

BOLLOBÁS-RIORDAN AND RELATIVE TUTTE POLYNOMIALS

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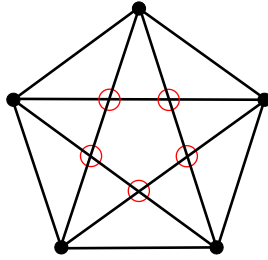
ABSTRACT. We establish a relation between the Bollobás-Riordan polynomial of a ribbon graph with the relative Tutte polynomial of a plane graph obtained from the ribbon graph using its projection to the plane in a nontrivial way. Also we give a duality formula for the relative Tutte polynomial of dual plane graphs and an expression of the Kauffman bracket of a virtual link as a specialization of the relative Tutte polynomial.

INTRODUCTION

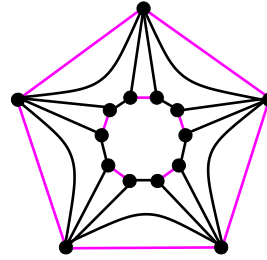
Graphs on surfaces can be studied in terms of plane graphs via a projection preserving the rotation system. These plane graphs are constructed in a nontrivial way in order to preserve the topological information of the graph. These constructed plane graphs usually have some extra (distinguished) edges and extra vertices. They are *relative* plane graphs.

Definition 0.1. A *relative plane graph* is a plane graph G with a distinguished subset $H \subseteq E(G)$ of edges. The edges H are called the *0-edges* of G .

For example, the presence of artificial virtual crossings, as in the picture of K_5 below, may be captured by 0-edges and some additional vertices.



Virtual crossings in plane pictures of non-planar graphs



Corresponding relative plane graph; 0-edges are dark

The motivation of our work comes from the knot theory. The classical Thistlethwaite theorem [Th] relates the Jones polynomial of an alternating link to the Tutte polynomial of plane graph obtained from a checkerboard coloring of the regions of the link diagram. This theorem has two different kinds of generalizations to virtual links. One [Ch, ChPa, ChVo, DFKLS, Mof] involves graphs on surfaces and a topological version of the Tutte polynomial due to B. Bollobás and O. Riordan [BR]. Another generalization is based on a relative version of the Tutte polynomial found by Y. Diao and G. Heteyi [DH]. In this paper we establish a direct relation between the Bollobás-Riordan and relative Tutte polynomials that explains how these two generalizations are connected.

In Section 1 we explain the construction of a relative plane graph from a ribbon graph as well as how to recover a ribbon graph from a relative plane graph. Our main theorem is formulated

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in Section 2 and proved in Section 3. In Section 4 we describe the relation between relative Tutte polynomials of dual plane graphs generalizing the classical relation $T_G(x, y) = T_{G^*}(y, x)$. In Section 5 we obtain the Kauffman bracket of a virtual link in terms of the relative Tutte polynomial, improving the theorem of [DH].

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<http://www.math.ohio-state.edu/~chmutov/wor-gr-su10/wor-gr.htm>

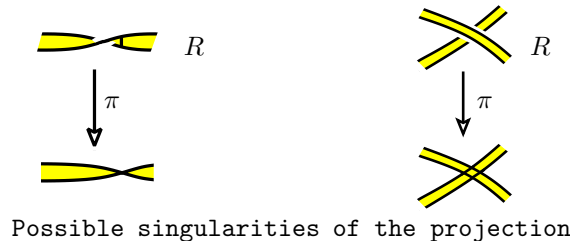
“Knots and Graphs” at the Ohio State University. We are grateful to all participants of the group for valuable discussions.

1. RIBBON GRAPHS AND RELATIVE PLANE GRAPHS

By a *ribbon graph* we mean an abstract (not necessarily orientable) surface with boundary decomposed into topological discs of two types, *vertex-discs* and *edge-ribbons*, satisfying some natural conditions; see [BR, Ch] for precise definition and [GT, LZ, MT] for the general notions and terminology of topological graph theory. Ribbon graphs are considered up to homeomorphisms of the underlying surfaces preserving the decomposition. A ribbon graph can be regarded as a regular neighborhood of a graph cellularly embedded into a surface. Thus the language of ribbon graph is essentially the same as for cellularly embedded graphs.

1.1. From ribbon graphs to relative plane graphs.

Let R be a ribbon graph. Consider a projection $\pi : R \rightarrow \mathbb{R}^2$ which is 1-to-1 except the points of singularities. We restrict these singularities to two types. The first occurs when a ribbon twists over itself; in this case a whole line interval on the ribbon is projected to a single point. The second type occurs when the images of two edge ribbons cross. In this case, the projection is 2-to-1 over the disc of the intersection.



Consequently, the restriction of the projection to the boundary of the ribbon graph is an immersion with only double points as singularities.

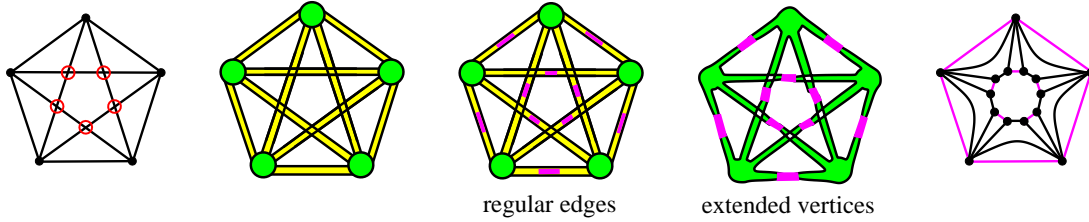
The construction of the relative plane graph G consists of four steps.

- (1) On each edge of R we choose a portion of the ribbon on which the projection is 1-to-1. We call it and its image on \mathbb{R}^2 a *regular edge*. The regular edges will be the non-zero edges of the relative plane graph G .
- (2) Extend the vertex discs of R through to the regular edge of each ribbon.
- (3) Each of these extended vertices is segmented by the regular edges and the singularities of the projection. These segments become the vertices of G .
- (4) The *0-edges* of G correspond to the double points of the restriction of the projection to the boundary of R . They connect the vertices of G which correspond to the extended regions sharing the same double point in a checkerboard manner.

Examples.

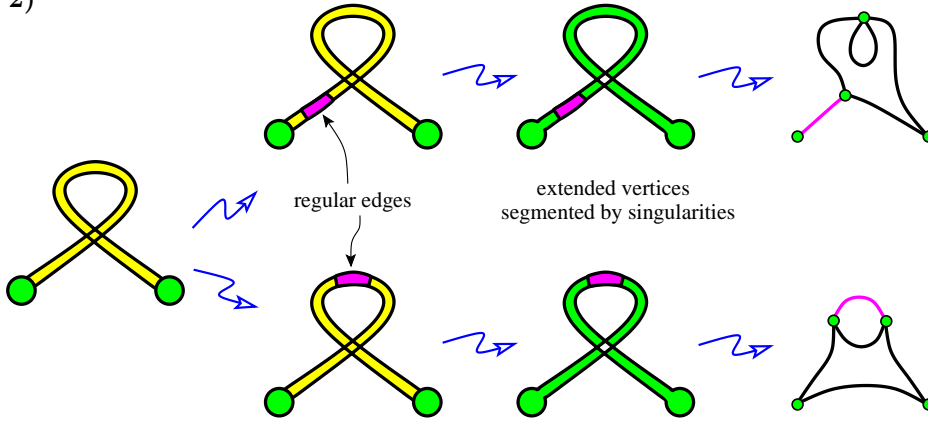
- 1) The picture of the graph K_5 from the introduction gives such a projection. The next figure

shows the steps in construction of the corresponding graph G

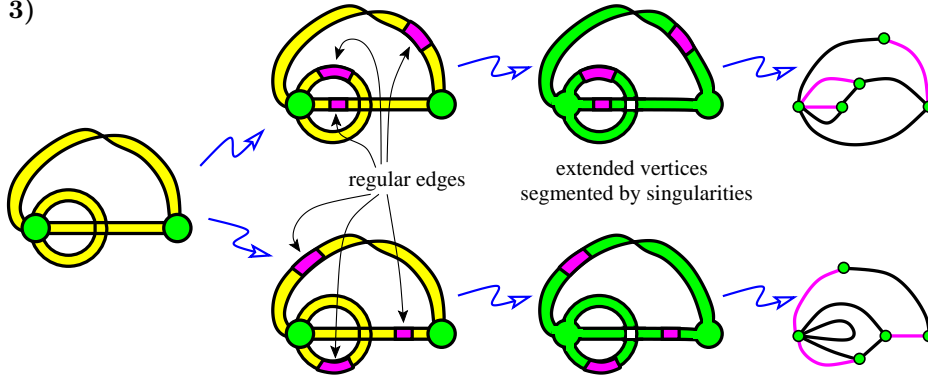


Here are two more examples showing the dependence of G on the choice of regular edges.

2)



3)



It is a remarkable consequence of the main theorem that the specialization of the relative Tutte polynomial does not depend on all these choices. It is not difficult to describe a sequence of moves on relative plane graphs relating the graphs with different choices of the regular edges. It would be interesting to find such moves for different choices of the projection π and, more generally, the moves preserving the relative Tutte polynomial.

1.2. From relative plane graphs to ribbon graphs.

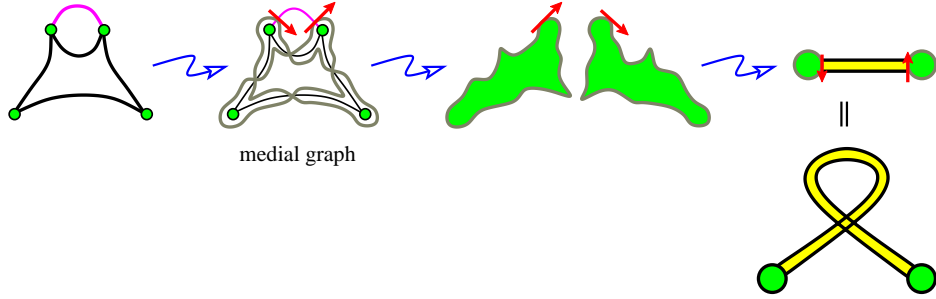
Conversely, from a relative plane graph G we may construct a ribbon graph R in the following way.

- (1) Delete all regular edges of G and for the obtained plane graph H construct its medial graph.
- (2) Consider the medial graph as an immersion of a union of circles with double points on its vertices, midpoints of 0-edges of G . Each such circle gives a *vertex* of R .

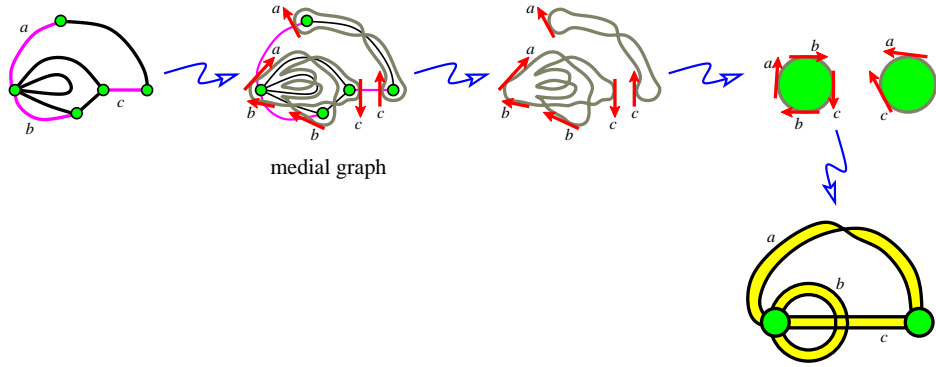
- (3) Each regular edge of $G \setminus H$ makes a trace on the circles which we denote by two arrows according to a counterclockwise orientation of the plane.
- (4) Pull apart and untwist the circles. Here we may reverse the orientations of certain arrows.
- (5) The *edges* of R are formed by attaching ribbons to the vertex discs of R according to the arrows. Thus we get the arrow presentation of the ribbon graph R in sense of [Ch].

Examples.

1)



2)



Observe that this construction depends on the planar embedding of G , as different embeddings can change the cyclic order of edges on a vertex of R . However, if we fix a projection of R , the regular edge portions of ribbons of R , and the planar embedding of G arising from this projection, then this construction is inverse to the construction of a relative plane graph from a ribbon graph.

1.3. The Bollobás–Riordan polynomial of ribbon graphs.

The Bollobás–Riordan polynomial, originally defined in [BR], was generalized to a multivariable polynomial of weighted ribbon graphs in [Mof, VT]. We will use a slightly more general doubly weighted Bollobás–Riordan polynomial of a ribbon graph R with weights (x_e, y_e) of an edge $e \in R$.

Definition 1.1.

$$B_R(X, Y, Z) := \sum_{F \subseteq R} \left(\prod_{e \in F} x_e \right) \left(\prod_{e \in R \setminus F} y_e \right) X^{k(F) - k(R)} Y^{n(F)} Z^{k(F) - bc(F) + n(F)},$$

where the sum runs over all spanning subgraphs F , $k(F)$ is the number of connected components of F , $n(F) = |E(F)| - v(F) + k(F)$ is the nullity of F , and $bc(F)$ is the number of boundary components of F .

1.4. The relative Tutte polynomial.

Definition 1.2. Let G be a relative plane graph with the distinguished set of 0-edges H .

$$T_{G,H}(G) := \sum_{F \subseteq G \setminus H} \left(\prod_{e \in F} x_e \right) \left(\prod_{e \in \overline{F}} y_e \right) X^{k(F \cup H) - k(G)} Y^{n(F)} \psi(H_F),$$

where $\overline{F} = G \setminus (F \cup H)$, ψ is a block-invariant function on graphs, and H_F is the plane graph obtained from $F \cup H$ by contracting all edges of F . Our choice of ψ is

$$\psi(H_F) := d^{\delta(H_F) - k(H_F)} w^{v(H_F) - k(H_F)},$$

$\delta(H_F)$ is the number of circles that immerse to the medial graph of H_F .

Remarks.

1. The relative Tutte polynomial was introduced by Y. Diao and G. Hetyei in [DH], who use the notion of *activities* to produce the most general form of it. The all subset formula we use was discovered by a group of undergraduate students (M. Carnovale, Y. Dong, J. Jeffries) at the OSU summer program “Knots and Graphs” in 2009. However, the similar expressions may be traced back to L. Traldi [Tr] for non relative case, and to S. Chaiken [Cha] for relative case of matroids.

2. The function ψ in [DH] can be obtained from ours by substitution $w = 1$.

3. Another difference with [DH] is that we are using a doubly weighted version of the relative Tutte polynomial with weights (x_e, y_e) of an edge $e \in G \setminus H$.

4. In the process of constructing the graph H_F by contracting the edges of F in $F \cup H$, we may come to a situation when we have to contract a loop. Then under contraction of a loop we actually mean its deletion. Since G and $F \cup H$ are plane graphs, then the graph H_F is also embedded into the plane.

2. MAIN THEOREM

Theorem 2.1. Suppose R is a ribbon graph, and G is a relative plane graph of a projection of R . Or, equivalently, G is a relative plane graph and R is the ribbon graph arising from G .

Then under the substitution $w = \sqrt{\frac{X}{Y}}, d = \sqrt{XY}$,

$$X^\alpha Y^\beta T_{G,H}(X, Y) = B_R(X, Y, \frac{1}{\sqrt{XY}}),$$

where $\alpha := k(G) - k(R) - \beta$ and $\beta := -\frac{1}{2}(v(R) - v(G))$.

Remark. The construction of G from R and backward can be generalized to a wider class of projections π . We can require only the restriction of π to the boundary of R to be an immersion with only ordinary double points as singularities. The theorem holds in this topologically more general situation. However, from the point of view of graph theory it is more natural to restrict ourselves to the class of projections which we use, as the more general projections are not necessarily invertible.

3. PROOF

Our constructions of G from R and backward from Sections 1.1 and 1.2 give a bijection between (regular) non-zero edges of G and the edge-ribbons of R . We denote the corresponding edges by the same letter e for both $e \in G \setminus H$ and for $e \in R$ since this will not lead to confusion. Moreover, in the theorem we assume that this bijection respects the weights of the doubly weighted polynomials.

The bijection can be naturally extended to the bijection between spanning subgraphs $F \subseteq G \setminus H$ and $F' \subseteq R$ so that the weights of F and F' are equal to each other:

$$\left(\prod_{e \in F} x_e\right) \left(\prod_{e \in \overline{F}} y_e\right) = \left(\prod_{e \in F'} x_e\right) \left(\prod_{e \in R \setminus F'} y_e\right)$$

Thus the theorem can be checked only on monomials in X and Y corresponding to $F \subseteq G \setminus H$ and $F' \subseteq R$. In other words, we have to prove that

$$(1) \quad X^{k(G)-k(R)-\frac{1}{2}(v(R)-v(G))} Y^{-\frac{1}{2}(v(R)-v(G))} X^{k(F \cup H)-k(G)} Y^{n(F)} d^{\delta(H_F)-k(H_F)} w^{v(H_F)-k(H_F)} \\ = X^{k(F')-k(R)} Y^{n(F')} Z^{k(F')-bc(F')+n(F')}$$

for $d = \sqrt{XY}$, $w = \sqrt{\frac{X}{Y}}$, and $Z = \frac{1}{\sqrt{XY}}$.

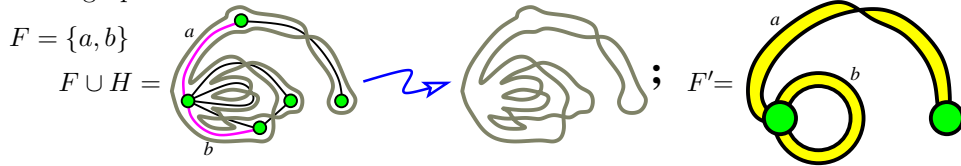
We need the following combinatorial equalities:

- (2) $|E(F)| = |E(F')|$
- (3) $k(H_F) = k(F \cup H)$
- (4) $bc(F') = n(F) + \delta(H_F)$
- (5) $v(H_F) = k(F)$

(2) is clear from the subgraph correspondence. Since contracting edges of a graph cannot disconnect it or connect disconnected components, (3) is immediate.

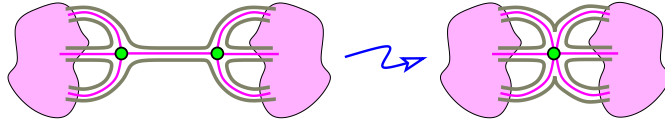
3.1. Proof of (4).

In the plane subgraph $F \cup H$ of G , trace the edges two times each in the following manner: the edges of F are traced by parallel lines and the edges of H are traced by crossed lines as in the medial graph:

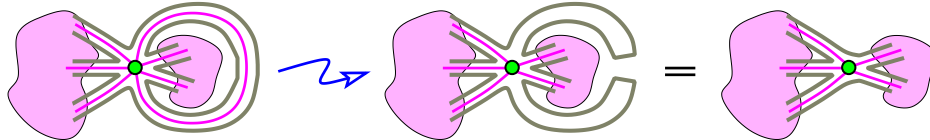


The number of immersed circles of the tracing is precisely $bc(F')$, by our construction.

Now consider contracting edges of F . The contraction of a non-loop does not change the number of curves of the tracing.



However, the contraction of a loop, i.e. deletion of the loop, fuses two disjoint curves together, one from the outside of the loop and one from the inside of the loop. So it reduces the number of curves by 1.



The result of contracting all the edges of F is the graph H_F , for which the number of curves will be $\delta(H_F)$. Since the number of loops contracted is $n(F)$, we have

$$bc(F') = n(F) + \delta(H_F) .$$

3.2. Proof of (5).

Let us consider $F \cup H$ as a spanning subgraph of G and let us remove the edges of H from it for a moment. Then we get the spanning subgraph F . Its edges are supposed to be contracted, so each connected component of F gives a vertex of the resulting graph. Now restoring the edges of H does not change the number of vertices of graph obtained by contracting by F . Thus $v(H_F) = k(F)$.

3.3. Proof of the theorem.

We deal with the exponents of X and Y separately. The exponent of X in the left hand side of equation (1) is

$$\frac{1}{2}(v(R) - v(G)) + k(G) - k(R) + k(F \cup H) - k(G) + \frac{1}{2}(\delta(H_F) - k(H_F) + v(H_F) - k(H_F))$$

Substituting the equalities above and making appropriate cancellations,

$$\begin{aligned} &= \frac{1}{2}(v(R) - v(G)) - k(R) + \frac{1}{2}(\delta(H_F) + k(F)) \\ &= \frac{1}{2}(v(R) - v(G)) - k(R) + \frac{1}{2}(bc(F') - n(F) + k(F)) \\ &= \frac{1}{2}(v(R) - v(G)) - k(R) + \frac{1}{2}(bc(F') - |E(F')| + v(G)) \\ &= -k(R) + \frac{1}{2}(bc(F') + v(R) - |E(F')|) \\ &= k(F') - k(R) + \frac{1}{2}(bc(F') - n(F') + k(F')) \end{aligned}$$

which is the exponent of X in B_R .

For Y , the exponent in the left hand side of equation (1) is

$$-\frac{1}{2}(v(R) - v(G)) + n(F) + \frac{1}{2}(\delta(H_F) - k(H_F) - v(H_F) + k(H_F))$$

This is equivalent to,

$$\begin{aligned} &= |E(F)| - v(G) + k(F) + \frac{1}{2}(bc(F') - n(F) - v(H_F) - v(R) + v(G)) \\ &= \frac{1}{2}(bc(F') - |E(F)| + 2v(G) - v(R) - 2k(F)) + |E(F)| - v(G) + k(F) \\ &= \frac{1}{2}(|E(F')| - v(R) + bc(F')) \\ &= n(F') + \frac{1}{2}(bc(F') - n(F') - k(F')) \end{aligned}$$

which is the exponent of Y in B_R .

4. DUAL RELATIVE PLANE GRAPHS

Let G be a relative plane graph. For an embedding of G in the plane, the *dual* of G , denoted G^* is formed by taking the dual of G as a plane graph, and labeling the edges of G^* which intersect 0-edges of G as the 0-edges of G^* . Note that $(G^*)^* = G$, as with usual planar duality.

Theorem 4.1. *Under the substitution $w = \sqrt{\frac{X}{Y}}$, $d = \sqrt{XY}$, we have*

$$X^{a(G,H)} Y^{b(G)} T_{G,H}(X, Y) = Y^{a(G^*,H^*)} X^{b(G^*)} T_{G^*,H^*}(Y, X)$$

with the correspondence on the edge weights being $x_e = y_{e^*}$, $y_e = x_{e^*}$, where e^* is the edge of G^* that intersects e , and $a(G, H) = (|E(G \setminus H)| - v(G))/2 + k(G)$, $b(G) = v(G)/2$.

Proof of the Theorem. The equality is on monomials of $T_{G,H}, T_{G^*,H^*}$ in the edge weights variables (x_e, y_e) which establishes the correspondence between spanning subgraphs F of $G \setminus H$ and F^* of $G^* \setminus H^*$. Namely, F^* consists of those regular edges of G^* which do not intersect the regular edges of F .

We prove the equality on monomials for the exponent of X . Equality for Y then follows from duality. The exponent of X on the left is

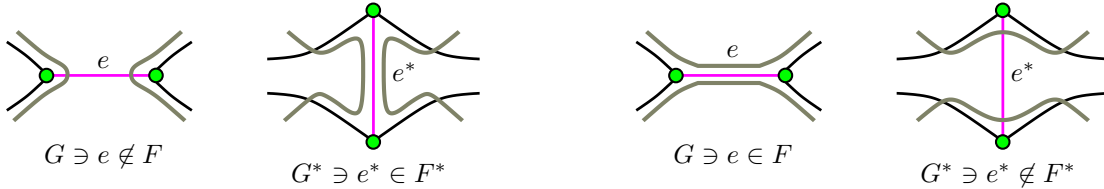
$$\begin{aligned} & \frac{1}{2}(|E(G \setminus H)| - v(G)) + k(G) + k(F \cup H) - k(G) + \frac{1}{2}(\delta(H_F) - k(H_F) + v(H_F) - k(H_F)) \\ &= \frac{1}{2}(|E(G \setminus H)| - v(G) + bc(F_R) - n(F) + k(F)) \\ &= \frac{1}{2}(|E(G \setminus H)| + bc(F_R) - |E(F)|) \\ &= \frac{1}{2}(|E(\overline{F})| + bc(F_R)) \end{aligned}$$

where F_R is the ribbon graph constructed from the relative plane graph $F \cup H$ in the manner of Section 1.2.

On the right, let F^* denote the subgraph of G^* corresponding to F . Then the exponent of X is

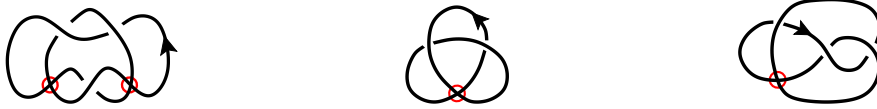
$$\begin{aligned} & n(F^*) + \frac{1}{2}(\delta(H_{F^*}) - k(H_{F^*}) - v(H_{F^*}) + k(H_{F^*}) + v(G^*)) \\ &= n(F^*) + \frac{1}{2}(bc(F_R^*) - n(F^*) - k(F^*) + v(G^*)) \\ &= \frac{1}{2}(bc(F_R^*) + |E(F^*)|) \end{aligned}$$

Now, $|E(F^*)| = |E(\overline{F})|$ by the subgraph correspondence. The equality $bc(F_R) = bc(F_R^*)$ follows from the fact that the ribbon graphs F_R and F_R^* have the same boundary. Also it can be seen from the following figures:



5. KAUFFMAN BRACKET OF VIRTUAL LINKS

In this section we generalize the result of [DH] which extends the Thistlethwaite theorem to virtual links. Virtual links are represented by diagrams similar to ordinary knot diagrams, except some crossings are designated as *virtual*. Here are some examples of virtual knots.



Virtual link diagrams are considered up to plane isotopy, the *classical* Reidemeister moves:



and the *virtual* Reidemeister moves:



The Kauffman bracket for virtual links is defined in the same way as for classical links. Let L be a virtual link diagram. Consider two ways of resolving a classical crossing. The A -splitting, $\frac{\text{+}}{\text{+}} \rightsquigarrow \text{+}$, is obtained by joining the two vertical angles swept out by the overcrossing arc when it is rotated counterclockwise toward the undercrossing arc. Similarly, the B -splitting, $\frac{\text{+}}{\text{+}} \rightsquigarrow \text{+}$, is obtained by joining the other two vertical angles. A *state* s of a link diagram L is a choice of either an A or B -splitting at each classical crossing. Denote by $\mathcal{S}(L)$ the set of states of L . A diagram L with n crossings has $|\mathcal{S}(L)| = 2^n$ different states.

Denote by $\alpha(s)$ and $\beta(s)$ the numbers of A -splittings and B -splittings in a state s , respectively, and by $\delta(s)$ the number of components of the curve obtained from the link diagram L by splitting according to the state $s \in \mathcal{S}(L)$. Note that virtual crossings do not connect components.

Definition 5.1. *The Kauffman bracket of a diagram L is a polynomial in three variables A, B, d defined by the formula*

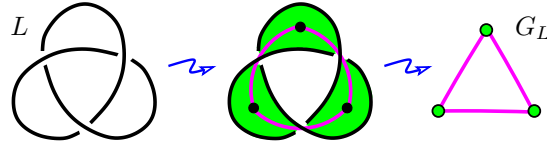
$$[L](A, B, d) := \sum_{s \in \mathcal{S}(L)} A^{\alpha(s)} B^{\beta(s)} d^{\delta(s)-1}.$$

Note that $[L]$ is *not* a topological invariant of the link; it depends on the link diagram and changes with Reidemeister moves. However, it determines the *Jones polynomial* $J_L(t)$ by a simple substitution:

$$A = t^{-1/4}, \quad B = t^{1/4}, \quad d = -t^{1/2} - t^{-1/2};$$

$$J_L(t) := (-1)^{w(L)} t^{3w(L)/4} [L](t^{-1/4}, t^{1/4}, -t^{1/2} - t^{-1/2}).$$

In 1987 Thistlethwaite [Th] (see also [K1]) proved that up to a sign and a power of t the Jones polynomial $V_L(t)$ of an alternating link L is equal to the Tutte polynomial $T_{G_L}(-t, -t^{-1})$ of the Tait graph G_L obtained from a checkerboard coloring of the regions of a link diagram.



L. Kauffman [K2] generalized the theorem to arbitrary (classical) links using signed graphs. To virtual links this theorem was extended in [Ch, ChPa, ChVo] using ribbon graphs. Another extension, using the relative Tutte polynomial, is due Y. Diao and G. Hetyei [DH]. In their construction the relative plane graph is the Tait graph of a virtual link diagram whose 0-edges correspond to virtual crossings. They expressed $[L](A, A^{-1}, -A^2 - A^{-2})$ as a specialization of the relative Tutte polynomial. We think that the whole Kauffman bracket $[L](A, B, d)$, although not a link invariant, is of some interest as a pure combinatorial invariant of link diagrams. It turns out that it also can be expressed as a specialization of the relative Tutte polynomial.

Following [DH], we assign signs to the edges of the Tait graph G depending on whether the edge connects A - or B -splitting regions:



Theorem 5.2. *Let L be a virtual link diagram, and G the relative plane Tait graph of L . Then, under the substitution*

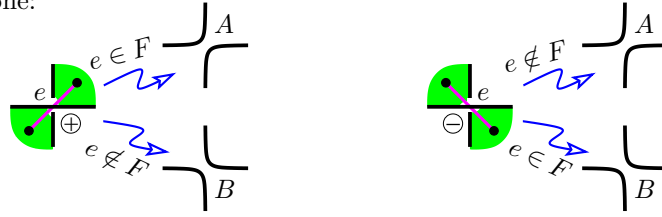
$$X = \frac{Bd}{A}, \quad Y = \frac{Ad}{B}, \quad w = \frac{B}{A}, \quad x_+ = y_+ = 1, \quad x_- = \sqrt{\frac{X}{Y}} = \frac{B}{A}, \quad y_- = \sqrt{\frac{Y}{X}} = \frac{A}{B}$$

we have,

$$[L](A, B, d) = A^{v(G)-k(G)} B^{|E(G \setminus H)|-v(G)+k(G)} d^{k(G)-1} T_{G,H}.$$

Proof.

The equality is on monomials, with the correspondence between subgraphs F and states S being the natural one:



Let $|E_-(F)|$ (resp. $|E_+(F)|$) be the number of negative (resp. positive) edges in the graph F . The power of B on the right is

$$\begin{aligned} & |E(G \setminus H)| - v(G) + k(G) + |E_-(F)| - |E_-(\bar{F})| + k(F \cup H) - k(G) - n(F) + v(H_F) - k(H_F) \\ &= |E_-(F)| - |E_-(\bar{F})| + |E(G \setminus H)| - |E(F)| \\ &= |E_-(F)| - |E_-(\bar{F})| + |E(\bar{F})| = |E_-(F)| + |E_+(\bar{F})| = \beta(S), \end{aligned}$$

as it can be easily seen from the picture above. The proof of equality on the exponent of A is similar. For d , the exponent on the right is

$$k(G) - 1 + k(F \cup H) - k(G) + n(F) + \delta(H_F) - k(H_F) = n(F) + \delta(H_F) - 1 = bc(F_R) - 1 = \delta(S) - 1.$$

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