DIASSOCIATIVE ALGEBRAS AND MILNOR'S INVARIANTS FOR TANGLES

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Abstract. We extend Milnor's μ-invariants of link homotopy to ordered (classical or virtual) tangles. Simple combinatorial formulas for μ-invariants are given in terms of counting trees in Gauss diagrams. Invariance under Reidemeister moves corresponds to axioms of Loday's diassociative algebra. The relation of tangles to diassociative algebras is formulated in terms of a morphism of corresponding operads.

1. Introduction

The theory of links studies embedding of several disjoint copies of S^1 into $\mathbb{R}^3$ and thus has to deal with a mixture of linking and self-knotting phenomena. The theory of link-homotopy, initiated by Milnor [7] is a useful notion to isolate the linking phenomena from the self-knotting ones and to study it separately. A fundamental set of link-homotopy invariants is given by Milnor's $\bar{\mu}_{i_1\dots i_r,j}$ invariants [7] with non-repeating indices $1 \le i_1, \dots i_r, j \le n$. Roughly speaking, these describe the dependence of j-th parallel on the meridians of i_1-th, ..., i_r-th components. The simplest invariant $\bar{\mu}_{i,j}$ is just the linking number of the corresponding components. The next one, $\bar{\mu}_{i_1 i_2,j}$, detects the Borromean-type linking of the corresponding 3 components and, together with the linking numbers, classify 3-component links up to link-homotopy.

Multi-component links lack a semi-group structure, present for knots. Namely, a connected sum, while well-defined for knots, is not defined for links. On the level of invariants, this is reflected in a complicated self-recurrent indeterminacy in the definition of $\bar{\mu}$-invariants (reflected in the use of notation $\bar{\mu}$, rather than μ). The introduction of string links [3] remedied this situation, since a connected sum is well-defined for string links. A version of $\bar{\mu}$-invariants modified for string links is thus free of the original indeterminacy; to stress this fact, we will further use the notation μ for these invariants. Milnor's invariants classify string links up to link-homotopy [3].

1.1. **Brief statement of results.** The notion of tangles generalizes that of links, braids and string links. We define Milnor's $\mu-$invariants for tangles with ordered components along the same lines as Milnor's original definition, i.e. in terms of generators of the (reduced) fundamental group of the complement of a tangle in a cylinder, using the Magnus expansion.

On the other hand, tangles may be encoded by Gauss diagrams (see [11, 2]). We follow the philosophy of [11] to define invariants of classical or virtual tangles

2010 *Mathematics Subject Classification.* 57M25; 57M27; 18D50; 16S37.
Key words and phrases. tangles, μ-invariants, planar trees, dialgebras, operads.
The second author was partially supported by the ISF grant 1343/10.

by counting (with appropriate weights and signs) certain subdiagrams of a Gauss diagram. Since subdiagrams used in computing these invariants correspond to rooted planar binary trees, we call the resulting invariants Z_j *tree invariants*.

The invariance under Reidemester moves defines equivalence relations among the corresponding trees. We study these relations and find (Theorem 3.3) that they could be interpreted as relations of the diassociative algebra introduced by Loday. Diassociative algebra is a vector space with two associative operations - left and right multiplications. The five defining axioms (equation 2) of a diassociative algebra describe the invariance under the third Reidemeister move – a crossing in a tangle diagram corresponds a diassociative algebra operation: the upper string being to the left (right) from the lower string in the Gauss diagram corresponds to the left (right) multiplication.

We explicitly write out the linear combinations of trees used in computing invariants of degrees 2,3 and 4. In particular, tree invariants $Z_{12,3}$ and $Z_{123,4}$ are computed and are later shown to coincide with the corresponding Milnor $\mu-$invariants.

We then discuss the properties of tree invariants of (classical or virtual) tangles. In particular, we study their dependence on orderings and orientations of strings. Moreover, we show that these invariants satisfy some skein relations, reminicent of those for the Conway polynomial and the Kauffman bracket. The skein relations for Milnor invariants were found in [8]. Similarity of these skein relations allows us to show that our tree invariants $Z_{i_1 \dots i_r, j}$ coincide with Milnor's μ-invariants $\mu_{i_1 \dots i_r, j}$ when either $1 \leq i_1 < \dots < i_r < j \leq n$ or $1 \leq j < i_1 < \dots < i_r \leq n$. This also allows us to extend Milnor's μ-invariants to virtual tangles.

We then switch to algebraic/operadic properties of tangles. We introduce a notion of a tree tangle as a tangle with only one string – the *t*runk – going all the way down, and all others starting and ending on the top. For this type of tangles there is an appropriate operation of grafting, which allows us to define the operad of tree tangles. We show that any tangle could be mapped to a tree tangle by an operation called *c*apping. This requires choosing a preferred string as the trunk. On the other hand, tree invariants are also defined with respect to a particular string playing the role of the trunk. The tree invariant Z_j takes values in the equivalence classes of trees with the trunk on the j-th component. It turns out that Z_j defines an operad morphism between the operad of tree tangles and the diassociative algebra operad $Dias$.

The paper is organized in the following way. In Section 2 the main objects and tools are introduced: tangles, Milnor's μ-invariants, and Gauss diagram formulas. In Section 3 we review diassociative algebras and introduce tree invariants of tangles and prove their invariance under Reidemeister moves. Section 4 is devoted to the properties of the invariants and their identification with the μ-invariants. Finally, in Section 5 we discuss the operadic structure on tree tangles and the corresponding morphism of operads.

The authors are grateful to Frédéric Chapoton and Jean-Louis Loday for stimulating discussions, and to the French Embassy in Israel for a generous travel support.

2. Preliminaries

2.1. **Tangles and string links.** Let D^2 be a disk in xy-plane. An (ordered, oriented) (k,l)-*tangle* without closed components in $C = D^2 \times [0,1]$ is an ordered collection of $n = \frac{k+l}{2}$ disjoint oriented arcs, properly embedded in C in such a way, that the endpoints of each interval belong to the set $P = \{p_i\}_{i=1}^k \times \{1\} \cup \{p_i\}_{i=1}^l \times \{0\}$ in C, where p_i are some prescribed points in the interior of D^2. See Figure 1a. Tangles are considered up to an oriented isotopy in C, fixed on the boundary. We will always assume that the only singularities of the projection of a tangle to the xz-plane are transversal double points. Such a projection, equipped with the indication of over- and underpasses in each double point, is called a *tangle diagram*. See Figure 1b.

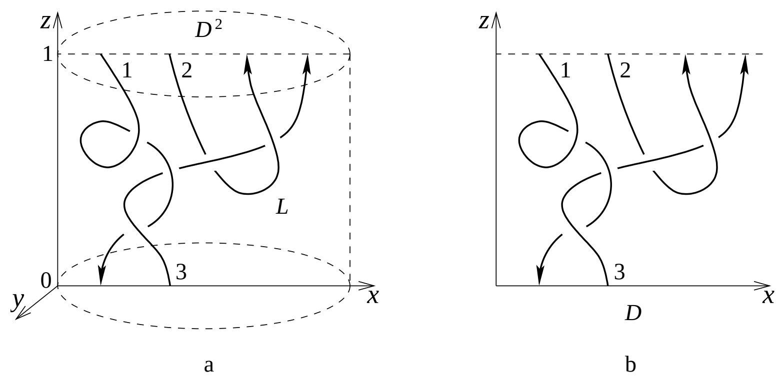


FIGURE 1. A $(4, 2)$-tangle and its diagram

An important class of tangles are *string links*, which are (n, n)-tangles such that the i-th arc ends in the points $p_i \times \{0, 1\}$. By the *closure* $\widehat{L}$ of a string link L we mean the braid closure of L. It is an n-component link obtained from L by an addition of n disjoint arcs in the plane $\{y = 0\}$, each of which meets C only at the endpoints $p_i \times \{0, 1\}$ of L, as illustrated in Figure 2a. The linking number lk of two components of L is their linking number in $\widehat{L}$. Two tangles are *link-homotopic*, if

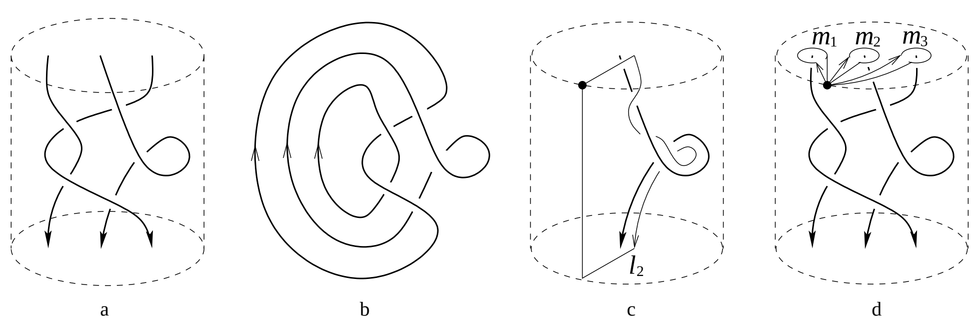


FIGURE 2. A string link, its closure, and canonical meridians and parallels

one can be transformed into the other by homotopy, which fails to be isotopy only in a finite number of instants, when a (generic) self-intersection point appears on one of the arcs.

2.2. **Milnor's μ-invariants.** Let us briefly recall the notion of Milnor's link-homotopy μ-invariants (see [7] for details, [5] for a modification to string links, and [8] for the case of tangles). We will first describe the well-studied case of string links, and then indicate modifications needed for the general case of tangles.

Let $L = \cup_{i=1}^{n} L_i$ be an n-component string link and consider the link group $\pi = \pi_1(C \smallsetminus L)$ with the base point $(0,1,1)$ on the upper boundary disc $D^2 \times \{1\}$. Choose *canonical parallels* $l_j \in \pi$, $j = 1, \dots, n$ represented by curves going parallel to L_j and then closed up by standard non-intersecting curves on the boundary of C so that $\mathrm{lk}(l_j, L_j) = 0$; see Figure 2c. Also, denote by $m_i \in \pi$, $i = 1, \dots, n$ the *canonical meridians* represented by the standard non-intersecting curves in $D^2 \times \{1\}$ with $\mathrm{lk}(m_i, L_i) = +1$, as shown in Figure 2d. If L is a braid, these meridians freely generate π, with any other meridian of L_i in π being a conjugate of m_i. For general string links, similar results hold for the reduced link group $\tilde{\pi}$.

Here by the reduced link group $\tilde{\pi}$ we mean the following. For any finitely-generated group G, the *reduced group* $\tilde{G}$ is the factor group of G by relations $[g, w^{-1}gw] = 1$, for any $g, w \in G$. Proceeding similarly to the usual construction of Wirtinger's presentation, one can show (see [3]) that $\tilde{\pi}(L)$ is generated by m_i, $i = 1, \dots, n$. Let F be the free group on n generators $x_1, \dots x_n$. The map $F \to \pi$ defined by $x_i \mapsto m_i$ induces the isomorphism $\widetilde{F} \cong \tilde{\pi}$ of the reduced groups [3]. We will use the same notation for the elements of π and their images in $\tilde{\pi} \cong \widetilde{F}$.

Now, let $\mathbb{Z}[[X_1, \dots, X_n]]$ be the ring of power series in n non-commuting variables X_i and denote by $\widetilde{Z}$ its factor by all the monomials, where at least one of the generators appears more than once. The *Magnus expansion* is a ring homomorphism of the group ring $\mathbb{Z}F$ into $\mathbb{Z}[[X_1, \dots, X_n]]$, defined by $x_i \mapsto \exp(X_i)$. It induces the homomorphisms $\theta : \mathbb{Z}\widetilde{F} \to \widetilde{Z}$ and $\theta_L : \mathbb{Z}\tilde{\pi} \to \widetilde{Z}$ of the corresponding reduced group rings.

Milnor's invariants $\mu_{i_1 \dots i_r, j}(L)$ of the string link L are defined as coefficients of the Magnus expansion $\theta_L(l_j)$ of the parallel l_j:

$$\theta_L(l_j) = \sum \mu_{i_1 \dots i_r, j} X_{i_1} X_{i_2} \dots X_{i_r} \ .$$

In particular, if L_j passes everywhere in front of the other components, all the invariants $\mu_{i_1 \dots i_r, j}$ vanish. Modulo lower degree invariants $\mu_{i_1 \dots i_r, j}(L) \equiv \bar{\mu}_{i_1 \dots i_r, j}(\bar{L})$, where $\bar{\mu}_{i_1 \dots i_r, j}(\bar{L})$ are the original Milnor's link invariants [7].

The above definition of invariants $\mu_{i_1 \dots i_r, j}(L)$ may be adapted to ordered oriented tangles without closed components in a straightforward way. A canonical meridian m_i of L_i is defined as a standard curve on the boundary of C, making a small loop around the starting point of L_i (with $\mathrm{lk}(m_i, L_i) = +1$), see Figure 3a. A canonical parallel l_j of L_j is a standard closure of a pushed-off copy of L_j (with $\mathrm{lk}(l_j, L_j) = 0$), see Figure 3b. The only difference from the string link case is that for general tangles there is no well-defined canonical closure (some additional choices – e.g. of a marked component – are needed), so in general one cannot directly compare values of $\mu_{i_1 \dots i_r, j}(L)$ to the original Milnor invariants.

Remark 2.1. Note that the invariants $\mu_{i_1 \dots i_r, j}$ significantly depend on the order of indices $i_1, i_2, \dots, i_r$ and j (e.g., in general $\mu_{i_2 i_1 \dots i_r, j}(L) \neq \mu_{i_2 i_1 \dots i_r, j})$. Under a transposition $\sigma \in S_n$ $\sigma : i \to \sigma(i)$ μ-invariants change in in an obvious way: $\mu_{i_1 i_2 \dots i_r, j}(L') = \mu_{\sigma(i_1)\sigma(i_2) \dots \sigma(i_r), \sigma(j)}(L)$, where L' is the tangle L with a transposed ordering: $L'_i = L_{\sigma(i)}$.

2.3. **Gauss diagrams.** Gauss diagrams provide a simple combinatorial way to present links and tangles. Consider a tangle diagram D as an immersion of n

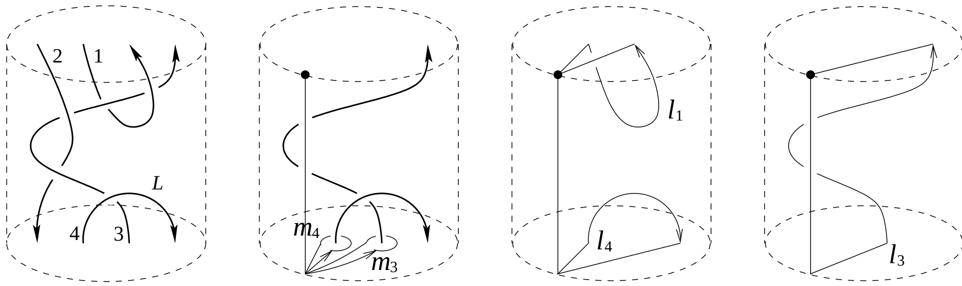


FIGURE 3. A choice of canonical meridians and parallels for a tangle

intervals into the plane, equipped with an information about the overpass and the underpass in each crossing. The *Gauss diagram* G corresponding to D is the ordered collection of n immersing intervals with the preimages of each crossing connected with a chord. We will usually depict these intervals as vertical lines, assuming that they are oriented downwards and ordered from left to right. Each chord a is oriented from the over- to the underpassing branch and equipped with the sign $\mathrm{sign}(a) = \pm 1$ of the corresponding crossing (its local writhe). See Figure 4.

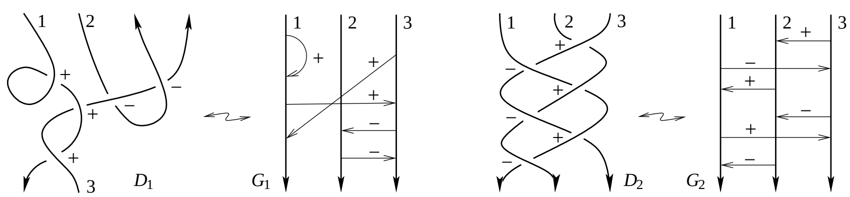


FIGURE 4. Gauss diagrams

The Gauss diagram encodes all the information about the crossings, and thus all the essential information contained in D, in a sense that, given endpoints of each string, D can be uniquely reconstructed from G. Reidemeister moves of tangle diagrams may be easily translated to the language of Gauss diagrams, see Figure 5. Here fragments participating in a move may be parts of the same or different strings, ordered in an arbitrary fashion, and the fragments in $\Omega 1$ and $\Omega 2$ may have different orientations. It suffices to consider only one oriented move of type three, see [1, 10].

2.4. **Virtual tangles.** Note that some collections of arrows connecting a set of n strings, while they look like Gauss diagrams, cannot be realized as a Gauss diagram of some tangle. Dropping this realization requirement leads to the theory of virtual tangles, see [4, 9]. We may simply define a virtual tangle as an equivalence class of virtual (i.e., not necessary realizable) Gauss diagrams modulo the Reidemeister moves of Figure 5.

The calculation of the fundamental group $\pi_1(C \setminus L)$ may be explicitly done from a Gauss diagram of a tangle L and it is easy to check its invariance under these Reidemeister moves. Thus the construction of Section 2.2 may be carried as well for virtual tangles, resulting in a definition of μ-invariants of virtual tangles.

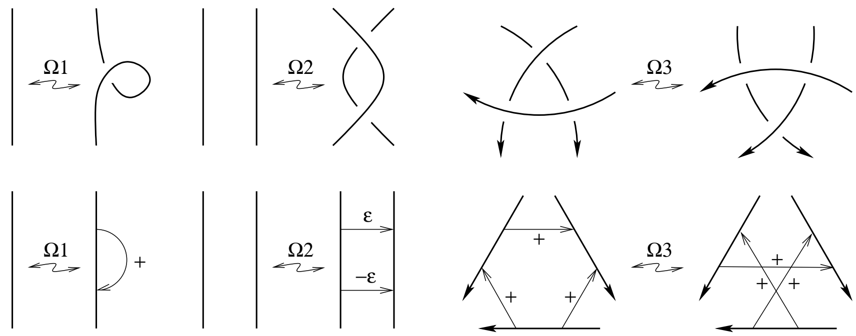


FIGURE 5. Reidemeister moves for diagrams and Gauss diagrams

The only new feature in the virtual case is the existence of two tangle groups. This is related to a possibility to choose the base point for the computation of the fundamental group $\pi = \pi_1(C \setminus L)$ either in the front half-space $y > 0$ (see Figure 2 and Section 2.2), or in the back half-space $y < 0$. While for classical tangles Wirtinger presentations obtained using one of these base points are two different presentations of the same group π, for virtual tangles we get two different - so-called the upper and the lower - tangle groups. See [2] for details. The passage from the upper to the lower group corresponds to a reversal of directions (but not signs!) of all arrows in a Gauss diagram. Using the lower group in the construction of Section 2.2, we would end up with another definition of μ-invariants, leading to a different set of "lower μ-invariants" in the virtual case. We will return to this discussion in Remark 4.9 below.

2.5. **Gauss diagram formulas.** An *arrow diagram* is a virtual Gauss diagram in which we forget about realizability and signs of arrows. In other words, an arrow diagram consists of an ordered set of oriented intervals (strings), with several arrows connecting pairs of distinct points on them, see Figure 6. We consider these diagrams up to orientation preserving diffeomorphisms of the intervals.

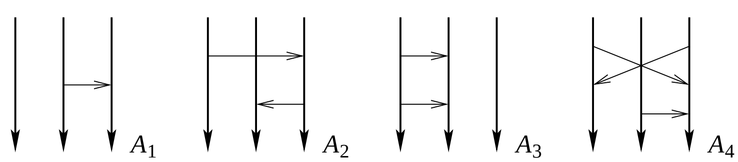


FIGURE 6. Arrow diagrams

Given an arrow diagram A on n intervals and a Gauss diagram G with the same number of intervals, we define a map $\phi : A \to G$ as a map from A to G which maps intervals to intervals and arrows to arrows, preserving their orientations and ordering of intervals. The sign of ϕ is defined as $\operatorname{sign}(\phi) = \prod_{a \in A} \operatorname{sign}(\phi(a))$. Finally, define a pairing $\langle A, G \rangle$ as

$$\langle A, G \rangle = \sum_{\phi : A \to G} \operatorname{sign}(\phi)$$

For example, for arrow diagrams $A_1 - A_4$ of Figure 6 and Gauss diagrams G_1, G_2 shown in Figure 4, we have $\langle A_1, G_1\rangle = \langle A_2, G_1\rangle = \langle A_4, G_1\rangle = -1$, $\langle A_2, G_2\rangle = 1$ and $\langle A_3, G_1\rangle = \langle A_1, G_2\rangle = \langle A_3, G_2\rangle = \langle A_4, G_2\rangle = 0$. We extend $\langle A, G\rangle$ to a vector space generated by all arrow diagrams on n strings by linearity.

If one picks A in an arbitrary fashion, $\langle A, G\rangle$ will change under the Reidemeister moves of G. However, for some special linear combinations A of arrow diagrams it will be preserved under the Reidemeister moves, thus resulting in an invariant of (ordered) tangles. See [11] and [2] for details and a general discussion on this type of formulas. The simplest example of such an invariant is a well-known formula for the linking number of two components:

$$\mathrm{lk}(L_1, L_2) = \langle\, |\!\!\leftarrow\!\!|\,, G\rangle. \tag{1}$$

The right hand side is the sum $\sum_{\phi: A\to G} \mathrm{sign}(\phi)$ over all maps of $A = |\!\!\leftarrow\!\!|$ to G. In other words, it is just the sum of signs of all crossings of D, where L_1 passes under L_2.

Remark 2.2. Note that for string links one has

$$\mathrm{lk}(L_1, L_2) = \langle\, |\!\!\leftarrow\!\!|\,, G\rangle = \langle\, |\!\!\rightarrow\!\!|\,, G\rangle = \mathrm{lk}(L_2, L_1).$$

For general tangles, however, these two invariants may differ. E.g., for a tangle diagram with just one crossing, where L_1 passses in front of L_2, we have $\langle\, |\!\!\leftarrow\!\!|\,, G\rangle = 0$ and $\langle\, |\!\!\rightarrow\!\!|\,, G\rangle = \pm 1$ depending on the sign of the crossing. This is a simple illustration of a general phenomenon: symmetries, which usually hold for classical links and string links, break down for tangles and virtual links. We will return to this observation in Section 3.

In the next section we introduce such Gauss diagram formulas for a family of tangle invariants which includes all Milnor's link-homotopy μ-invariants.

3. Tangle invariants by counting trees

Throughout the paper, let $I = \{i_1, i_2 \dots, i_r\}$, $1 \le i_1 < i_2 < \dots < i_r \le n$ and $j \in \{1, 2, \dots, n\} \smallsetminus I$.

3.1. **Tree diagrams.**

Definition 3.1. A *tree arrow diagram* A with leaves on I and a trunk on j is an arrow diagram which satisfies the following conditions:

- An arrowtail and an arrowhead of an arrow belong to different strings;
- There is exactly one arrow with an arrowtail on i-th string, if $i \in I$, and no such arrows if $i \notin I$;
- All arrows have arrowheads on $I \cup \{j\}$ strings;
- As we follow an i-th string, $i \in I$, all arrowheads precede the (unique) arrowtail.

Note that the total number of arrows in a tree arrow diagram is $r = |I|$; we will call this number the *degree* of A. Our choice of the term tree arrow diagram is explained by the following. Consider A as a graph (with vertices being heads and tails of arrows and endpoints of the strings). Removing all strings except for $I \cup \{j\}$, and cutting off the part of each of the remaining strings after the corresponding arrowtail, we obtain a tree T_A with $r + 1$ leaves on the beginning of each string in

$I \cup \{j\}$ and the root in the endpoint of j-th string. We will also say that T_A is a tree with leaves on I and a trunk on j. See Figure 7, where some tree arrow diagrams with $r = 2$, $j = 1$, $I = \{2, 3\}$ are shown together with corresponding trees.

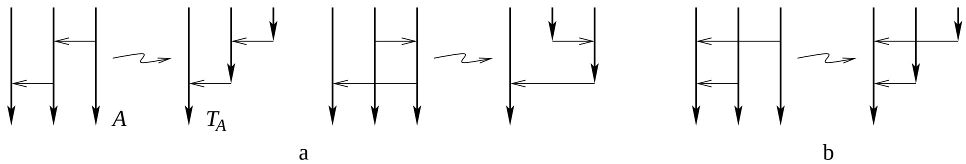


Figure 7. Planar and non-planar tree diagrams

Note that every tree T_A could be realized as a planar graph. The tree arrow diagram A is called *planar*, if this can be done so that the order of the leaves of the planar realization coincides, as we count the leaves starting clockwise from the root clockwise, with the initial ordering $i_1 < i_2 < \cdots < i_l < j < i_{l+1} < \cdots < i_r$ of the strings. E.g., diagrams in Figure 7a are planar, while the one in Figure 7b is not. Denote by $\mathcal{A}_{I,j}$ the set of all planar tree arrow diagrams with leaves on I and a trunk on j and by $\mathcal{A}_j$ the union $\mathcal{A}_j = \cup_I \mathcal{A}_{I,j}$.

3.2. **Diassociative algebras and trees.** Define the sign of an arrow diagram A to be $\operatorname{sign}(A) = (-1)^q$, where q is the number of right-oriented arrows in A. Given a Gauss diagram G of a tangle with a distinguished j-th component, we define the following quantity, taking value in a free abelian group generated by planar rooted trees[1]:

$$\sum_{A \in \mathcal{A}_j} \operatorname{sign}(A) \langle A, G \rangle \cdot T_A$$

While this formal sum of trees fails to be a tangle invariant, it becomes one after factorizing it by certain equivalence relations on trees. Appropriate relations turn out to be the ones of a diassociative algebra (or, as it was called in earlier literature, an associative dialgebra; the new name was suggested by Loday to avoid confusion with dialgebras of Gan and other structures with cobrackets):

Definition 3.2. ([6]) A diassociative algebra over a ground field k is a k-space V equipped with two k-linear maps

$$\vdash : V \otimes V \to V \quad \text{and} \quad \dashv : V \otimes V \to v,$$

called left and right products and satisfying the following five axioms:

$$\left\{ \begin{array}{ll} (1) & (x \dashv y) \dashv z = x \dashv (y \vdash z) \\ (2) & (x \dashv y) \dashv z = x \dashv (y \dashv z) \\ (3) & (x \vdash y) \dashv z = x \vdash (y \dashv z) \\ (4) & (x \dashv y) \vdash z = x \vdash (y \vdash z) \\ (5) & (x \vdash y) \vdash z = x \vdash (y \vdash z) \end{array} \right. \tag{2}$$

Diagrammatically, one can think about a free diassociative algebra as follows. Depict products $a \vdash b$ and $a \dashv b$ as elementary trees shown in Figure 8a, respectively. Composition of these operations corresponds then to grafting of trees, see Figure 8b,c.

[1]Note that this sum is always finite, since the Gauss diagram contains a fixed number of strings.

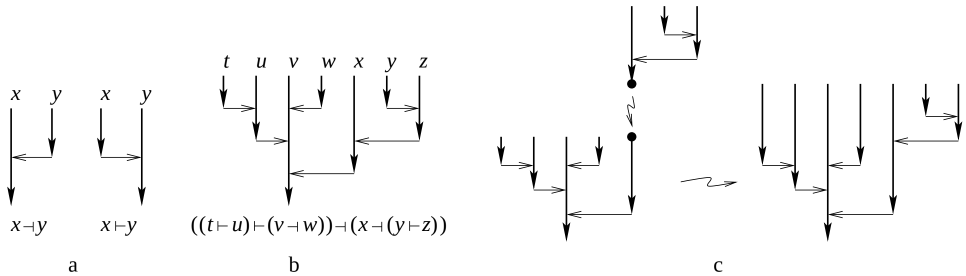


FIGURE 8. Diassociative operations as trees and their compositions

Axioms (2) correspond to relations on trees shown in Figure 9.

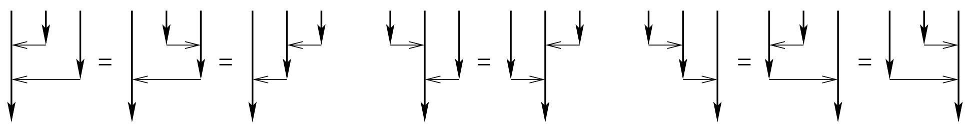


FIGURE 9. Diassociative algebra relations on trees

Denote by $Dias(n)$ the quotient of a vector space generated by planar rooted trees with n leaves by the relations of the diassociative algebra and let $Dias = \cup_n Dias(n)$. The operad structure on $Dias$ corresponds to grafting of trees, as illustrated in Figure 8c. See [6] for details.

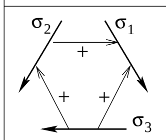
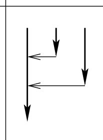
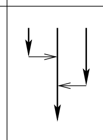
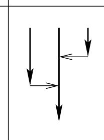
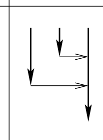
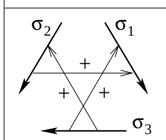
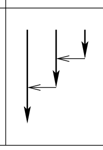
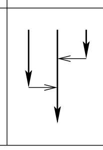
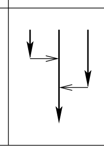
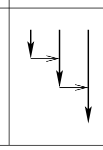
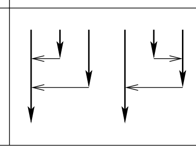
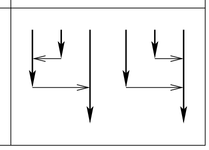
3.3. **tree invariants.** Denote by $[T]$ the equivalence class of a planar tree T in $Dias$ and define $Z_j(G) \in Dias$ by

$$Z_j(G) = \sum_{A \in \mathcal{A}_j} \operatorname{sign}(A) \langle A, G \rangle [T_A] \tag{3}$$

We call $Z_j(G)$ the *tree invariant* of a tangle which has G as its Gauss diagram, since it satisfies the following

Theorem 3.3. *Let L be an ordered (classical or virtual) tangle and let G be a Gauss diagram of L. Then $Z_j(L) = Z_j(G)$ is an invariant of ordered tangles.*

Proof. It suffices to prove that $Z_j(G)$ is preserved under the Reidemeister moves $\Omega 1$–$\Omega 3$ for Gauss diagrams, shown in Figure 5. The invariance under $\Omega 1$ and $\Omega 2$ follows immediately from the definition of $\mathcal{A}$. Indeed, a new arrow appearing in $\Omega 1$ has both its arrowhead and its arrowtail on the same string, so cannot be in the image of a tree diagram. Similarly, two new arrows which appear in $\Omega 2$ have their arrowtails on the same string, so cannot simultaneously be in an image of a tree diagram, while maps which contain one of them cancel out in pairs due to opposite signs of the two arrows. It remains to verify the invariance under the third Reidemeister move $\Omega 3$ depicted in Figure 5. Denote by G and G' Gauss diagrams related by $\Omega 3$. Note that there is a bijective correspondence between terms of $Z_j(G)$ and $Z_j(G')$. Indeed, since only the relative position of the three arrows participating in the move changes, all terms which involve only one of these arrows do not change. No terms involve all three arrows, since such a diagram can not be a tree diagram. It remains to compare terms which involve exactly two arrows. Note that a diagram which involves two arrows can be a tree diagram only if the fragments participating in the move belong to different strings. There is a number of cases, depending on the ordering $\sigma_1, \sigma_2, \sigma_3$ of these three strings. Using for simplicity indices $1, 2, 3$ for such an ordering, we can summarize the correspondence of these terms in the table below.

$\sigma_1\,\sigma_2\,\sigma_3$	1 2 3	2 1 3	3 1 2	3 2 1	1 3 2	2 3 1
					–	–
						

We see that the invariance is assured exactly by the diassociative algebra relations, see Figure 9. For four orderings out of six the correspondence is bijective, while for the two last orderings, pairs of trees appearing in the bottom row have opposite signs (due to a different number of right-oriented arrows), so their contributions to $Z_j(G')$ cancel out. $\square$

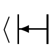
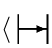
4. Properties of the tree invariants

The tree invariant $Z_j(L)$ takes values in the quotient *Dias* of the free abelian group generated by trees by the diassociative algebra relations. An equivalence class $[T_A]$ of a tree T_A with the trunk on j depends only on the set of its leaves, so it is the same for all arrow diagrams A in the set $\mathcal{A}_{I,j}$ of all planar tree arrow diagrams with leaves on I and a trunk on j.

Let $Z_{I,j}$ be the coefficient of Z_j corresponding to trees with leaves on I, i.e., $Z_{I,j} = \sum_{A \in \mathcal{A}_{I,j}} \operatorname{sign}(A)\langle A, G\rangle$. For $I = \emptyset$ we set $Z_{\emptyset,j} = 1$.

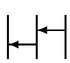
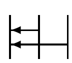
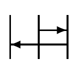
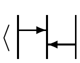
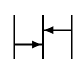
4.1. **Invariants of low degrees.** Let us start with invariants $Z_{I,j}$ for small values of $r = |I|$.

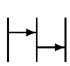
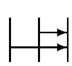
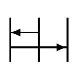
Counting tree diagrams with one arrow we get

(4) $Z_{2,1}(L) = \langle$  $, G\rangle$, $Z_{1,2}(L) = \langle$  $, G\rangle$.

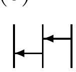
Note that for string links $Z_{2,1}(L) = Z_{1,2}(L) = \operatorname{lk}(L_1, L_2)$.

For digrams with two arrows we obtain

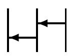
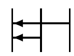
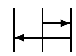
(5) $Z_{23,1}(L) = \langle$  $+$  $-$  $, G\rangle$, $Z_{13,2}(L) = \langle$  $+$  $, G\rangle$,

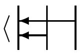
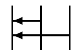
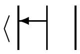
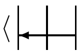
$Z_{12,3}(L) = \langle$  $+$  $-$  $, G\rangle$

In particular, $Z_{13,2}(L) = Z_{1,2}(L) \cdot Z_{3,2}(L)$. Also, $Z_{12,3}(L) = Z_{23,1}(\bar{L})$, where $\bar{L}$ is the tangle L with a reflected ordering $\bar{L}_i = L_{4-i}$ of strings.

Example 4.1. Consider a tangle L with a diagram D_2 depicted in Figure 4 and let us compute $Z_{23,1}(L)$ using formula (5). The corresponding Gauss diagram G_2 contains three subdiagrams of the type , two of which cancel out, while the remaining one contributes +1; there are no subdiagrams of other types appearing in (5). Hence $Z_{23,1}(L) = 1$.

When an orientation of a component is reversed, invariants $Z_{I,j}$ change sign and jump by a combination of lower degree invariants. E.g., denote by L' a 3-string tangle obtained from L by the reversal of orientations of L_1. Then

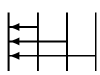
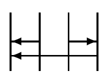
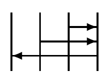
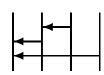
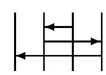
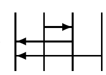
$Z_{23,1}(-L) = \langle -$  $+$  $+$  $, G\rangle.$

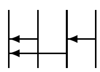
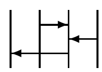
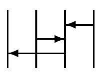
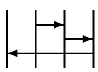
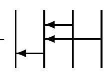
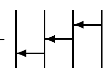
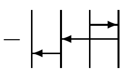
But it is easy to see that $\langle$  $+$  $, G\rangle = \langle$  $, G\rangle \cdot \langle$  $, G\rangle$, thus we obtain

$$Z_{23,1}(L') = -Z_{23,1}(L) + Z_{2,1}(L) \cdot Z_{3,1}(L).$$

Due to a large number of tree diagrams with 3 arrows, let us write down explicitly only diagrams with the trunk on the first string:

(6)

$Z_{234,1}(L) = \langle$  $-$  $+$  $+$  $-$  $-$ 

$+$  $-$  $-$  $+$  $+$  $+$  $-$  $, G\rangle$

For diagrams with the trunk on the second or third strings we have $Z_{134,2}(L) = Z_{1,2}(L) \cdot Z_{34,2}(L)$, $Z_{124,3}(L) = Z_{12,3}(L) \cdot Z_{4,3}(L)$. Finally, for $j = 4$ we again have $Z_{123,4}(L) = Z_{432,1}(\bar{L})$, where $\bar{L}$ is obtained from L by a reflection $\bar{L}_i = L_{5-i}$ of the ordering.

4.2. **Elementary properties of the tree invariants.** Unlike μ-invariants discussed in Section 2.2, which had a simple behavior under a change of ordering (see Remark 2.1), tree invariants $Z_{I,j}(L)$ significantly depend on the order of $i_1, \dots, i_r$ and j. I.e., if $L'_i = L_{\sigma(i)}$ for some $\sigma \in S_n$, $\sigma : i \to \sigma(i)$, then in general $Z_{I,j}(L')$ is not directly related to $Z_{\sigma(I),\sigma(j)}(L)$. However, in some simple cases their dependence on orderings and their behavior under simple changes of ordering and reflections of orientation can be deduced directly from their definition via planar trees:

Proposition 4.2. *Let L be an ordered (classical or virtual) tangle on n strings and let $I = \{i_1, i_2, \dots, i_r\}$, with $1 \le i_1 < i_2 < \dots < i_r \le n$.*

(1) *For $1 < k < r$ we have*
$$Z_{I \smallsetminus i_k, i_k}(L) = Z_{I_k^-, i_k}(L) \cdot Z_{I_k^+, i_k}(L)$$
where $I_k^- = I \cap [1, i_k - 1] = \{i_1, \dots, i_{k-1}\}$ and $I_k^+ = I \cap [i_k + 1, n] = \{i_{k+1}, \dots, i_r\}$.

(2) *Denote by $\bar{L}$ the tangle L with a reflected ordering: $\bar{L}_i = L_{\bar{i}}$, $i = 1, \dots, n$, where $\bar{i} = n + 1 - i$, so $\bar{I} = \{\bar{i_r}, \dots, \bar{i_2}, \bar{i_1}\}$. Then*
$$Z_{I,j}(\bar{L}) = (-1)^r Z_{\bar{I},\bar{j}}(L)$$

(3) *Finally, denote by L^σ the tangle L with an ordering shifted by a transposition $\sigma = (i_1 i_2 \dots i_r)$ (i.e.: $L^\sigma_{i_k} = L_{i_{k+1}}$ for $k = 1, \dots, r-1$ and $L^\sigma_{i_r} = L_{i_1}$), followed by a reflection of orientation of $L^\sigma_{i_r} = L_{i_1}$. Then*
$$Z_{I \smallsetminus i_r, i_r}(L^\sigma) = Z_{I \smallsetminus i_1, i_1}(L)$$

Proof. Indeed, a planar tree with a trunk on j consists of the "left half-tree" with leaves in $I \cap [1, j-1]$ and the "right half-tree" with leaves in $I \cap [j+1, n]$. Thus the first equality follows directly from the definition of the invariants.

Also, the reflection $i \to \bar{i}$ of ordering simply reflects a planar tree with respect to its trunk, exchanging the left and the right half-trees and changing all right-oriented arrows into left-oriented ones and vice versa, so the second equality follows (since the total number of arrows is r).

Finally, let us compare planar tree subdiagrams in a Gauss diagram G of L and in the corresponding Gauss diagram G^σ of L^σ. An application of the shift σ of ordering, followed by the reversal of orientation of the trunk, establishes a bijective correspondence between planar tree dagrams with leaves on $I \smallsetminus i_i$ and a trunk on i_1 and planar tree diagrams with leaves on $I \smallsetminus i_r$ and a trunk on i_r. Given a diagram $A \in \mathcal{A}_{i_1}$, we can get the corresponding diagram $A^\sigma \in \mathcal{A}_{i_r}$ in two steps: (1) redraw the trunk i_1 of A on the right of all strings, with an upwards orientation; (2) reverse the orientation of the trunk so that it is directed downwards. See Figure 10. Signs of these diagrams are related as follows: $\operatorname{sign}(A) = (-1)^q \operatorname{sign}(A^\sigma)$, where

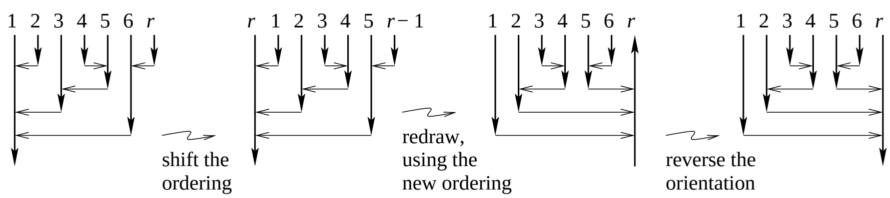


FIGURE 10. Reordering strings and reversing the orientation of the trunk

q is the number of arrows with arrowheads on the trunk (since all such arrows become right-oriented instead of left-oriented). Now note, that when we pass from G to G^σ, the reflection of orientation of $L^\sigma_{i_r}$ has a similar effect on signs of arrows, namely the sign of each arrow in G^σ with one end on the trunk (and the other end on some other string) is reversed, so $\langle A, G\rangle = (-1)^q \langle A^\sigma, G^\sigma\rangle$. These two factors of $(-1)^q$ cancel out to give $\operatorname{sign}(A)\langle A, G\rangle = \operatorname{sign}(A^\sigma)\langle A^\sigma, G^\sigma\rangle$ and the last statement follows. □

tree invariants $Z_{I,j}(L)$ satisfy the following skein relations. Let L_+, L_-, L_0 and L_∞ be four tangles which differ only in a neighborhood of a single crossing d, where they look as shown in Figure 11. In other words, L_+ has a positive crossing, L_- has a negative crossing, L_0 is obtained from $L_\pm$ by smoothing, and L_∞ is obtained from $L_\pm$ by the reflection of orientation of L_{i_k}, followed by smoothing. Orders of strings of $L_\pm$, L_0 and L_∞ coincide in the beginning of each string. See Figures 11 and 12. We will call $L_\pm$, L_0 and L_∞ a *skein quadruple*.

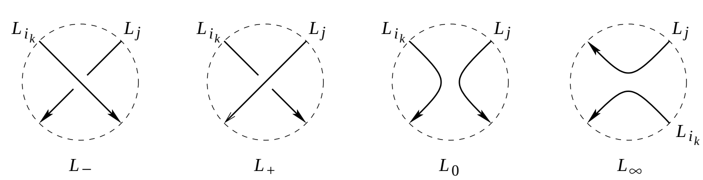


FIGURE 11. Skein quadruple of tangles

Theorem 4.3. *Let $j < i_1 < i_2 < \cdots < i_r$ and $1 \le k \le r$. Let L_+, L_-, L_0 and L_∞ be a skein quadruple of tangles on n strings which differ only in a neighborhood of a single crossing d of j-th and i_k-th components, see Figure 11. For $m = 1, \dots, k$ denote $I_m^- = \{i_1, \dots, i_{m-1}\}$, $I_m^+ = I \smallsetminus I_m^- \smallsetminus i_k = \{i_m, \dots, i_{k-1}, i_{k+1}, \dots, i_r\}$. Then*

$$Z_{I,j}(L_+) - Z_{I,j}(L_-) = Z_{I_k^-,j}(L_\infty) \cdot Z_{I_k^+,i_k}(L_0) \; ; \tag{7}$$

$$Z_{I,j}(L_+) - Z_{I,j}(L_-) = \sum_{m=1}^{k} Z_{I_m^-,j}(L_\pm) \cdot Z_{I_m^+,i_k}(L_0) \; . \tag{8}$$

Here we used the notation $Z_{I_m^-,j}(L_\pm)$ to stress that $Z_{I_m^-,j}(L_+) = Z_{I_m^-,j}(L_-)$.

Remark 4.4. Note that for $m = 1$ we have $I_1^- = \emptyset$ and $I_1^+ = I \smallsetminus i_k$, which corresponds to the summand $Z_{I \smallsetminus i_k, i_k}(L_0)$ in the right hand side of (8). Also, in a particular case $k = 1$ both equations (8), (7) simplify to

$$Z_{I,j}(L_+) - Z_{I,j}(L_-) = Z_{I \smallsetminus i_1, i_1}(L_0) \qquad (k = 1) \tag{9}$$

Finally, for and $k = r$ equation (7) becomes

$$Z_{I,j}(L_+) - Z_{I,j}(L_-) = Z_{I \smallsetminus i_r, j}(L_\infty) \qquad (k = r)$$

Example 4.5. Consider a tangle $L = L_+$ depicted in Figure 12 and let us compute $Z_{23,1}(L)$. Notice that if we switch the indicated crossing of L_1 with L_2 to the negative one, we get a link L_- with L_3 unlinked from L_1 and L_2, so $Z_{23,1}(L_-) = 0$. We have $i_1 = 2, i_2 = 3$ and $k = 1$, thus we can use equation (9) and get

$$Z_{23,1}(L) = Z_{23,1}(L) - Z_{23,1}(L_-) = Z_{3,2}(L_0) = 1,$$

in agreement with calculations of Exercise 4.1.

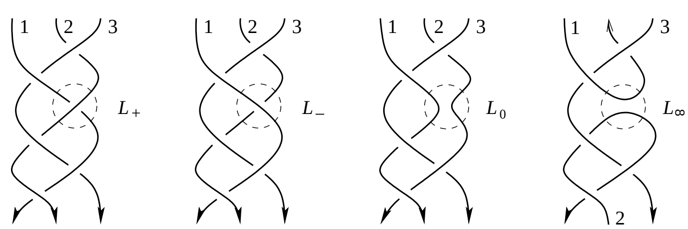
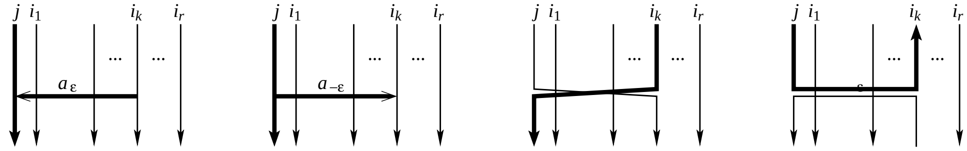


FIGURE 12. Computation of $\mu_{23,1}$ for Borromean rings

Proof. Consider Gauss diagrams G_ε of L_ε, $\varepsilon = \pm$ in a neighborhood of the arrow $a_\pm$ corresponding to the crossing d of $L_\pm$, see Figure 13a.



a b

FIGURE 13. Gauss diagrams which participate in skein relations

Here $\varepsilon = +$ if L_j passes under L_{i_k} in the crossing d of L_+, and $\varepsilon = -$ otherwise. There is an obvious bijective correspondence between tree subdiagrams of G_+ and G_- which do not include $a_\pm$, so these subdiagrams cancel out in pairs in $\langle A, G_+\rangle - \langle A, G_-\rangle$. Since we count only trees with the root on j-th string, the only subdiagrams which contribute to $Z_{I,j}(L_+) - Z_{I,j}(L_-)$ are subdiagrams of G_+ which contain a_+ if $\varepsilon = +$, and subdiagrams of G_- which contain a_- if $\varepsilon = -$. Note that in both cases, the arrow $a_\pm$ is counted with the positive sign (since if $\varepsilon = -1$, it appears in $-Z_{I,j}(L_-)$). Without a loss of generality we may assume that $\varepsilon = +$. Thus

$$Z_{I,j}(L_+) - Z_{I,j}(L_-) = \sum_{A \in \mathcal{A}_{I,j}} \langle A, G_+\rangle_{a_+} \ ,$$

where we used the notation $\langle A, G\rangle_a$ for the sum of all maps $\phi : A \to G$ such that $a \in \mathrm{Im}(\phi)$. See the left hand side of Figure 14.

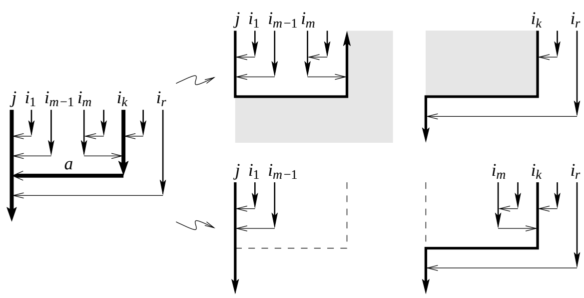


FIGURE 14. Skein relations on Gauss diagrams

Interpreting L_0 and L_∞ in terms of Gauss diagrams as shown in Figure 13b, and using Proposition 4.2, we immediately get equality (7). See the top row of Figure 14.

Subdiagrams which participate in equality (8) are shown in the botom row of Figure 14. To establish (8), it remains to understand why subdiagrams which contain arrows with arrowheads on j under a_+ cancel out in $\sum_{m=1}^{k} Z_{I_m^-,j}(L_\pm) \cdot Z_{I_m^+,i_k}(L_0)$. Fix $1 \geq m \leq k$ and let $G_1 \in \mathcal{A}_{I_m^-,j}$ and $A_2 \in \mathcal{A}_{I_m^+,i_k}$ be a pair of tree arrow diagrams together with maps $\phi_1 : A_1 \to G_+$, $\phi_2 : A_2 \to G_0$. Suppose that one of the subdiagrams $G_1 = \mathrm{Im}(\phi_1(A_1))$ and $G_2 = \mathrm{Im}(\phi_2(A_2))$ of G_+ contains an arrow, which ends on j under a. Denote by a_{bot} the lowest such arrow in $G_1 \cup G_2$ (as we follow j along the orientation). Without a loss of generality, suppose that it belongs to G_1. See Figure 15. Since a_{bot} ends on a common part of the trunk of G_+

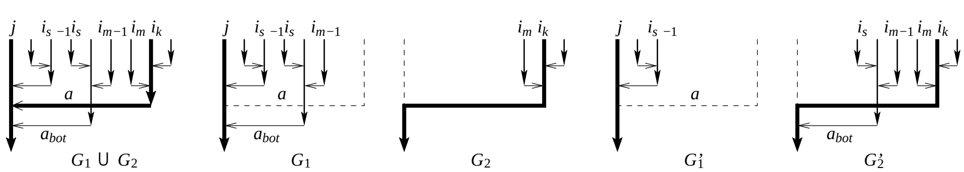


FIGURE 15. Cancellation of subdiagrams with arrows under a

and G_0, we may rearrange pieces of G_1 to get another pair of tree diagrams with the same set of arrows as $G_1 \cup G_2$. Namely, a removal of a_{bot} from G_1 splits it into two connected components G_1' and G_1'', so that G_1' contains strings $j, i_1, \dots, i_{s-1}$ and G_1'' contains strings $i_s, \dots, i_{m-1}$ for some $1 \le s \le m$. Then G_1' is a tree subdiagram of G_+ (with the trunk on j and leaves on I_s^-), and $G_2' := G_1'' \cup a_{bot} \cup G_2$ is a tree subdiagram of G_0 (with the trunk on i_k and leaves on I_s^+). See Figure 15. Their contribution to $Z_{I_s^-,j}(L_\pm) \cdot Z_{I_s^+,i_k}(L_0)$ cancels out with that of G_1 and G_2 to $Z_{I_m^-,j}(L_\pm) \cdot Z_{I_m^+,i_k}(L_0)$. Indeed, while $G_1' \cup G_2'$ contain the same set of arrows as $G_1 \cup G_2$, the arrow a_{bot} is now right-oriented, so is counted with an additional factor of -1. This completes the proof of the theorem. □

4.3. **Identification with Milnor's μ-invariants.** It turns out, that for j either smaller or larger than all indices in I, the tree invariant $Z_{I,j}$ coincides with a Milnor's μ-invariant:

Theorem 4.6. *Let L be an ordered (classical or virtual) tangle on n strings and let $1 \le i_1 < i_2 < \dots < i_r \le n$. Then for any j such that either $1 \le j < i_1$, or $i_r < j \le n$ we have*

$$Z_{I,j}(L) = \mu_{i_1 \dots i_r, j}(L)$$

Proof. Theorem 3.1 of [8] (together with Remark 2.1) implies that $\mu_{i_1 \dots i_r, j}(L)$ satisfies the same skein relation as (7), i.e.

$$\mu_{I,j}(L_+) - \mu_{I,j}(L_-) = \mu_{I_k^-,j}(L_\infty) \cdot \mu_{I_k^+,i_k}(L_0)\ .$$

Moreover, these invariants have the same normalization $Z_{I,j}(L) = \mu_{I,j}(L) = 0$ for any tangle L with j-th component passing everywhere in front of all other components. The skein relation, together with this normalization, completely determines the invariant. □

Corollary 4.7. *Formulas* (5) *and* (6) *define invariants $\mu_{23,1}$ and $\mu_{234,1}$ respectively.*

Example 4.8. If we return to the tangle L of Examples 4.1 and 4.5, shown in Figure 12, we get $\mu_{12,3}(L) = Z_{12,3}(L) = \mu_{23,1}(L) = Z_{23,1} = 1$, in agreement with the fact that the closure $\widehat{L}$ of L is the Borromean link.

Remark 4.9. Note that in the proof of Theorem 3.3 we did not use the realizability of Gauss diagrams in our verification of the invariance under Reidemeister moves of Figure 5, so Theorems 3.3 and 4.6 hold for virtual tangles as well. Recall, however, that in the virtual case there is an alternative definition of "lower" μ-invariants of virtual tangles via the lower tangle group, see Section 2.4. To recover these invariants using Gauss diagram formulas we simply reverse directions of all arrows in the definition of $\mathcal{A}_j$.

5. Operadic structure of the invariants

5.1. **Tree tangles.**

Definition 5.1. A tree tangle L is a $(k,1)$-tangle without closed components. The string ending on the bottom (i.e. $D^2 \times \{0\}$) is called the trunk of L.

We will assume that tree tangles are oriented in such a way, that the trunk starts on the top $D^2 \times \{1\}$ and ends on the bottom $D^2 \times \{0\}$ of C. To simplify the notation, for a tree tangle L with a trunk on j-th string we will denote $Z(L) = Z_j(L)$. There is a natural way to associate to a (k,l)-tangle with a distinguished component a tree tangle by pulling up all but one of its strings. Namely, suppose that the j-th string of a (k,l)-tangle L starts on the top and ends on the bottom. Then L can be made into a tree $(k+l-1,1)$-tangle $\widehat{L}_j$ with the trunk on j by an operation of $j-$capping shown in Figure 16.

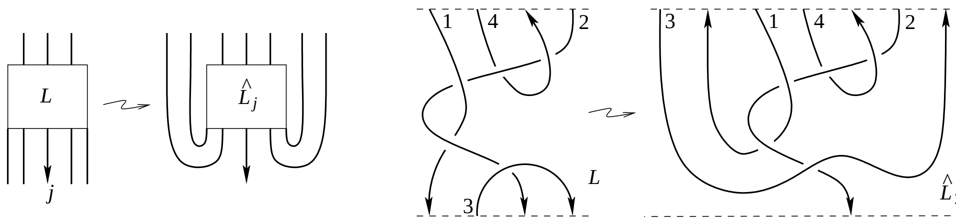


FIGURE 16. Capping a tangle

Gauss diagrams of L and $\widehat{L}_j$ are the same (since crossings of $\widehat{L}_j$ are the same as in L), so their tree invariants coincide: $Z_j(L) = Z(\widehat{L}_j)$.

5.2. **Operadic structure on tree tangles.** Denote by $\mathcal{T}(n)$ the set of tree tangles on n strings. Tree tangles form an operad $\mathcal{T}$. The operadic product

$$\mathcal{T}(n) \times \mathcal{T}(m_1) \times \cdots \times \mathcal{T}(m_n) \to \mathcal{T}(m_1 + \cdots + m_n)$$

is defined as follows. A partial composition $\circ_i : \mathcal{T}(n) \times \mathcal{T}(m) \to \mathcal{T}(n+m-1)$ corresponds to taking a satellite of the i-th component of a tangle:

Definition 5.2. Let $L \in \mathcal{T}(n)$ and $L' \in \mathcal{T}(m)$ be tree tangles, and let $1 \leq i \leq n$. Define a satellite tangle $L \circ_i L' \in \mathcal{T}(n+m-1)$ as follows. Cut out of $C = D^2 \times [0,1]$ a tubular neighborhood $N(L_i)$ of the i-th string L_i of L. Glue back into $C \smallsetminus N(L_i)$ a copy of a cylinder C which contains L', identifying the boundary $\partial D^2 \times [0,1]$ with the boundary of $N(L_i)$ in $C \smallsetminus N(L_i)$ using the zero framing[2] of L_i. See Figure 17. Reorder components of the resulting tree tangle appropriately.

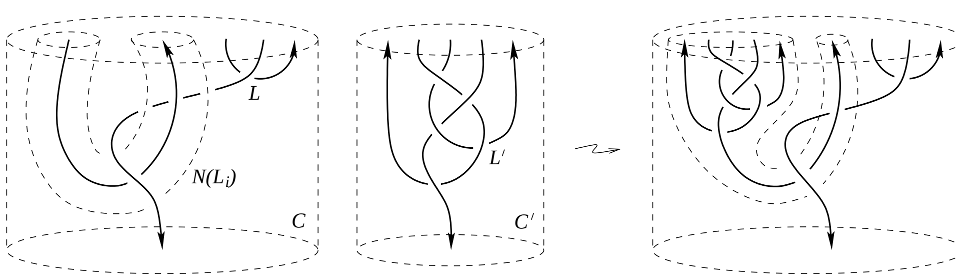


FIGURE 17. Taking a satellite $L \circ_i L'$ of the i-th component of a tree tangle L

Now, given a tangle $L \in \mathcal{T}(n)$ and a collection of n tree tangles $L^1 \in \mathcal{T}(m_1), \ldots, L^n \in \mathcal{T}(m_n)$, we define the composite tangle $L(L^1, \ldots, L^n) \in \mathcal{T}(m_1 + \cdots + m_n)$ by

[2]In fact, the result does not depend on the framing since only one component of L' ends on the bottom of the cylinder.

taking the relevant satellites of all components of L (and reordering the components of the resulting tangle appropriately).

The following theorem directly follows from the definition of the operadic structure on $\mathcal{T}$ and the construction of the map Z from tangles to diassociative trees given by equation (3), Section 3.3.

Theorem 5.3. *The map $Z : \mathcal{T} \to Dias$ is a morphism of operads.*

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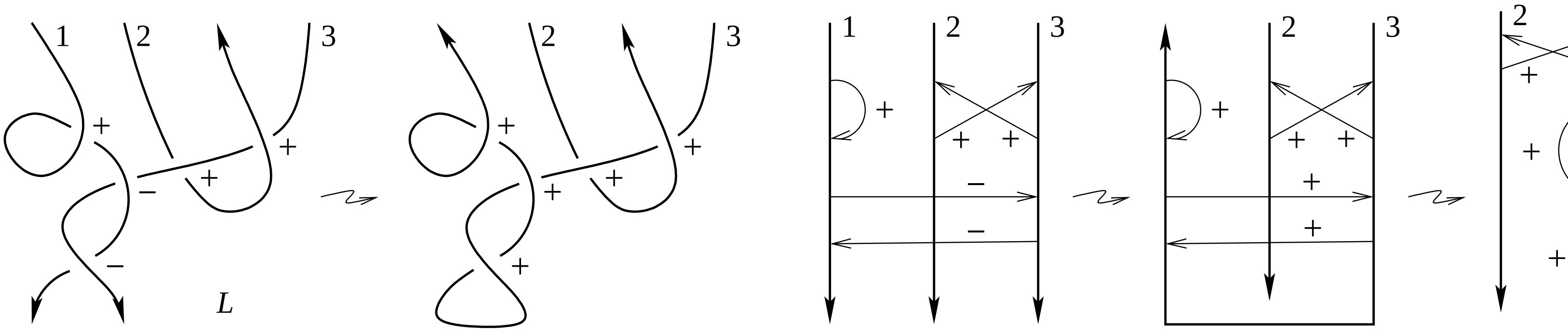

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