \in \mathcal{I}_{\mathsf{CSRA}}$, that satisfies $\boldsymbol{I}^*(\mu) \in \{0,1\}^{N\times K\times M}$. Moreover, if $\mu_1, \mu_2 \in (\tilde{\mu}-\delta, \tilde{\mu})$, then there exists $\boldsymbol{I}^*(\mu_1), \boldsymbol{I}^*(\mu_2) \in \{0,1\}^{N\times K\times M}$ such that $\boldsymbol{I}^*(\mu_1) = \boldsymbol{I}^*(\mu_2)$. The same property holds if both $\mu_1, \mu_2 \in (\tilde{\mu}, \tilde{\mu}+\delta)$.*

Proof: A proof sketch is given in Appendix D. For the full proof, see [15]. ■

In conjunction with (12), the above lemma implies that the discontinuities in Fig. 2 are isolated and that around every point on the horizontal axis, there is a small region over which $X^*_{\mathsf{tot}}(\mu)$ is continuous.

IV. Scheduling and Resource Allocation without subchannel sharing

In this section, we will solve the scheduling and resource allocation (SRA) problem (1) under the constraint that $I_{n,k,m} \in \{0,1\}$, i.e., each subchannel can be allocated to at most one combination of user and MCS per time slot. We will refer to this problem as the discrete scheduling and resource allocation (DSRA) problem. Storing the values of $I_{n,k,m}$ in the $N \times K \times M$ matrix $\boldsymbol{I}$, the DSRA subchannel constraint can be expressed as $\boldsymbol{I} \in \mathcal{I}_{\mathsf{DSRA}}$, where

$$\mathcal{I}_{\mathsf{DSRA}} := \left\{ \boldsymbol{I} : \boldsymbol{I} \in \{0,1\}^{N\times K\times M}, \sum_{k,m} I_{n,k,m} \le 1 \ \forall n \right\}.$$

Then, using (1), the DSRA problem can be stated as

$$\text{DSRA} := \max_{\substack{\{p_{n,k,m}\ge 0\} \\ \boldsymbol{I}\in\mathcal{I}_{\mathsf{DSRA}}}} \sum_{n,k,m} I_{n,k,m} \,\mathrm{E}\left\{ U_{n,k,m}\big((1-a_m e^{-b_m p_{n,k,m}\gamma_{n,k}}) r_m\big) \right\} \text{ s.t.} \sum_{n,k,m} I_{n,k,m} p_{n,k,m} \le P_{\mathsf{con}}. \tag{33}$$

Let us denote the optimal $\boldsymbol{I}$ and $\boldsymbol{p}$ for (33) by $\boldsymbol{I}^*_{\mathsf{DSRA}}$ and $\boldsymbol{p}^*_{\mathsf{DSRA}}$, respectively.

The DSRA problem is a mixed-integer programming problem. Mixed-integer programming problems are generally NP-hard, meaning that polynomial-time solutions do not exist [16]. Fortunately, in some cases as in ours, one can exploit the problem structure to come up with polynomial complexity algorithms that reach solutions in close vicinity of the optimal solution. We first describe the naive approach of solving the mixed-integer programming problem, DSRA, by exhaustively searching over all possible user-MCS allocations in order to arrive at the optimal user, rate, and power allocation. We will see that this approach (termed brute-force) has complexity that is exponential in the number of subchannels.

Later, we will exploit the DSRA problem structure, and its relation to the CSRA problem, to propose an algorithm with polynomial complexity.

A. Brute-Force Algorithm

Consider that, if we attempted to solve our DSRA problem via brute-force (i.e., by solving the power allocation sub-problem for every possible choice of $\boldsymbol{I} \in \mathcal{I}_{\mathsf{DSRA}}$), we would solve the following sub-problem for every given $\boldsymbol{I}$.

$$\max_{\{p_{n,k,m}\geq 0\}} \sum_{n,k,m} I_{n,k,m}\,\mathrm{E}\left\{U_{n,k,m}\big((1-a_m e^{-b_m p_{n,k,m}\gamma_{n,k}})r_m\big)\right\} \;\text{ s.t.} \sum_{n,k,m} I_{n,k,m} p_{n,k,m} \leq P_{\mathsf{con}}. \tag{34}$$

Using the same approach as taken for the CSRA problem, we transform the variable $p_{n,k,m}$ into $x_{n,k,m}$ via the relation: $x_{n,k,m} = I_{n,k,m}\,p_{n,k,m}$. The problem in (34) can, therefore, be written as:

$$\min_{\{x_{n,k,m}\geq 0\}} \sum_{n,k,m} I_{n,k,m}\,F_{n,k,m}(I_{n,k,m}, x_{n,k,m}) \;\text{ s.t.} \sum_{n,k,m} x_{n,k,m} \leq P_{\mathsf{con}}, \tag{35}$$

where $F_{n,k,m}(I_{n,k,m}, x_{n,k,m})$ is defined in (4). This problem is a convex optimization problem that satisfies Slater's condition [14] when $x_{n,k,m} = P_{\mathsf{con}}/2NKM$ for all n, k, m. Therefore, its solution is equal to the solution of its dual problem (i.e., zero duality gap) [14]. To formulate the dual problem, we write the Lagrangian of the primal problem (35) as

$$L_{\boldsymbol{I}}(\mu, \boldsymbol{x}) = \sum_{n,k,m} I_{n,k,m} F_{n,k,m}(I_{n,k,m}, x_{n,k,m}) + \Big(\sum_{n,k,m} x_{n,k,m} - P_{\mathsf{con}}\Big)\mu, \tag{36}$$

where μ is the dual variable and $\boldsymbol{x}$ is the $N \times K \times M$ matrix containing actual powers allocated to all (n, k, m) combinations. Note that the Lagrangian in (36) is exactly the same as the Lagrangian for the CSRA problem in (5). Using (36), the dual of the brute-force problem can be written as

$$\max_{\mu\geq 0} \min_{\boldsymbol{x}\succeq 0} L_{\boldsymbol{I}}(\mu, \boldsymbol{x}) = \max_{\mu \geq 0} L_{\boldsymbol{I}}(\mu, \boldsymbol{x}^*(\mu)) = L_{\boldsymbol{I}}(\mu^*_{\boldsymbol{I}}, \boldsymbol{x}^*(\mu^*_{\boldsymbol{I}})), \tag{37}$$

for optimal solutions $\mu^*_{\boldsymbol{I}}$ and $\boldsymbol{x}^*(\mu^*_{\boldsymbol{I}})$. Minimizing $L_{\boldsymbol{I}}(\mu, \boldsymbol{x})$ over $\{\boldsymbol{x} \succeq 0\}$ by equating the differential of $L_{\boldsymbol{I}}(\mu, \boldsymbol{x})$ w.r.t. $x_{n,k,m}$ to zero (identically to the approach taken in Section III-A for the CSRA problem), we get that, for any subchannel n,

$$x^*_{n,k,m}(\mu) = I_{n,k,m}\,p^*_{n,k,m}(\mu). \tag{38}$$

Here,

$$p^*_{n,k,m}(\mu) = \begin{cases} \tilde{p}_{n,k,m}(\mu) & \text{if } 0 \leq \mu \leq a_m b_m r_m U'_{n,k,m}\big((1-a_m)r_m\big)\,\mathrm{E}\{\gamma_{n,k}\} \\ 0 & \text{otherwise,} \end{cases} \tag{39}$$

and $\tilde{p}_{n,k,m}(\mu)$ is the unique[4] value satisfying (11), repeated as (40) for convenience.

$$\mu = a_m b_m r_m \operatorname{E}\left\{U'_{n,k,m}\big((1-a_m e^{-b_m \tilde{p}_{n,k,m}(\mu)\gamma_{n,k}})r_m\big)\gamma_{n,k} e^{-b_m \tilde{p}_{n,k,m}(\mu)\gamma_{n,k}}\right\}. \tag{40}$$

Note that the Lagrangian as well as the power allocation in (36) and (38) are identical to that obtained for the CSRA problem in (5) and (12), respectively. Also recall that (19)-(21) hold even when $\boldsymbol{I}^*(\mu)$ is replaced by arbitrary $\boldsymbol{I}$. Thus, we have $\mu^*_{\boldsymbol{I}} \in [\mu_{\mathsf{min}}, \mu_{\mathsf{max}}]$, where μ_{min} and μ_{min} are defined in (20) and (21), respectively.

As discussed in Section III-A, $\tilde{p}_{n,k,m}(\mu)$ is a strictly-decreasing continuous function of μ, which makes $p^*_{n,k,m}(\mu)$ a decreasing continuous function of μ. Let us now define

$$X^*_{\mathsf{tot}}(\boldsymbol{I},\mu) \triangleq \sum_{n,k,m} x^*_{n,k,m}(\mu) = \sum_{n,k,m} I_{n,k,m} p^*_{n,k,m}(\mu) \tag{41}$$

as the total optimal power allocation for allocation $\boldsymbol{I}$ at μ. Therefore, $X^*_{\mathsf{tot}}(\boldsymbol{I},\mu)$ is also a decreasing continuous function of μ. This reduces our problem to finding the minimum value of $\mu \in [\mu_{\mathsf{min}}, \mu_{\mathsf{max}}]$ for which $X^*_{\mathsf{tot}}(\boldsymbol{I},\mu) = P_{\mathsf{con}}$. Such a problem structure (i.e., finding the minimum Lagrange multiplier satisfying a sum-power constraint) yields a *water-filling* solution (e.g., [2], [10]). To obtain such a solution (in our case, $\mu^*_{\boldsymbol{I}}$) one can use the bisection-search algorithm given in Table I.

While there are other ways to find μ, we focus on bisection-search for easy comparison to the CSRA algorithm. Thus, to solve the resource allocation problem for a given $\boldsymbol{I} \in \mathcal{I}_{\mathsf{DSRA}}$, the complexity in terms of the number of times (40) (or (11)) is solved to yield $\hat{\mu}_{\boldsymbol{I}}$ such that $|\hat{\mu}_{\boldsymbol{I}} - \mu^*_{\boldsymbol{I}}| < \kappa$, is $\left(\sum_{n,k,m} I_{n,k,m}\right)\left\lceil \log_2\left(\frac{\mu_{\mathsf{max}}-\mu_{\mathsf{min}}}{\kappa}\right)\right\rceil$. Since, the brute-force algorithm examines $|\mathcal{I}_{\mathsf{DSRA}}| = (KM+1)^N$ hypotheses of $\boldsymbol{I}$, the corresponding complexity needed to find the optimal DSRA solution is $\left\lceil \log_2\left(\frac{\mu_{\mathsf{max}}-\mu_{\mathsf{min}}}{\kappa}\right)\right\rceil \times \sum_{n=1}^{N} n\binom{N}{n}(KM)^n$, which equals

$$\left\lceil \log_2\left(\frac{\mu_{\mathsf{max}}-\mu_{\mathsf{min}}}{\kappa}\right)\right\rceil \times (KM+1)^{N-1} NKM. \tag{42}$$

This may be impractical to implement for typical values of K, M, and N, as it grows exponentially with the number of subchannels N. In the sequel, using insights from the CSRA problem, we will find approximations to the DSRA solution with much lower complexity.

[4] By assumption, $U'_{n,k,m}(\cdot)$ is a decreasing positive function and $e^{-b_m \tilde{p}_{n,k,m}(\mu)\gamma_{n,k}}$ is a strictly-decreasing positive function of $\tilde{p}_{n,k,m}(\mu)$, which makes the right side of (40) a strictly-decreasing positive function of $\tilde{p}_{n,k,m}(\mu)$.

B. Proposed DSRA Algorithm

Equation (30) in Section III-B demonstrated that there exists an optimal user-MCS allocation for the CSRA problem that either lies in the domain of DSRA problem, i.e., $\boldsymbol{I}^*(\mu^*) \in \mathcal{I}_{\textsf{DSRA}}$ or is a convex combination of two points from the domain of DSRA problem, i.e., $\boldsymbol{I}^*(\mu^*) = \lambda \boldsymbol{I}^{\textsf{min}}(\mu^*) + (1-\lambda)\boldsymbol{I}^{\textsf{max}}(\mu^*)$, where $\boldsymbol{I}^{\textsf{min}}(\mu^*) \neq \boldsymbol{I}^{\textsf{max}}(\mu^*)$ and $\boldsymbol{I}^{\textsf{min}}(\mu^*), \boldsymbol{I}^{\textsf{max}}(\mu^*) \in \mathcal{I}_{\textsf{DSRA}}$ (note that if $\boldsymbol{I} \in \mathcal{I}_{\textsf{CSRA}}$ and $\boldsymbol{I} \in \{0,1\}^{N\times K\times M}$, then $\boldsymbol{I} \in \mathcal{I}_{\textsf{DSRA}}$). This motivates us to attack the DSRA problem using the CSRA algorithm. In this section, we provide details of the proposed approximate DSRA solution using the above observation.

The following lemma will be instrumental in understanding the relationship between the CSRA and DSRA problems and will serve as the basis for allocating resources in the DSRA problem setup.

Lemma 4. *If the solution of the Lagrangian dual of the CSRA problem (6) for a given μ is such that $\boldsymbol{I}^*(\mu) \in \{0,1\}^{N\times K\times M}$, and the corresponding total power is $X^*_{\textsf{tot}}(\mu)$ as in (31), then the solution to the optimization problem*

$$(\mathbb{P}^*, \mathbb{I}^*) = \underset{\substack{\{\mathbb{P}\succeq 0\} \\ \boldsymbol{I}\in\mathcal{I}_{\textsf{DSRA}}}}{\operatorname{argmax}} \sum_{n,k,m} \mathbb{I}_{n,k,m} \operatorname{E}\left\{U_{n,k,m}\Big((1-a_m e^{-b_m \mathbb{P}_{n,k,m}\gamma_{n,k}})r_m\Big)\right\} \text{ s.t.} \sum_{n,k,m} \mathbb{I}_{n,k,m}\mathbb{P}_{n,k,m} \leq X^*_{\textsf{tot}}(\mu)$$

satisfies $\mathbb{I}^* = \boldsymbol{I}^*(\mu)$ and, for every (n,k,m), $\mathbb{P}^*_{n,k,m} = \begin{cases} \frac{x^*_{n,k,m}(\mu, \boldsymbol{I}^*(\mu))}{I^*_{n,k,m}(\mu)} & \textit{if } I^*_{n,k,m}(\mu) \neq 0 \\ 0 & \textit{otherwise}. \end{cases}$

Proof: A proof sketch is given in Appendix E. For the full proof, see [15]. ■

From the above lemma, we conclude that if a μ exists such that $\boldsymbol{I}^*(\mu) \in \mathcal{I}_{\textsf{DSRA}}$ and $X^*_{\textsf{tot}}(\mu) = P_{\textsf{con}}$, then the DSRA problem is solved optimally by the CSRA solution set $(\boldsymbol{I}^*(\mu), \boldsymbol{x}^*(\mu, \boldsymbol{I}^*(\mu)))$, i.e., the optimal user-MCS allocation $\boldsymbol{I}^*_{\textsf{DSRA}}$ equals $\boldsymbol{I}^*(\mu)$ and the optimal power allocation, $\boldsymbol{p}^*_{\textsf{DSRA}}$, for any (n,k,m) is given by

$$p^*_{n,k,m,\textsf{DSRA}} = \begin{cases} \frac{x^*_{n,k,m}(\mu, \boldsymbol{I}^*(\mu)))}{I^*_{n,k,m}(\mu)} & \text{if } I^*_{n,k,m}(\mu) \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{43}$$

Recall that the optimal total power achieved for a given value of Lagrange multiplier μ, i.e., $X^*_{\textsf{tot}}(\mu) = \sum_{n,k,m} x^*_{n,k,m}(\mu, \boldsymbol{I}^*(\mu))$, is piece-wise continuous and a discontinuity (or "gap") occurs at μ when multiple allocations achieving the same optimal value of Lagrangian exist. When the sum-power constraint, $P_{\textsf{con}}$, lies in one of such "gaps", the optimal allocation for the CSRA problem is given by a convex combination of two elements from the set $\mathcal{I}_{\textsf{DSRA}}$, and the CSRA solution is not admissible for DSRA. In such cases, we are motivated to choose the sub-optimal DSRA solution $\hat{\boldsymbol{I}}_{\textsf{DSRA}} \in \{\boldsymbol{I}^{\textsf{min}}(\mu), \boldsymbol{I}^{\textsf{max}}(\mu)\}$ that yields highest utility. In Table I, we provide details of the implementation of the proposed sub-optimal

DSRA algorithm that has a significantly lower complexity compared to the brute-force algorithm. We also show using numerical simulations in Section V that its performance is very close to optimal.

The following lemma bounds the asymptotic difference in utilities achieved by the optimal DSRA allocation and the proposed DSRA algorithm.

Lemma 5. *Let μ^* be the optimal μ for the CSRA problem and $\underline{\mu}, \bar{\mu}$ be such that $\mu^* \in [\underline{\mu}, \bar{\mu}]$. Let U^*_{DSRA} and $\hat{U}_{\mathsf{DSRA}}(\underline{\mu}, \bar{\mu})$ be the utilities achieved by the optimal DSRA solution and the proposed DSRA algorithm, respectively. Then,*

$$0 \;\le\; U^*_{\mathsf{DSRA}} - \lim_{\underline{\mu} \to \bar{\mu}} \hat{U}_{\mathsf{DSRA}}(\underline{\mu}, \bar{\mu}) \;\le\; (\mu^* - \mu_{\mathsf{min}})\big(P_{\mathsf{con}} - X^*_{\mathsf{tot}}(\boldsymbol{I}^{\mathsf{min}}(\mu^*), \mu^*)\big) \tag{44}$$

$$\le \begin{cases} 0 & \textit{if } |S_n(\mu^*)| \le 1 \;\forall n \\ (\mu_{\mathsf{max}} - \mu_{\mathsf{min}}) P_{\mathsf{con}} & \textit{otherwise} \end{cases}. \tag{45}$$

Proof: The proof is given in Appendix F. ■

It may be interesting to note that the bound in (45) does not scale with number of users K or subchannels N.

The complexity of the proposed DSRA algorithm is marginally greater than that of the CSRA algorithm, since an additional comparison of two possible user-MCS allocation choices is involved. In units of solving (11) for a given (n, k, m, μ), the DSRA complexity is, at most,

$$N(KM+2)\left\lceil \log_2\left(\tfrac{\mu_{\mathsf{max}} - \mu_{\mathsf{min}}}{\kappa}\right)\right\rceil. \tag{46}$$

Comparing (42) and (46), we find that the complexity of the proposed DSRA algorithm is polynomial in N, K, M, which is considerably less than that of the brute-force algorithm (exponential in N).

V. Numerical Evaluation

In this section, we analyze the performance of an OFDMA downlink system that uses the proposed CSRA and DSRA algorithms for scheduling and resource allocation under different system parameters. Here, we choose the utility function $U_{n,k,m}(\cdot)$ in the primal objective to be the identity function, and thus the objective is to maximize sum-goodput of the system.

For downlink transmission, the BS employs a 2^{m+1}-QAM signaling scheme with MCS index $m \in \{1, \ldots, 15\}$. In this case, we have $r_m = m+1$ bits per symbol. In the symbol error rate model $\epsilon_m(p\gamma) = a_m e^{-b_m p\gamma}$, we choose $a_m = 1$ and $b_m = 1.5/((m+1)^2 - 1)$ because the actual symbol error rate of a 2^{m+1}-QAM system is proportional to $\exp(-1.5p\gamma/((m+1)^2-1))$ in the high-$(p\gamma)$ regime and is equal to 1 when $p\gamma = 0$. We use the standard OFDM model [17] to describe the (instantaneous) frequency-domain observation made by the k^{th} mobile user on the n^{th} subchannel:

$$y_{n,k} = h_{n,k} x_n + \nu_{n,k}, \quad \text{for } n \in \{1, \ldots, N\} \text{ and } k \in \{1, \ldots, K\} \tag{47}$$

In (47), x_n denotes the QAM symbol broadcast by the BS on the n^{th} subchannel, $h_{n,k}$ the gain of the n^{th} subchannel between the k^{th} user and the BS, and $\nu_{n,k}$ a corresponding complex Gaussian noise sample. We assume that $\{\nu_{n,k}\}$ is unit variance and white across (n,k), and we recall that the exogenous subchannel-SNR satisfies $\gamma_{n,k} = |h_{n,k}|^2$. We furthermore assume that the k^{th} user's frequency-domain channel gains $\boldsymbol{h}_k = (h_{1,k}, \ldots, h_{N,k})^T \in \mathbb{C}^N$ are related to the channel impulse response $\boldsymbol{g}_k = (g_{1,k}, \ldots, g_{L,k})^T \in \mathbb{C}^L$ via $\boldsymbol{h}_k = \boldsymbol{F}\boldsymbol{g}_k$, where $\boldsymbol{F} \in \mathbb{C}^{N \times L}$ contains the first $L(< N)$ columns of the N-DFT matrix, and where $\{g_{l,k}\}$ are i.i.d. over (l,k) and drawn from a zero-mean complex Gaussian distribution with variance σ_g^2 chosen so that $\mathrm{E}\{\gamma_{n,k}\} = 1$. Since the total available power for all subchannels at the base-station is P_{con}, the average available SNR per subchannel can be denoted by $\mathsf{SNR} = \frac{P_{\mathsf{con}}}{N} \mathrm{E}\{\gamma_{n,k}\}$.

To model imperfect CSI, we assume that there is a channel-estimation period during which the mobiles take turns to broadcast one pilot OFDM symbol, from which the BS estimates the corresponding subchannel gains. Furthermore, we assume that the channels do not vary between pilot and data periods. To estimate $\boldsymbol{h}_k$, we assume that the BS observes $\tilde{\boldsymbol{y}}_k = \sqrt{p_{\mathsf{pilot}}}\,\boldsymbol{h}_k + \tilde{\boldsymbol{\nu}}_k \in \mathbb{C}^N$. Note that the average SNR per subchannel under pilot transmission is $\mathsf{SNR}_{\mathsf{pilot}} = p_{\mathsf{pilot}}\,\mathrm{E}\{\gamma_{n,k}\}$. The channel $\boldsymbol{h}_k$ and the pilot observations $\tilde{\boldsymbol{y}}_k$ are jointly Gaussian, and furthermore $\boldsymbol{h}_k|\tilde{\boldsymbol{y}}_k$ is Gaussian with mean $\mathrm{E}\{\boldsymbol{h}_k|\tilde{\boldsymbol{y}}_k\} = \boldsymbol{R}_{\boldsymbol{h}_k,\tilde{\boldsymbol{y}}_k}\boldsymbol{R}^{-1}_{\tilde{\boldsymbol{y}}_k,\tilde{\boldsymbol{y}}_k}\tilde{\boldsymbol{y}}_k$ and covariance $\mathrm{Cov}(\boldsymbol{h}_k|\tilde{\boldsymbol{y}}_k) = \boldsymbol{R}_{\boldsymbol{h}_k,\boldsymbol{h}_k} - \boldsymbol{R}_{\boldsymbol{h}_k,\tilde{\boldsymbol{y}}_k}\boldsymbol{R}^{-1}_{\tilde{\boldsymbol{y}}_k,\tilde{\boldsymbol{y}}_k}\boldsymbol{R}_{\tilde{\boldsymbol{y}}_k\boldsymbol{h}_k}$, where $\boldsymbol{R}_{\boldsymbol{z}_1,\boldsymbol{z}_2}$ denotes the cross-correlation of random vectors $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ [18, pp. 155]. Since $\boldsymbol{R}_{\boldsymbol{h}_k,\boldsymbol{h}_k} = \sigma_g^2\boldsymbol{F}\boldsymbol{F}'$, $\boldsymbol{R}_{\boldsymbol{h}_k,\tilde{\boldsymbol{y}}_k} = \sqrt{p_{\mathsf{pilot}}}\sigma_g^2\boldsymbol{F}\boldsymbol{F}'$, and $\boldsymbol{R}_{\tilde{\boldsymbol{y}}_k,\tilde{\boldsymbol{y}}_k} = p_{\mathsf{pilot}}\sigma_g^2\boldsymbol{F}\boldsymbol{F}' + \mathbf{I}$ (where $\mathbf{I}$ denotes the identity matrix), it is straightforward to show that the elements on the diagonal of $\mathrm{Cov}(\boldsymbol{h}_k|\tilde{\boldsymbol{y}}_k)$ are equal. Furthermore, $\mathrm{E}\{\boldsymbol{h}_k|\tilde{\boldsymbol{y}}_k\}$ can be recognized as the pilot-aided MMSE estimate of $\boldsymbol{h}_k$. In summary, conditioned on the pilot observations, $h_{n,k}$ is Gaussian with mean $\hat{h}_{n,k}$ given by the n^{th} element of $\mathrm{E}\{\boldsymbol{h}_k|\tilde{\boldsymbol{y}}_k\}$, and with variance σ_e^2 given by the first diagonal element of $\mathrm{Cov}(\boldsymbol{h}_k|\tilde{\boldsymbol{y}}_k)$. Thus, conditioned on the pilot observations, $\gamma_{n,k}$ has a non-central chi-squared distribution with two degrees of freedom.

We will refer to the proposed CSRA and DSRA algorithms implemented under imperfect CSI as "CSRA-ICSI" and "DSRA-ICSI," respectively. Their performances will be compared to that of "CSRA-PCSI," i.e., CSRA implemented under perfect CSI, which serves as a performance upper bound, and *fixed-power random-user scheduling* (FP-RUS), which serves as a performance lower bound. FP-RUS schedules, on each subchannel, one user selected uniformly from $\{1, \ldots, K\}$, to which it allocates power P_{con}/N and the fixed MCS m that maximizes expected sum-goodput. Unless specified, the number of OFDM subchannels was $N = 64$, the number of users was $K = 16$, the impulse response length was $L = 2$, the average SNR per subchannel was $\mathsf{SNR} = 10$ dB, the pilot SNR was $\mathsf{SNR}_{\mathsf{pilot}} = -10$ dB, and the DSRA/CSRA tuning parameter was $\kappa = 0.3/P_{\mathsf{con}}$ (recall Table I). In all plots, goodput values were empirically averaged over 1000 realizations.

Figure 3 plots the subchannel-averaged goodput achieved by the above-described scheduling and resource-allocation schemes for different grades of CSI. In this curve, $\mathsf{SNR}_{\mathsf{pilot}}$ is varied so as to obtain estimates of subchannel SNR with different grades of accuracy. All other parameters remain unchanged. The plot shows that as $\mathsf{SNR}_{\mathsf{pilot}}$ is increased, the performance of the proposed schemes under the availability of imperfect CSI increases from that of FP-RUS to that achieved by the CSRA-PCSI scheme. This is expected because with increasing $\mathsf{SNR}_{\mathsf{pilot}}$, the BS uses more accurate channel-state information for scheduling and resource allocation, and thus achieves higher goodput. The plot also shows that, even though the proposed CSRA algorithm optimally solves the CSRA problem and the proposed DSRA algorithm sub-optimally solves the DSRA problem, their performances almost coincide. In particular, although the goodput achieved by CSRA-ICSI scheme exceeds that of DSRA-ICSI scheme in up-to 49% of the realizations, the maximum difference in the subchannel-averaged goodput is merely 4×10^{-3} bits/channel-use. Since the DSRA-ICSI schemes cannot achieve a sum-goodput higher than that achieved by the CSRA-ICSI scheme, it can be deduced that the proposed DSRA algorithm is near-optimal in this case.

Figure 4 plots the sum-goodput over all subchannels as a function of N, ranging between 16 and 128. In this numerical evaluation, P_{con} is fixed such that $\mathsf{SNR} = 10$ dB for a 64-subchannel OFDM system. The plot shows that the sum-goodput increases with N. This is expected since more subchannels allows more scheduling flexibility and availability of even stronger subchannels, which can be effectively exploited by BS to achieve higher goodput. It can be seen that the performances of CSRA-ICSI and DSRA-ICSI schemes are almost identical, regardless of N. In particular, although the goodput achieved by CSRA-ICSI scheme exceeds that of DSRA-ICSI scheme in up-to 27% of the realizations, the maximum difference in sum-goodput is merely 3×10^{-3} bits/channel-use.

Figure 5 plots the subchannel-averaged goodput as a function of K (number of active users) varying between 1 and 32. It shows that, as K increases, the goodput per subchannel achieved by the proposed schemes under perfect and imperfect CSI increase, whereas that achieved by the FP-RUS scheme remains constant. This is because, in the former schemes, the availability of more users can be exploited to schedule users with stronger subchannels, whereas in the FP-RUS scheme, this advantage is lost due to the lack of information about the users' instantaneous channel conditions. Similar to the observations in the previous plots, the performance of the proposed algorithms under imperfect CSI remain almost identical. In particular, although the goodput achieved by CSRA-ICSI scheme exceeds that of DSRA-ICSI scheme in up-to 29% of the realizations, the maximum difference in the subchannel-averaged goodput is merely 7×10^{-4} bits/channel-use.

In Figure 6, the top plot shows the subchannel-averaged goodput and the bottom plot shows the subchannel and realization-averaged value of the bound (in (44)) on the optimality gap for the DSRA problem as a function of SNR. In the top plot, it can be seen that as SNR increases, the difference between CSRA-PCSI and CSRA-ICSI (or, DSRA-ICSI) schemes increases. However, the difference grows slower than the difference between CSRA-PCSI and FP-RUS schemes. Interestingly, even for high values of SNR, the performance of CSRA-ICSI and DSRA-ICSI remain almost identical. In particular, although the goodput achieved by CSRA-ICSI scheme exceeds that of DSRA-ICSI scheme in up-to 28% of the realizations, the maximum difference in the subchannel-averaged goodput is merely 4×10^{-5} bits/channel-use. The bottom plot, which illustrates the average value of $(\mu^* - \mu_{\mathsf{min}})\big(P_{\mathsf{con}} - X^*_{\mathsf{tot}}(\boldsymbol{I}^{\mathsf{min}}, \mu^*)\big)$ over all realizations and subchannels w.r.t. SNR shows that the loss in sum-goodput over all subchannels due to sub-optimality of proposed DSRA solution under imperfect CSI is bounded by 7×10^{-3} bits/channel-use even when the subchannel-averaged goodput of DSRA-ICSI is of the order of tens of bits/channel-use. This suggests that the proposed bound in Lemma 5 on the optimality gap of proposed DSRA solution is quite tight at high values of SNR.

From the above plots, we conclude that even with imperfect CSI, the proposed CSRA and DSRA algorithms for scheduling and resource allocation can greatly enhance the system's goodput performance compared to non CSI-based schemes. It was seen that the algorithms' performances are very close to each other, confirming that the proposed DSRA algorithm is near optimal in the tested scenarios.

VI. Conclusion

In this paper, we considered the problem of joint scheduling and resource allocation (SRA) in downlink OFDMA systems under imperfect channel-state information. We considered two scenarios: 1) when subchannel sharing is allowed, and 2) when it is not. Both cases were framed as optimization problems that maximize a utility function subject to a sum-power constraint. Although the optimization problem in the first scenario (the so-called "continuous" SRA problem) was found to be non-convex, we showed that it can be converted to a convex optimization problem and solved using a dual optimization approach with zero duality gap. An algorithmic implementation of the CSRA solution was also provided. The optimization problem faced in the second scenario (the so-called "discrete" SRA problem) was found to be a mixed-integer programming problem. To attack it, we linked the DSRA problem to the CSRA problem, and showed that, in some cases, the DSRA solution coincides with the CSRA solution. For the case that the solutions do not coincide, we proposed a practical DSRA algorithm and bounded its performance. Numerical results were then presented under a variety of settings. The performance of the proposed CSRA and DSRA algorithms schemes under imperfect CSI were compared to those under perfect CSI and no CSI (i.e., fixed-power random scheduling). In all cases, it was found that the proposed imperfect-CSI-based algorithms offer a significant advantage over schemes that do not use any CSI. Moreover, the performance of DSRA was nearly equal to CSRA. Therefore, we conclude that, in OFDMA-based downlink communication systems under imperfect CSI, it is unlikely that the performance gains that result from time-sharing of multiple user-MCS combinations within a single subchannel would justify the additional system-level complexity that would be required to implement such time-sharing.

Appendix A
Sketch of proof for convexity of CSRA problem

First, we show that $I_{n,k,m}F_{n,k,m}(I_{n,k,m},x_{n,k,m})$ is convex in $I_{n,k,m}$ and $x_{n,k,m}$. For this, consider the case when $I_{n,k,m}>0$. In this case, the Hessian of $I_{n,k,m}F_{n,k,m}(I_{n,k,m},x_{n,k,m})$ w.r.t. $I_{n,k,m}$ and $x_{n,k,m}$ can be calculated and found to be positive semi-definite. Next, consider the case when $I_{n,k,m}=0$. To prove convexity in this case, we apply the definition of convexity, i.e., for any two points $(I^{(1)}_{n,k,m},x^{(1)}_{n,k,m})$ and $(I^{(2)}_{n,k,m},x^{(2)}_{n,k,m})$ in the domain of CSRA problem and for any $\lambda\in[0,1]$, convexity means

$$\begin{aligned}&\lambda I^{(1)}_{n,k,m}F_{n,k,m}\Big(I^{(1)}_{n,k,m},x^{(1)}_{n,k,m}\Big)+(1-\lambda)I^{(2)}_{n,k,m}F_{n,k,m}\Big(I^{(2)}_{n,k,m},x^{(2)}_{n,k,m}\Big)\\ &\geq\Big[\lambda I^{(1)}_{n,k,m}+(1-\lambda)I^{(2)}_{n,k,m}\Big]F_{n,k,m}\Big(\lambda I^{(1)}_{n,k,m}+(1-\lambda)I^{(2)}_{n,k,m},\lambda x^{(1)}_{n,k,m}+(1-\lambda)x^{(2)}_{n,k,m}\Big).\end{aligned}\tag{48}$$

When one or both of $\{I^{(1)}_{n,k,m},I^{(2)}_{n,k,m}\}$ are zero, it is straightforward to show that the above equation holds. Therefore, $I_{n,k,m}F_{n,k,m}(I_{n,k,m},x_{n,k,m})$ is convex in $I_{n,k,m}$ and $x_{n,k,m}$. Consequently, it is a convex function of $\boldsymbol{I}$ and $\boldsymbol{x}$. Since the primal objective function of the CSRA problem, i.e., $\sum_{n,k,m}I_{n,k,m}F_{n,k,m}(I_{n,k,m},x_{n,k,m})$, is a sum of functions that are convex in $\boldsymbol{I}$ and $\boldsymbol{x}$, it is also convex in $\boldsymbol{I}$ and $\boldsymbol{x}$.

Appendix B
Sketch of proof of Lemma 1

Suppose that $\mu_1 < \mu_2$, where $\mu_1, \mu_2 \in [\mu_{\mathsf{min}}, \mu_{\mathsf{max}}]$. With μ fixed, the minimization problem becomes

$$\begin{aligned} & L(\mu, \boldsymbol{I}^*(\mu), \boldsymbol{x}^*(\mu, \boldsymbol{I}^*(\mu))) \\ &= \min_{\substack{\{\boldsymbol{x}\succeq 0\} \\ \boldsymbol{I}\in\mathcal{I}_{\mathsf{CSRA}}}} L(\mu, \boldsymbol{I}, \boldsymbol{x}) = \min_{\substack{\{\boldsymbol{x}\succeq 0\} \\ \boldsymbol{I}\in\mathcal{I}_{\mathsf{CSRA}}}} \Big(\sum_{n,k,m} x_{n,k,m} - P_{\mathsf{con}}\Big)\mu + \sum_{n,k,m} I_{n,k,m} F_{n,k,m}(I_{n,k,m}, x_{n,k,m}) \end{aligned} \tag{49}$$

recalling (6). At $\mu = \mu_1$, $\boldsymbol{I}^*(\mu_2)$ and $\boldsymbol{x}^*(\mu_2, \boldsymbol{I}^*(\mu_2))$ are suboptimal values of $\boldsymbol{I}^*(\mu)$ and $\boldsymbol{x}^*(\mu, \boldsymbol{I}^*(\mu))$, and at $\mu = \mu_2$, $\boldsymbol{I}^*(\mu_1)$ and $\boldsymbol{x}^*(\mu_1, \boldsymbol{I}^*(\mu_1))$ are suboptimal values of $\boldsymbol{I}^*(\mu)$ and $\boldsymbol{x}^*(\mu, \boldsymbol{I}^*(\mu))$. Therefore,

$$L(\mu_1, \boldsymbol{I}^*(\mu_1), \boldsymbol{x}^*(\mu_1, \boldsymbol{I}^*(\mu_1))) \le L(\mu_1, \boldsymbol{I}^*(\mu_2), \boldsymbol{x}^*(\mu_2, \boldsymbol{I}^*(\mu_2))), \text{ and} \tag{50}$$

$$L(\mu_2, \boldsymbol{I}^*(\mu_2), \boldsymbol{x}^*(\mu_2, \boldsymbol{I}^*(\mu_2))) \le L(\mu_2, \boldsymbol{I}^*(\mu_1), \boldsymbol{x}^*(\mu_1, \boldsymbol{I}^*(\mu_1))). \tag{51}$$

Adding (50) and (51), and evaluating the result, we get

$$(\mu_1 - \mu_2)\Big(\sum_{n,k,m} x^*_{n,k,m}(\mu_1, \boldsymbol{I}^*(\mu_1)) - x^*_{n,k,m}(\mu_2, \boldsymbol{I}^*(\mu_2))\Big) \le 0. \tag{52}$$

Since $\mu_1 < \mu_2$, we have $X^*_{\mathsf{tot}}(\mu_1) \ge X^*_{\mathsf{tot}}(\mu_2)$. Therefore, $X^*_{\mathsf{tot}}(\mu)$ is monotonically decreasing in μ.

Appendix C
Proof of Lemma 2

Proof: To compare the utilities obtained by the proposed CSRA algorithm and the optimal CSRA solution, we compare the Lagrangian values achieved by the two solutions. Recall $\mu^* \in [\underline{\mu}, \bar{\mu}] \subset [\mu_{\mathsf{min}}, \mu_{\mathsf{max}}]$. Therefore,

$$\begin{aligned} & L(\mu^*, \boldsymbol{I}^*(\mu^*), \boldsymbol{x}^*(\mu^*, \boldsymbol{I}^*(\mu^*))) - L(\underline{\mu}, \boldsymbol{I}^*(\underline{\mu}), \boldsymbol{x}^*(\underline{\mu}, \boldsymbol{I}^*(\underline{\mu}))) \ge 0, \text{ and} \\ & L(\mu^*, \boldsymbol{I}^*(\mu^*), \boldsymbol{x}^*(\mu^*, \boldsymbol{I}^*(\mu^*))) - L(\bar{\mu}, \boldsymbol{I}^*(\bar{\mu}), \boldsymbol{x}^*(\bar{\mu}, \boldsymbol{I}^*(\bar{\mu}))) \ge 0. \end{aligned} \tag{53}$$

The solution of the proposed CSRA algorithm allocates resources such that the sum-power constraint is satisfied while achieving a Lagrangian value of

$$\hat{L}_{\mathsf{CSRA}} \triangleq \lambda L(\bar{\mu}, \boldsymbol{I}^*(\bar{\mu}), \boldsymbol{x}^*(\bar{\mu}, \boldsymbol{I}^*(\bar{\mu}))) + (1-\lambda) L(\underline{\mu}, \boldsymbol{I}^*(\underline{\mu}), \boldsymbol{x}^*(\underline{\mu}, \boldsymbol{I}^*(\underline{\mu}))).$$

For any μ, notice that $L(\mu, \boldsymbol{I}^*(\mu), \boldsymbol{x}^*(\mu, \boldsymbol{I}^*(\mu))) = -U^*(\mu) + (X^*_{\mathsf{tot}}(\mu) - P_{\mathsf{con}})\mu$, where $U^*(\mu)$ is the total utility achieved due to optimal power allocation at that μ. Since the resource allocation obtained by the proposed CSRA algorithm and the optimal CSRA solution satisfy the sum-power constraint with equality, we have

$$U^*_{\mathsf{CSRA}} = -L(\mu^*, \boldsymbol{I}^*(\mu^*), \boldsymbol{x}^*(\mu^*, \boldsymbol{I}^*(\mu^*))), \text{ and} \tag{54}$$

$$\begin{aligned}\hat{L}_{\mathsf{CSRA}} &= -\hat{U}_{\mathsf{CSRA}}(\underline{\mu}, \bar{\mu}) + (X^*_{\mathsf{tot}}(\bar{\mu}) - P_{\mathsf{con}})\lambda\bar{\mu} + (X^*_{\mathsf{tot}}(\underline{\mu}) - P_{\mathsf{con}})(1-\lambda)\underline{\mu} \\ &= -\hat{U}_{\mathsf{CSRA}}(\underline{\mu}, \bar{\mu}) + (X^*_{\mathsf{tot}}(\bar{\mu}) - P_{\mathsf{con}})(\bar{\mu} - \underline{\mu})\lambda. \end{aligned} \tag{55}$$

Equation (55) holds since $\lambda X^*_{\mathsf{tot}}(\bar{\mu}) + (1-\lambda)X^*_{\mathsf{tot}}(\underline{\mu}) = P_{\mathsf{con}}$. From (54) and (55), we get

$$0 \le U^*_{\mathsf{CSRA}} - \hat{U}_{\mathsf{CSRA}}(\underline{\mu}, \bar{\mu}) = -L(\mu^*, \boldsymbol{I}^*(\mu^*), \boldsymbol{x}^*(\mu^*, \boldsymbol{I}^*(\mu^*))) + \hat{L}_{\mathsf{CSRA}} - (X^*_{\mathsf{tot}}(\bar{\mu}) - P_{\mathsf{con}})(\bar{\mu} - \underline{\mu})\lambda.$$

From the above equation and (53), we have

$$0 \le U^*_{\mathsf{CSRA}} - \hat{U}_{\mathsf{CSRA}}(\underline{\mu}, \bar{\mu}) \le (P_{\mathsf{con}} - X^*_{\mathsf{tot}}(\bar{\mu}))(\bar{\mu} - \underline{\mu})\lambda \le (\bar{\mu} - \underline{\mu})P_{\mathsf{con}}. \tag{56}$$

■

APPENDIX D
SKETCH OF PROOF OF LEMMA 3

Let $\tilde{\mu} \in [\mu_{\mathsf{min}}, \mu_{\mathsf{max}}]$ be any value of the Lagrangian dual variable for the CSRA problem. Then, at $\tilde{\mu}$, one of the following three cases holds.

1) $|S_n(\tilde{\mu})| \le 1 \ \forall n$.
2) For some n, $|S_n(\tilde{\mu})| > 1$ but no two combinations in $S_n(\tilde{\mu})$ have the same allocated power.
3) For some n, $|S_n(\tilde{\mu})| > 1$ and at least two combinations in $S_n(\tilde{\mu})$ have the same allocated power.

We make use of two properties in the proof. Firstly, $V_{n,k,m}(\mu, p^*_{n,k,m}(\mu))$ is a continuous function of μ. Therefore, by definition of continuous functions, if $V_{n,k,m}(\tilde{\mu}, p^*_{n,k,m}(\tilde{\mu})) > 0$, then we can fix a $\delta_{n,k,m}\,(> 0)$ such that $V_{n,k,m}(\mu, p^*_{n,k,m}(\mu)) > 0$ whenever $|\mu - \tilde{\mu}| < \delta_{n,k,m}$. Secondly, for all values of μ, we know $\frac{\partial V_{n,k,m}(\mu, p^*_{n,k,m}(\mu))}{\partial \mu} = p^*_{n,k,m}(\mu)$. We now apply these properties to each of the three cases to determine $S_n(\mu)\ \forall n$. When μ is sufficiently close to $\tilde{\mu}$, we show that, in cases $1)$ and $2)$, one can fix a δ such that $|S_n(\mu)| \le 1\ \forall n$ whenever $0 < |\mu - \tilde{\mu}| < \delta$. When this happens, it can be shown that, for all $\mu_1, \mu_2 \in (\tilde{\mu} - \delta, \tilde{\mu})$, one has $\boldsymbol{I}^*(\mu_1), \boldsymbol{I}^*(\mu_2) \in \{0,1\}^{N \times K \times M}$ and $S_n(\mu_1) = S_n(\mu_2)\ \forall n$. The same property holds when $\mu_1, \mu_2 \in (\tilde{\mu}, \tilde{\mu} + \delta)$. In case $3)$, we establish that all combinations with the same allocated power contribute equally to the total power allocated, as well as the total optimal value of Lagrangian. Therefore, all but any one combination can be ignored safely, implying that there exists a fixed δ such that $\boldsymbol{I}^*(\mu) \in \{0,1\}^{N \times K \times M}$ whenever $|\mu - \tilde{\mu}| < \delta$. After ignoring the redundant combinations, it follows from cases 1) and 2) that, for all $\mu_1, \mu_2 \in (\tilde{\mu}-\delta, \tilde{\mu})$ and $\mu_1, \mu_2 \in (\tilde{\mu}, \tilde{\mu}+\delta)$, there exists $\boldsymbol{I}^*(\mu_1), \boldsymbol{I}^*(\mu_2) \in \{0,1\}^{N\times K\times M}$ such that $\boldsymbol{I}^*(\mu_1) = \boldsymbol{I}^*(\mu_2)$.

APPENDIX E
SKETCH OF PROOF OF LEMMA 4

From (6) and the stated assumptions, we have $\boldsymbol{I}^*(\mu) \in \mathcal{I}_{\mathsf{DSRA}} \subset \mathcal{I}_{\mathsf{CSRA}}$ and

$$\Big(\boldsymbol{I}^*(\mu), \boldsymbol{x}^*(\mu, \boldsymbol{I}^*(\mu))\Big) = \underset{\substack{\boldsymbol{x}\succeq 0 \\ \boldsymbol{I}\in\mathcal{I}_{\mathsf{DSRA}}}}{\operatorname{argmin}} \sum_{n,k,m} I_{n,k,m}\, F_{n,k,m}(I_{n,k,m}, x_{n,k,m}) + \Big(\sum_{n,k,m} x_{n,k,m} - P_{\mathsf{con}}\Big)\mu, \quad (57)$$

where $F_{n,k,m}(\cdot,\cdot)$ was defined in (4). Then, applying the concept of generalized Lagrange multiplier method from [19, Theorem 1], we conclude that

$$(\mathbb{I}^*, \mathbb{X}^*) = \underset{\substack{\{\mathbb{X}\succeq 0\} \\ \mathbb{I}\in\mathcal{I}_{\mathsf{DSRA}}}}{\operatorname{argmin}} \sum_{n,k,m} \mathbb{I}_{n,k,m} F_{n,k,m}(\mathbb{I}_{n,k,m}, \mathbb{X}_{n,k,m}) \ \text{ s.t.} \sum_{n,k,m} \mathbb{X}_{n,k,m} \le \sum_{n,k,m} x^*_{n,k,m}(\mu, \boldsymbol{I}^*(\mu)). \quad (58)$$

Substituting $\mathbb{X}_{n,k,m} = \mathbb{I}_{n,k,m}\mathbb{P}_{n,k,m}$ back into the above equation, we obtain the desired result.

APPENDIX F
PROOF OF LEMMA 5

Proof: Let us denote $\lim_{\underline{\mu}\to\bar{\mu}} \hat{U}_{\mathsf{DSRA}}(\underline{\mu}, \bar{\mu})$ by $\hat{U}_{\mathsf{DSRA}}$. The left inequality in the lemma is straightforward since $U^*_{\mathsf{DSRA}} \ge \hat{U}_{\mathsf{DSRA}}(\underline{\mu}, \bar{\mu})\ \forall \underline{\mu}, \bar{\mu}$. Now, if $|S_n(\mu^*)| \le 1\ \forall n$, then we have $U^*_{\mathsf{DSRA}} = U^*_{\mathsf{CSRA}} = \hat{U}_{\mathsf{DSRA}}$, ensuring that the solution obtained via the proposed DSRA algorithm is optimal in the limit $\underline{\mu}, \bar{\mu} \to \mu^*$. However, when $|S_n(\mu^*)| > 1$ for some n, P_{con} lies in one of the "gaps" as mentioned in Fig. 2 and $\boldsymbol{I}^*_{\mathsf{CSRA}} \notin \mathcal{I}_{\mathsf{DSRA}}$. In this case, we have $0 \le U^*_{\mathsf{DSRA}} - \hat{U}_{\mathsf{DSRA}} \le U^*_{\mathsf{CSRA}} - \hat{U}_{\mathsf{DSRA}}$. Let $U^*(\boldsymbol{I})$ be the optimal utility achieved for user-MCS allocation matrix $\boldsymbol{I} \in \mathcal{I}_{\mathsf{DSRA}}$. We recall from Section III-C that, at μ^*, the allocation $\boldsymbol{I}^{\mathsf{min}}(\mu^*)$ is one of possibly many values of $\boldsymbol{I}$ minimizing $L(\mu^*, \boldsymbol{I}, \boldsymbol{x}^*(\mu^*, \boldsymbol{I}))$. Thus, $U^*_{\mathsf{CSRA}} = -L(\mu^*, \boldsymbol{I}^{\mathsf{min}}(\mu^*), \boldsymbol{x}^*(\mu^*, \boldsymbol{I}^{\mathsf{min}}(\mu^*)))$. For brevity in this proof, let us denote $\boldsymbol{I}^{\mathsf{min}}(\mu^*)$ and $\boldsymbol{I}^{\mathsf{max}}(\mu^*)$ $(\in \mathcal{I}_{\mathsf{DSRA}})$, defined in (30), by $\boldsymbol{I}^{\mathsf{min}}$ and $\boldsymbol{I}^{\mathsf{max}}$, respectively. Therefore, $\hat{U}_{\mathsf{DSRA}} = \max\{U^*(\boldsymbol{I}^{\mathsf{min}}), U^*(\boldsymbol{I}^{\mathsf{max}})\}$. This gives us

$$\begin{aligned} U^*_{\mathsf{CSRA}} - \hat{U}_{\mathsf{DSRA}} &\le U^*_{\mathsf{CSRA}} - U^*(\boldsymbol{I}^{\mathsf{min}}) \\ &= -L(\mu^*, \boldsymbol{I}^{\mathsf{min}}, \boldsymbol{x}^*(\mu^*, \boldsymbol{I}^{\mathsf{min}})) + L_{\boldsymbol{I}^{\mathsf{min}}}(\mu^*_{\boldsymbol{I}^{\mathsf{min}}}, \boldsymbol{x}^*(\mu^*_{\boldsymbol{I}^{\mathsf{min}}}))) \\ &= -L(\mu^*, \boldsymbol{I}^{\mathsf{min}}, \boldsymbol{x}^*(\mu^*, \boldsymbol{I}^{\mathsf{min}})) + L(\mu^*_{\boldsymbol{I}^{\mathsf{min}}}, \boldsymbol{I}^{\mathsf{min}}, \boldsymbol{x}^*(\mu^*_{\boldsymbol{I}^{\mathsf{min}}}, \boldsymbol{I}^{\mathsf{min}})), \end{aligned} \quad (59)$$

where, for (59), we use the equivalence between $L(\mu, \boldsymbol{I}, \boldsymbol{x})$ in (5) and $L_{\boldsymbol{I}}(\mu, \boldsymbol{x})$ in (36). Note that $\mu^*_{\boldsymbol{I}^{\mathsf{min}}} \le \mu^*$, since the total optimally allocated power for $\boldsymbol{I}^{\mathsf{min}}$ at $\mu = \mu^*$ is less than or equal to P_{con} and the total optimally allocated power for any given $\boldsymbol{I}$ is a decreasing function of μ. Plugging $L(\cdot,\cdot,\cdot)$ from (5) into (59), we get

$$U^*_{\mathsf{CSRA}} - \hat{U}_{\mathsf{DSRA}} \leq -\Big[-\mu^* P_{\mathsf{con}} + \sum_{n,k,m} I^{\mathsf{min}}_{n,k,m}\Big(-\bar{U}_{n,k,m}(p^*_{n,k,m}(\mu^*)) + \mu^* p^*_{n,k,m}(\mu^*)\Big)\Big]$$
$$+\Big[-\mu^*_{\boldsymbol{I}^{\mathsf{min}}} P_{\mathsf{con}} + \sum_{n,k,m} I^{\mathsf{min}}_{n,k,m}\Big(-\bar{U}_{n,k,m}(p^*_{n,k,m}(\mu^*_{\boldsymbol{I}^{\mathsf{min}}})) + \mu^*(\boldsymbol{I}^{\mathsf{min}}) p^*_{n,k,m}(\mu^*_{\boldsymbol{I}^{\mathsf{min}}})\Big)\Big], \tag{60}$$

where, $\bar{U}_{n,k,m}(x) = \mathrm{E}\left\{U_{n,k,m}\big((1-a_m e^{-b_m x \gamma_{n,k}})r_m\big)\right\}$. Using the definition of $X^*_{\mathsf{tot}}(\boldsymbol{I},\mu)$ in (41), we have $X^*_{\mathsf{tot}}(\boldsymbol{I}^{\mathsf{min}},\mu^*) \leq P_{\mathsf{con}}$ and $X^*_{\mathsf{tot}}(\boldsymbol{I}^{\mathsf{min}},\mu^*_{\boldsymbol{I}^{\mathsf{min}}}) = P_{\mathsf{con}}$. Therefore, (60) can be re-written as

$$U^*_{\mathsf{CSRA}} - \hat{U}_{\mathsf{DSRA}}$$
$$\leq \mu^*\big(P_{\mathsf{con}} - X^*_{\mathsf{tot}}(\boldsymbol{I}^{\mathsf{min}},\mu^*)\big) - \sum_{n,k,m} I^{\mathsf{min}}_{n,k,m}\Big[\bar{U}_{n,k,m}(p^*_{n,k,m}(\mu^*_{\boldsymbol{I}^{\mathsf{min}}})) - \bar{U}_{n,k,m}(p^*_{n,k,m}(\mu^*))\Big]. \tag{61}$$

Calculating the first two derivatives of $\bar{U}_{n,k,m}(x)$ with respect to x, we find that it is a strictly-increasing concave function of x. Therefore, if $x_1 \leq x_2$, one can write that $\bar{U}_{n,k,m}(x_2) - \bar{U}_{n,k,m}(x_1) \geq (x_2 - x_1)\bar{U}'_{n,k,m}(x_2)$. Plugging $x_1 = p^*_{n,k,m}(\mu^*)$ and $x_2 = p^*_{n,k,m}(\mu^*_{\boldsymbol{I}^{\mathsf{min}}})$ into this inequality, we get

$$\bar{U}_{n,k,m}(p^*_{n,k,m}(\mu^*_{\boldsymbol{I}^{\mathsf{min}}})) - \bar{U}_{n,k,m}(p^*_{n,k,m}(\mu^*)) \geq \Big(p^*_{n,k,m}(\mu^*_{\boldsymbol{I}^{\mathsf{min}}}) - p^*_{n,k,m}(\mu^*)\Big)\frac{\partial \bar{U}_{n,k,m}(x)}{\partial x}\bigg|_{x=p^*_{n,k,m}(\mu^*_{\boldsymbol{I}^{\mathsf{min}}})} \tag{62}$$

From (61) and (62), we then get

$$U^*_{\mathsf{CSRA}} - \hat{U}_{\mathsf{DSRA}}$$
$$\leq \mu^*\big(P_{\mathsf{con}} - X^*_{\mathsf{tot}}(\boldsymbol{I}^{\mathsf{min}},\mu^*)\big) - \sum_{n,k,m} I^{\mathsf{min}}_{n,k,m}\bar{U}'_{n,k,m}\big(p^*_{n,k,m}(\mu^*_{\boldsymbol{I}^{\mathsf{min}}})\big)\Big(p^*_{n,k,m}(\mu^*_{\boldsymbol{I}^{\mathsf{min}}}) - p^*_{n,k,m}(\mu^*)\Big). \tag{63}$$

Evaluating $\bar{U}'_{n,k,m}\big(p^*_{n,k,m}(\mu^*_{\boldsymbol{I}^{\mathsf{min}}})\big)$, we find

$$\frac{\partial \bar{U}_{n,k,m}(x)}{\partial x}\bigg|_{x=p^*_{n,k,m}(\mu^*_{\boldsymbol{I}^{\mathsf{min}}})} = a_m b_m r_m \,\mathrm{E}\left\{U'_{n,k,m}\big((1-a_m e^{-b_m p^*_{n,k,m}(\mu^*_{\boldsymbol{I}^{\mathsf{min}}})\gamma_{n,k}})r_m\big)\gamma_{n,k} e^{-b_m p^*_{n,k,m}(\mu^*_{\boldsymbol{I}^{\mathsf{min}}})\gamma_{n,k}}\right\}$$
$$\geq \mu_{\mathsf{min}}. \tag{64}$$

From (63) and (64), we finally obtain

$$U^*_{\mathsf{CSRA}} - \hat{U}_{\mathsf{DSRA}} \leq (\mu^* - \mu_{\mathsf{min}})\big(P_{\mathsf{con}} - X^*_{\mathsf{tot}}(\boldsymbol{I}^{\mathsf{min}},\mu^*)\big) \leq (\mu_{\mathsf{max}} - \mu_{\mathsf{min}})P_{\mathsf{con}}. \tag{65}$$

■

REFERENCES

[1] C. Y. Wong, R. S. Cheng, K. B. Letaief, and R. D. Murch, “Multiuser OFDM with adaptive subcarrier, bit and power allocation,” *IEEE J. Select. Areas In Commun.*, vol. 17, pp. 1747–1758, Oct. 1999.

[2] J. Jang and K. B. Lee, “Transmit power adaptation for multiuser OFDM systems,” *IEEE J. Select. Areas In Commun.*, vol. 21, pp. 171–178, Feb. 2003.

[3] T. J. Willink and P. H. Wittke, “Optimization and performance evaluation of multicarrier transmission,” *IEEE Trans. Inform. Theory*, vol. 43, pp. 426–440, Mar. 1997.

[4] L. M. C. Hoo, B. Halder, J. Tellado, and J. M. Cioffi, “Multiuser transmit optimization for multicarrier broadcast channels: Asymptotic FDMA capacity region and algorithms,” *IEEE Trans. Commun.*, vol. 52, pp. 922–930, Jun. 2004.

[5] I. Wong and B. Evans, “Optimal downlink OFDMA resource allocation with linear complexity to maximize ergodic rates,” *IEEE Trans. Wireless Commun.*, vol. 7, pp. 962 –971, Mar. 2008.

[6] K. Seong, M. Mohseni, and J. M. Cioffi, “Optimal resource allocation for OFDMA downlink systems,” in *Proc. IEEE Int. Symposium Inform. Theory*, vol. 3, pp. 1394–1398, Jul. 2006.

[7] M. R. Souryal and R. L. Pickholtz, “Adaptive modulation with imperfect channel information in OFDM,” in *Proc. IEEE Int. Conf. Commun.*, vol. 6, pp. 1861–1865, Jun. 2001.

[8] S. Ye, R. S. Blum, and L. J. Cimini, “Adaptive modulation for variable rate OFDM systems with imperfect channel information,” in *Proc. IEEE Veh. Tech. Conf.*, vol. 2, pp. 767–771, 2002.

[9] Y. Yao and G. B. Giannakis, “Rate-maximizing power allocation in OFDM based on partial channel knowledge,” *IEEE Trans. Wireless Commun.*, vol. 4, pp. 1073–1083, May. 2005.

[10] I. C. Wong and B. L. Evans, “Optimal resource allocation in the OFDMA downlink with imperfect channel knowledge,” *IEEE Trans. Commun.*, vol. 57, pp. 232–241, Jan. 2009.

[11] D. Hui and V. Lau, “Delay-sensitive cross-layer designs for OFDMA systems with outdated CSIT,” in *Proc. IEEE Wireless Commun. and Networking Conf.*, pp. 264 –269, Mar. 2007.

[12] A. Ahmad and M. Assaad, “Margin adaptive resource allocation in downlink OFDMA system with outdated channel state information,” *Proc. IEEE Int. Symposium Personal Indoor Mobile Radio Commun.*, pp. 1868–1872, 2009.

[13] Y. J. Zhang and S. C. Liew, “Proportional fairness in multi-channel multi-rate wireless networks–Part II: The case of time-varying channels with application to OFDM systems,” *IEEE Trans. Wireless Commun.*, vol. 7, pp. 3457–3467, Sept. 2008.

[14] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[15] R. Aggarwal, *On Scheduling and Resource Allocation using Imperfect Channel State Information*. PhD thesis, The Ohio State University, Columbus, OH, under preparation.

[16] J. Mitchell and E. K. Lee, “Branch-and-bound methods for integer programming,” in *Encyclopedia of Optimization* (A. Floudas and P. M. Pardalos, eds.), vol. 2, ch. 4, pp. 509–519, The Netherlands: Kluwer Academic Publishers, 2001.

[17] J.-J. van de Beek, O. Edfors, M. Sandell, S. Wilson, and P. Borjesson, “On channel estimation in OFDM systems,” in *Proc. IEEE Veh. Tech. Conf.*, vol. 2, pp. 815–819, 1995.

[18] H. V. Poor, *An Introduction to Signal Detection and Estimation*. New York: Springer-Verlag, 2nd ed., 1994.

[19] H. Everett, “Generalized Lagrange multiplier method for solving problems of optimum allocation of resources,” *Operations Research*, vol. 11, pp. 399–417, May-June 1963.

TABLE I
ALGORITHMIC IMPLEMENTATIONS OF THE PROPOSED ALGORITHMS

Proposed CSRA algorithm

1) Set $\underline{\mu} = \mu_{\mathsf{min}}$, $\bar{\mu} = \mu_{\mathsf{max}}$, and $\mu = \frac{\underline{\mu}+\bar{\mu}}{2}$.
2) For each subchannel $n = 1, \ldots, N$:
   a) For each (k, m),
      i) Use (11) and (13) to calculate $p^{*}_{n,k,m}(\mu)$.
      ii) Use (14) to calculate $V_{n,k,m}(\mu, p^{*}_{n,k,m}(\mu))$.
   b) Calculate $S_n(\mu)$ using (15).
3) If $|S_n(\mu)| \leq 1 \; \forall n$, then find $\boldsymbol{I}^{*}(\mu)$ using (16), else use (30) and set $\boldsymbol{I}^{*}(\mu) = \boldsymbol{I}^{\mathsf{min}}(\mu)$.
4) Find $\boldsymbol{x}^{*}(\mu, \boldsymbol{I}^{*}(\mu))$ using (12) and calculate $X^{*}_{\mathsf{tot}}(\mu) = \sum_{n,k,m} x^{*}_{n,k,m}(\mu, \boldsymbol{I}^{*}(\mu))$.
5) If $X^{*}_{\mathsf{tot}}(\mu) \geq P_{\mathsf{con}}$, set $\underline{\mu} = \mu$, otherwise set $\bar{\mu} = \mu$.
6) If $\bar{\mu} - \underline{\mu} > \kappa$, go to step 2), else proceed.
7) Now we have $\mu^{*} \in [\underline{\mu}, \bar{\mu}]$ and $\bar{\mu} - \underline{\mu} < \kappa$. If $X^{*}_{\mathsf{tot}}(\underline{\mu}) \neq X^{*}_{\mathsf{tot}}(\bar{\mu})$, set $\lambda = \frac{X^{*}_{\mathsf{tot}}(\underline{\mu}) - P_{\mathsf{con}}}{X^{*}_{\mathsf{tot}}(\underline{\mu}) - X^{*}_{\mathsf{tot}}(\bar{\mu})}$, else set $\lambda = 0$.
8) The optimal user-MCS allocation is given by $\hat{\boldsymbol{I}}_{\mathsf{CSRA}} = \lambda \boldsymbol{I}^{*}(\bar{\mu}) + (1 - \lambda) \boldsymbol{I}^{*}(\underline{\mu})$ and the corresponding optimal $\boldsymbol{x}$ is given by $\hat{\boldsymbol{x}}_{\mathsf{CSRA}} = \lambda \boldsymbol{x}^{*}(\bar{\mu}, \boldsymbol{I}^{*}(\bar{\mu})) + (1 - \lambda) \boldsymbol{x}^{*}(\underline{\mu}, \boldsymbol{I}^{*}(\underline{\mu}))$. The optimal power allocation, $\hat{\boldsymbol{p}}_{\mathsf{CSRA}}$, then can be found using
$$\hat{p}_{n,k,m,\mathsf{CSRA}} = \begin{cases} \frac{\hat{x}_{n,k,m,\mathsf{CSRA}}}{\hat{I}_{n,k,m,\mathsf{CSRA}}} & \text{if } \hat{I}_{n,k,m,\mathsf{CSRA}} \neq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{66}$$
where $\hat{I}_{n,k,m,\mathsf{CSRA}}$ and $\hat{x}_{n,k,m,\mathsf{CSRA}}$ denote the $(n, k, m)^{\text{th}}$ component of $\hat{\boldsymbol{I}}_{\mathsf{CSRA}}$ and $\hat{\boldsymbol{x}}_{\mathsf{CSRA}}$, respectively. Notice that the obtained solution satisfies the sum-power constraint with equality.

Brute force algorithm for a given $\boldsymbol{I}$

1) Initialize $\underline{\mu} = \mu_{\mathsf{min}}$ and $\bar{\mu} = \mu_{\mathsf{max}}$.
2) Set $\mu = \frac{\underline{\mu}+\bar{\mu}}{2}$.
3) For each (n, k, m), use (38)-(40) to obtain $x^{*}_{n,k,m}(\mu)$.
4) Find $X^{*}_{\mathsf{tot}}(\boldsymbol{I}, \mu)$ using (41).
5) If $X^{*}_{\mathsf{tot}}(\boldsymbol{I}, \mu) > P_{\mathsf{con}}$, set $\underline{\mu} = \mu$, otherwise set $\bar{\mu} = \mu$.
6) If $\bar{\mu} - \underline{\mu} < \kappa$, go to step 7), otherwise go to step 2).
7) If $X^{*}_{\mathsf{tot}}(\boldsymbol{I}, \bar{\mu}) \neq X^{*}_{\mathsf{tot}}(\boldsymbol{I}, \underline{\mu})$, set $\lambda = \frac{X^{*}_{\mathsf{tot}}(\boldsymbol{I}, \underline{\mu}) - P_{\mathsf{con}}}{X^{*}_{\mathsf{tot}}(\boldsymbol{I}, \underline{\mu}) - X^{*}_{\mathsf{tot}}(\boldsymbol{I}, \bar{\mu})}$, otherwise set $\lambda = 0$.
8) Set $\hat{\mu}_{\boldsymbol{I}} = \bar{\mu}$. The best actual power allocation is given by $\hat{\boldsymbol{x}}_{\boldsymbol{I}} = \lambda \boldsymbol{x}^{*}(\bar{\mu}) + (1 - \lambda) \boldsymbol{x}^{*}(\underline{\mu})$ and the best power allocation, $\hat{\boldsymbol{p}}_{\boldsymbol{I}}$, is given by
$$\hat{p}_{n,k,m,\boldsymbol{I}} = \begin{cases} \frac{\hat{x}_{n,k,m,\boldsymbol{I}}}{I_{n,k,m}} & \text{if } I_{n,k,m} \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$
where $\hat{p}_{n,k,m,\boldsymbol{I}}$ and $\hat{x}_{n,k,m,\boldsymbol{I}}$ are the $(n, k, m)^{\text{th}}$ element of $\hat{\boldsymbol{p}}_{\boldsymbol{I}}$ and $\hat{\boldsymbol{x}}_{\boldsymbol{I}}$, respectively. The corresponding Lagrangian, found using $\hat{L}_{\boldsymbol{I}} = L_{\boldsymbol{I}}(\bar{\mu}, \boldsymbol{p}^{*}(\mu))$, gives the optimal Lagrangian value.

Proposed DSRA algorithm

1) Use the algorithmic implementation of the proposed CSRA solution in to find $\boldsymbol{I}^{*}(\underline{\mu})$ and $\boldsymbol{I}^{*}(\bar{\mu})$, where the optimal μ for the CSRA problem, i.e., μ^{*} lies in the set $[\underline{\mu}, \bar{\mu}]$, $\bar{\mu} - \underline{\mu} < \kappa$, and $\boldsymbol{I}^{*}(\underline{\mu}), \boldsymbol{I}^{*}(\bar{\mu}) \in \mathcal{I}_{\mathsf{DSRA}}$.
2) For both $\boldsymbol{I} = \boldsymbol{I}^{*}(\underline{\mu})$ and $\boldsymbol{I} = \boldsymbol{I}^{*}(\bar{\mu})$ (since they may differ), calculate $\hat{\boldsymbol{p}}_{\boldsymbol{I}}$ and $\hat{L}_{\boldsymbol{I}}$ as described for the brute force algorithm.
3) Choose $\hat{\boldsymbol{I}}_{\mathsf{DSRA}} = \operatorname{argmin}_{\boldsymbol{I} \in \{\boldsymbol{I}^{*}(\underline{\mu}), \boldsymbol{I}^{*}(\bar{\mu})\}} \hat{L}_{\boldsymbol{I}}$ as the user-MCS allocation and $\hat{\boldsymbol{p}}_{\mathsf{DSRA}} = \hat{\boldsymbol{p}}_{\hat{\boldsymbol{I}}_{\mathsf{DSRA}}}$ as the associated power allocation.

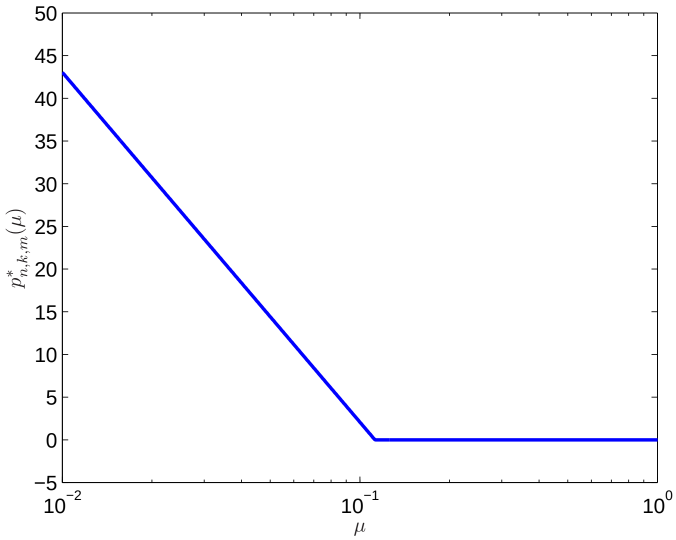

Fig. 1. Prototypical plot of $p^*_{n,k,m}(\mu)$ as a function of μ. The choice of system parameters are the same as those used in Section V.

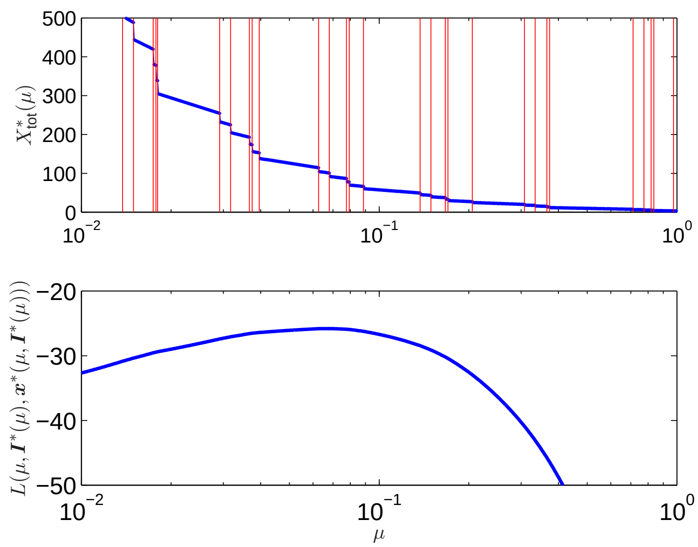

Fig. 2. Prototypical plot of $X^*_{\text{tot}}(\mu)$ and $L(\mu, \boldsymbol{I}^*(\mu), \boldsymbol{x}^*(\mu, \boldsymbol{I}^*(\mu)))$ as a function of μ for $N = K = 5$, and $P_{\text{con}} = 100$. Refer to Section V for other details. The red vertical lines in the top plot show that a change in $\boldsymbol{I}^*(\mu)$ occurs at that μ.

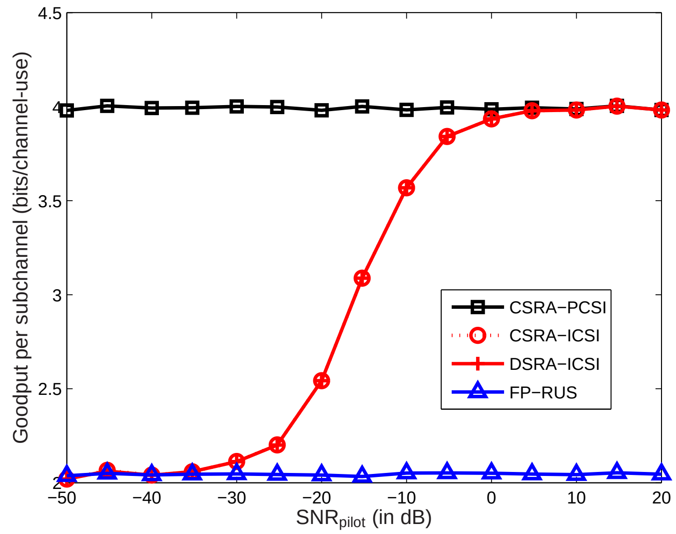

Fig. 3. Average goodput per subchannel versus $\mathsf{SNR}_{\mathsf{pilot}}$. Here, $N = 64$, $K = 16$, and $\mathsf{SNR} = 10$ dB.

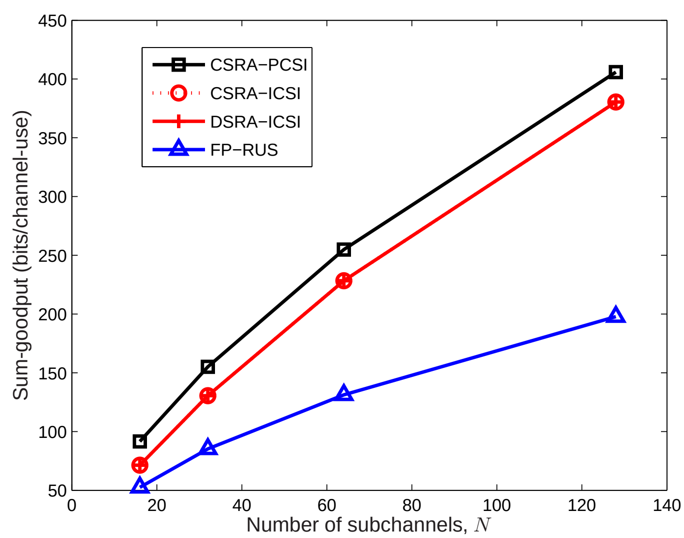

Fig. 4. Average sum-goodput versus number of subchannels N. Here, $K = 16$, P_{con} is fixed such that $\mathsf{SNR} = 10$ dB for a 64-subchannel OFDM system, and $\mathsf{SNR}_{\mathsf{pilot}} = -10$ dB.

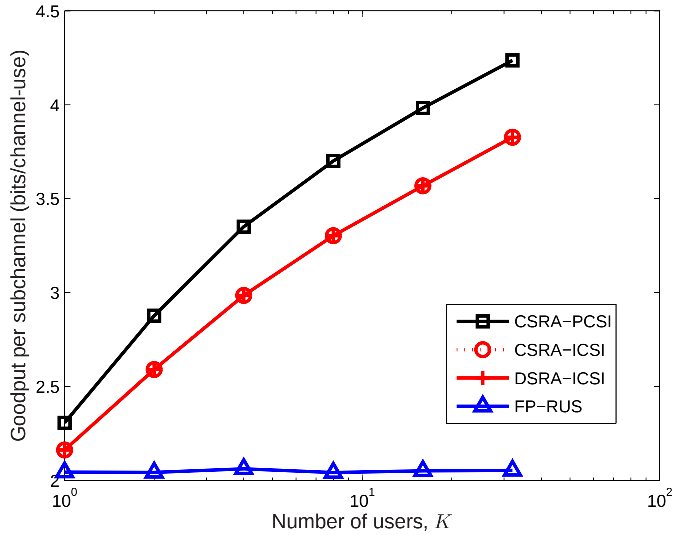

Fig. 5. Subchannel-averaged goodput versus number of users. In this plot, $N = 64$, SNR $= 10$ dB, and SNR$_{\text{pilot}} = -10$ dB.

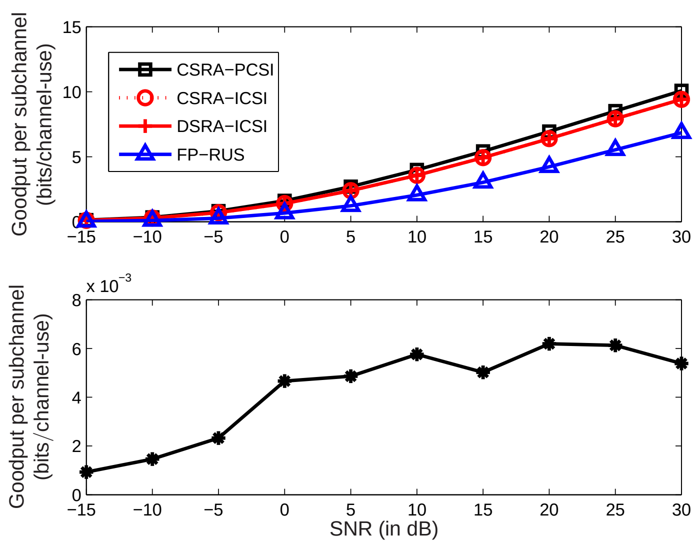

Fig. 6. The top plot shows the subchannel-averaged goodput as a function of SNR. The bottom plot shows the average bound on the optimality gap between the proposed and optimal DSRA solutions (given in (44)), i.e., the average value of $(\mu^* - \mu_{\min})(P_{\text{con}} - X^*_{\text{tot}}(\boldsymbol{I}^{\min}, \mu^*))/N$. In this plot, $N = 64$, $K = 16$, and SNR$_{\text{pilot}} = -10$ dB.