arXiv:1011.0120v1 [math.LO] 31 Oct 2010

# GENERIC EXPANSIONS OF COUNTABLE MODELS

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Abstract. We compare two different notions of generic expansions of countable saturated structures. One kind of genericity is related to model-companions and to amalgamation constructions à la Hrushovski–Fraïssé. Another notion of generic expansion is defined via topological properties and Baire category theory. The second type of genericity was first formulated by Truss for automorphisms. We work with a later generalization, due to Ivanov [Ivan], to finite tuples of predicates and functions.

Let $N$ be a countable saturated model of some complete theory $T$, and let $(N,\sigma)$ denote an expansion of $N$ to the signature $L_0$ which is a model of some universal theory $T_0$. Let $T_{\rm mc}$ be the model companion of $T_0$. We prove that $(N,\sigma)$ is Truss-generic if and only if $(N,\sigma)$ is an *e-atomic* model of $T_{\rm mc}$. This answers a question in [Tru2]. When $T$ is $\omega$–categorical and $T_{\rm mc}$ is model-complete, the e-atomic models are simply the atomic models of $T_{\rm mc}$.

## 1. Introduction

In model theory and descriptive set theory there are two main notions of a *generic expansion* of a model. In some cases, the expansions of a given model that one obtains through these notions are similar enough that it is natural to ask whether, and how, they are related.

Let $T$ be a theory with quantifier elimination in a language $L$. Let $L_0 = L \cup \{f\}$, where $f$ is a unary function symbol. Let $T_0$ be $T$ together with the sentences which say that $f$ is an automorphism.

One notion of genericity was introduced by Lascar in [Lasc2]. Lascar constructs some models of $T_0$ that have certain properties of universality and homogeneity. The interpretations of $f$ in these models are called *beaux automorphismes* in [Lasc2], and *generic* automorphisms later on (e.g. [ChaHr] and [ChaPi]). When $T_0$ has a model companion $T_{\rm mc}$, $T_{\rm mc}$ turns out to be the theory of these universal homogeneous models and, in this case, all sufficiently saturated models of $T_{\rm mc}$ are generic automorphisms (see [ChaPi]).

A second notion of genericity was introduced by Truss in [Tru1]. The interpretation of $f$ in a countable model $M \models T_0$ is Truss-generic if its conjugacy class is comeagre in the canonical topology on $\mathrm{Aut}(M)$. More generally, a tuple $(f_1,\dots,f_n) \in \mathrm{Aut}(M)^n$ is generic in this sense if $\{(f_1^g,\dots,f_n^g) : g \in \mathrm{Aut}(M)\}$ is comeagre in the product space $\mathrm{Aut}(M)^n$. Truss-generic automorphisms populate rather different habitats: they are a useful tool in the two main techniques for reconstructing $\omega$–categorical structures from

2000 *Mathematics Subject Classification.* Primary 03C10; Secondary 20B27, 03C50.

*Key words and phrases.* Generic automorphism, homogeneous-universal structure, rich models.

The first author gratefully acknowledges support by the Commission of the European Union under contract MEIF-CT-2005-023302 'Reconstruction and generic automorphisms'.

their automorphism group, namely, the small index property [Lasc1] and Rubin's weak $\forall\exists$-interpretations [Rubin] (see e.g. [HHLS] and [BaMac] for specific applications of Truss generics). The existence of a comeagre conjugacy class is often interesting in its own right: for an $\omega$-categorical structure $M$, it implies that $\mathrm{Aut}(M)$ cannot be written non trivially as a free product with amalgamation [MacTh]. See also a recent paper by Kechris and Rosendal [KeRo] for a wealth of topological consequences in Polish groups.

Ivanov generalises Truss genericity so that it applies to predicates, and indeed to arbitrary finite signatures [Ivan]. His work concerns the relation of 'generic expansions' of $\omega$-categorical structures to generalized quantifiers in the context of second-order logic. Lascar's genericity also applies to predicates: in [ChaPi] the authors show that for a complete $L$-theory $T$, $L_0 = L \cup \{r\}$, where $r$ is a unary relation and $T_0 = T$, $T_0$ has a model companion if and only if $T$ eliminates the $\exists^\infty$ quantifier.

In [Ivan] the structures considered are models of $\omega$-categorical theories. In [KeRo] they are locally finite ultrahomogeneous structures. Both work within the framework of Fraïssé amalgamation classes. Our context is different: we require our base theory $T_0$ to be small and to have a model companion which is a complete theory. By well-known facts (see e.g. [BaZam] for a self-contained introduction), in the context of amalgamation classes our assumptions translate as follows: the existence of a model companion (which we require to have Lascar generics) is equivalent to the saturation of the Fraïssé limit (see for instance Theorem 5.1 in [BaZam]. The completeness of the model companion is a consequence of the joint embedding property and the right amount of amalgamation (see e.g. Corollary 3.10 in [BaZam]).

We work with a given countable saturated model $N \models T$ and we consider the set $\mathrm{Exp}(N, T_0)$ of expansions of $N$ that model $T_0$. We endow $\mathrm{Exp}(N, T_0)$ with a topology which makes it a Polish space. Our topology is the one in [Ivan], a natural generalisation of the canonical topology on $\mathrm{Aut}(N)$. Assume $T_0$ has a model companion $T_{\mathrm{mc}}$. We prove that the expansions of $N$ which model $T_{\mathrm{mc}}$ form a comeagre subset of $\mathrm{Exp}(N, T_0)$.

In Section 2 we also define a set of 'slightly saturated' expansions of $N$ which we call *smooth*. A smooth expansion of $N$ realizes all types of the form $(*)$ $p_{\restriction L}(x) \cup \{\varphi(x)\}$, where $p_{\restriction L}(x)$ is a type in the base language $L$ and $\varphi(x)$ is a quantifier-free formula in the expanded language $L_0$. We prove that smooth expansions are a comeagre subset of $\mathrm{Exp}(N, T_0)$. Finally, in Section 3 we define *e-atomic* expansions. An e-atomic expansion is existentially closed, smooth, and only realizes $p(x)$ if $p_{\restriction\forall}(x) \cup p_{\restriction\exists}(x)$ is isolated by types of the form $\exists y\, p(x, y)$, where $p(x, y)$ is as in $(*)$. We show that e-atomic expansions are exactly the expansions that are generic in the sense of [Tru1]. When $T$ is $\omega$-categorical and $T_{\mathrm{mc}}$ exists, this amounts to showing that the Truss-generic expansions are the atomic models of $T_{\mathrm{mc}}$.

As remarked by an anonymous referee, some of our results appear with different terminology in [Hodg1], where the approach is that of Robinson forcing, so that 'enforceable' corresponds to 'comeagre' in our context. Our original motivation was a comparison between more recent notions of generic automorphisms and led to a different approach. For a smoother comparison with [Hodg1] one should take our $L_0$ to be empty and let $T_0$ be the theory of a pure infinite set. The Henkin constants play the role of the model $N$ in our context. Then the notion of $\exists$-atomic model translates to our $e$-atomic. With this dictionary in mind the reader may compare Lemma 2.4 with Corollary 3.4.3 of [Hodg1] and Theorem 3.6 with Theorem 4.2.6 (cf. also Theorem 5.1.6) of [Hodg1].

The first author is grateful to Alexander Berenstein for helpful initial remarks, and to Enrique Casanovas and Dugald Macpherson for useful conversation.

## 2. Baire categories of first-order expansions

Let $T$ be a complete theory in the countable language $L$. Let $L_0$ be the language $L$ enriched with finitely many new relation and function symbols. We shall denote a structure of signature $L_0$ by a pair $(N, \sigma)$, where $N$ is a structure of signature $L$ and $\sigma$ is the interpretation of the symbols in $L_0 \smallsetminus L$.

Let $T_0$ be any theory of signature $L_0$ containing $T$. We define

$$\mathrm{Exp}(N, T_0) \quad := \quad \Big\{\sigma \ : \ (N, \sigma) \models T_0\Big\}.$$

We write $\mathrm{Exp}(N)$ for $\mathrm{Exp}(N, T)$.

There is a canonical topology on $\mathrm{Exp}(N)$, cf. [Ivan]. For a sentence $\varphi$ with parameters in $N$ we define $[\varphi]_N := \{\sigma : (N, \sigma) \models \varphi\}$. The topology on $\mathrm{Exp}(N)$ is generated by the open sets of the form $[\varphi]_N$ where $\varphi$ is quantifier-free. When $N$ is countable, this topology is completely metrizable: fix an enumeration $\{a_i : i \in \omega\}$ of $N$, define $d(\sigma, \tau) = 2^{-n}$, where $n$ is the largest natural number such that for every tuple $a$ in $\{a_0, \ldots, a_{n-1}\}$ and any symbols $r, f$ in $L_0 \smallsetminus L$,

$$a \in r^\sigma \Leftrightarrow a \in r^\tau \quad \text{and} \quad f^\sigma(a) = f^\tau(a),$$

where $r^\sigma$ is short for $r^{(N,\sigma)}$. When such an $n$ does not exist, $d(\sigma, \tau) = 0$.

The reader may easily verify that this metric is complete. We check that it induces the topology defined above. Fix $n$ and $\tau$. Let $\varphi$ be the conjunction of the formulas of the form $fa = b$ and $ra$ which hold in $(N, \tau)$ for some $b \in N$ and some tuple $a$ from $\{a_0, \ldots, a_n\}$. Then

$$[\varphi]_N \ = \ \{\sigma : d(\sigma, \tau) < 2^{-n}\}.$$

Vice versa, let $\varphi$ be a quantifier-free sentence with parameters in $N$, and take an arbitrary $\tau \in [\varphi]_N$. Let $A$ be the set of parameters occurring in $\varphi$. Let $n$ be large enough that

$$\{t^\tau(a) : a \subseteq A \text{ and } t \text{ is a subterm of a term appearing in } \varphi\} \quad \subseteq \quad \{a_0, \ldots, a_{n-1}\}.$$

Clearly $(N,\sigma) \models \varphi$ for any $\sigma$ at distance $< 2^{-n}$ from $\tau$ so

$$\{\sigma : d(\sigma,\tau) < 2^{-n}\} \subseteq\ [\varphi]_N$$

as required.

If $g : M \to N$ is an isomorphism and $\sigma \in \operatorname{Exp}(M)$ we write $\sigma^g$ for the unique expansion of $N$ that makes $g : (M,\sigma) \to (N,\sigma^g)$ an isomorphism. Explicitly, for every predicate $r$, every function $f$ in $L_0 \smallsetminus L$, and every tuple $a \in N$,

$$(N,\sigma^g) \models r\,a \qquad \Leftrightarrow \qquad (M,\sigma) \models r\,g^{-1}a$$

$$(N,\sigma^g) \models f\,a = b \qquad \Leftrightarrow \qquad (M,\sigma) \models g\,f\,g^{-1}a = b$$

We write $T_{0,\forall}$ for the set of consequences of $T_0$ that are universal modulo $T$. We have $\operatorname{Exp}(N,T_0) \subseteq \operatorname{Exp}(N,T_{0,\forall}) \subseteq \operatorname{Exp}(N)$.

**Notation 2.1.** For the rest of this section we assume $T$ to be small and fix some $N$, a countable saturated model of $T$. In many occasions we shall confuse $\sigma \in \operatorname{Exp}(N)$ with the model $(N,\sigma)$.

**Lemma 2.2.** *Let $T_0$ be an arbitrary expansion of $T$ to the signature $L_0$. Then Exp$(N,T_{0,\forall})$ is the closure of Exp$(N,T_0)$ in the above topology.*

*Proof.* Let $\tau \in \operatorname{Exp}(N,T_{0,\forall})$. We claim that $\tau$ is adherent to $\operatorname{Exp}(N,T_0)$. Let $[\varphi]_N$ be an arbitrary basic open set containing $\tau$. As $(N,\tau)$ models the universal consequences of $T_0$, there exists some $(N',\tau') \models T_0$ such that $(N,\tau) \subseteq (N',\tau')$. Let $A \subseteq N$ be the set of parameters occurring in $\varphi$. We may assume that $N'$ is countable and saturated (in $L$), therefore it is isomorphic to $N$ over $A$, and so $[\varphi]_N$ contains some element of $\operatorname{Exp}(N,T_0)$.

Vice versa, suppose that $\tau \notin \operatorname{Exp}(N,T_{0,\forall})$. Then for some parameter- and quantifier-free formula $\varphi(x)$ we have $T_0 \vdash \forall x\,\varphi(x)$ and $(N,\tau) \models \neg\varphi(a)$. Then the open set $[\neg\varphi(a)]_N$ separates $\tau$ from $\operatorname{Exp}(N,T_0)$. □

**Notation 2.3.** For the rest of this section we fix a theory $T_0$ that is universal modulo $T$ (i.e. equivalent to a universal theory in every model of $T$), so the lemma above $\operatorname{Exp}(N,T_0)$ is a closed subset of $\operatorname{Exp}(N)$, hence it is complete (as a metrizable space). If not otherwise specified, the expansions $\sigma$, $\tau$, etc. range over $\operatorname{Exp}(N,T_0)$.

We say that $\sigma$ is *existentially closed* if every quantifier-free $L_0$–formula with parameters in $N$ that has a solution in some $(U,\upsilon)$ such that $(N,\sigma) \subseteq (U,\upsilon) \models T_0$, has a solution in $(N,\sigma)$.

**Lemma 2.4.** *The set of existentially closed expansions is comeagre in Exp$(N,T_0)$.*

*Proof.* Let $\psi(x)$ be a quantifier-free formula with parameters in $N$. We show that the following set is open dense:

$$(\star)\ \Big\{\sigma\ :\ (N,\sigma)\models\exists x\psi(x)\Big\}\cup\Big\{\sigma\ :\ (U,\upsilon)\not\models\exists x\,\psi(x)\text{ for every }(N,\sigma)\subseteq(U,\upsilon)\models T_0\Big\}.$$

The set of existentially closed expansions is the intersection of these sets as $\psi(x)$ ranges over the quantifier-free formulas of $L_0$. So the lemma will follow.

Take a basic open $[\varphi]_N$ and assume it does not intersect the second set in $(\star)$. Then $(N,\sigma)\subseteq(U,\upsilon)\models T_0\wedge\varphi\wedge\exists x\,\psi(x)$. Let $A\subseteq N$ be a finite set containing the parametes the two formulas. As $U$ can be taken countable and $L$–saturated, it is $L$-isomorphic to $N$ over $A$. So $\psi(x)$ has a solution $b\in N$. Then $(\star)$ contains $[\varphi\wedge\psi(b)]_N$. □

**Definition 2.5.** We say that $\sigma$ is a *smooth expansion* if $(N,\sigma)$ realizes every finitely consistent type of the form $p_{\restriction L}(x)\wedge\psi(x)$ where $\psi(x)$ is quantifier-free and $p(x)_{\restriction L}$ is a type in $L$ with finitely many parameters.

When $T$ is $\omega$-categorical, any expansion is smooth. An example of an expansion that is *not* smooth. Suppose $T$ is the theory of the algebraically closed field of some fixed characteristic and let $N$ be an algebraically closed fields of infinite transcendence degree. The expansion where $r(x)$ holds exactly for the elements of $\mathrm{acl}(\varnothing)$ is not smooth.

**Lemma 2.6.** *The set of smooth expansions is comeagre in Exp$(N,T_0)$.*

*Proof.* The set of smooth expansions is the intersection of sets of the form

$$\Big\{\sigma\ :\ (N,\sigma)\models\exists x\,[p(x)\wedge\psi(x)]\Big\}\cup\Big\{\sigma\ :\ p_{\restriction L}(x)\wedge\psi(x)\text{ not finitely consistent in }(N,\sigma)\Big\},$$

where $p_{\restriction L}(x)\wedge\psi(x)$ is as in Definition 2.5. As $T$ is small, there are countably many of these sets, so suffices to show that they open dense. The argument proceeds as in Lemma 2.4. □

**Example 2.7.** Let $T$ be any complete small theory with quantifier elimination in the language $L$. Let $L_0\smallsetminus L$ contain only a unary relation symbol $r$ and let $T_0=T$. Let $\mathrm{acl}(A)$ denote the algebraic closure in $T$. In [ChaPi] it is proved that if $T$ eliminates the $\exists^\infty$ quantifier, then $T_0$ has a model companion: $T_{\mathrm{mc}}$. In this case, $\mathrm{Exp}(N,T_{\mathrm{mc}})$ is comeagre, by Lemma 2.4.

**Example 2.8.** Let $T$ and $L$ be as in Example 2.7. Let $L_0\smallsetminus L$ contain two unary function symbols $f$ and $f^{-1}$ and let $T_0$ be $T$ together with a sentence which says that $f$ is an automorphism with inverse $f^{-1}$. We need a symbol for the inverse of $f$ because we want $T_0$ to be universal. It is considerably more difficult to find a condition which guarantees the existence of a model-companion [BaShe]. An important example is the case where $T$ is the theory of algebraically closed fields [ChaHr]. Then $T_{\mathrm{mc}}$ is also kown as ACFA. Let $N$ be a countable algebraically closed field of infinite transcendence degree. By Lemma 2.4, $\mathrm{Exp}(N,T_{\mathrm{mc}})$ is comeagre.

## 3. Truss-generic expansions

The notation is as in 2.1 and 2.3. We also assume that $T_0$ has a model companion $T_{\rm mc}$ which is a complete theory. We shall write $Y$ for the set of existentially closed smooth expansions of $N$. From Fact 2.6 we know that $Y$ is a comeagre subset of $\mathrm{Exp}(N, T_0)$. We may consider $Y$ as a topological space in its own right with the topology inherited from $\mathrm{Exp}(N, T_0)$.

If $\varphi(x, y)$ is a quantifier-free formula in $L_0$ and $p(x, y)$ is a type in $L$ then in every smooth model the infinitary formula $\exists y\,[p_{\restriction L}(x, y) \wedge \varphi(x, y)]$ is equivalent to a type. Types of this form are called *existential quasifinite*.

Let $b$ be a finite tuple in $N$. For any $\alpha \in Y$ we define the *1-diagram* of $\alpha$ at $b$

$$\mathrm{diag}_{\restriction 1}(\alpha, b) \quad := \quad \big\{\varphi(b)\ :\ \varphi(x) \text{ universal or existential and } (N, \alpha) \models \varphi(b)\big\},$$

and write $D_b$ for the set of 1-diagrams at $b$. On $D_b$ we define a topology whose basic open sets are of the form

$$[\,\pi(b)\,]_D \quad = \quad \big\{\mathrm{diag}_{\restriction 1}(\alpha, b)\ :\ (N, \alpha) \models \pi(b)\big\},$$

where $\pi(x)$ is any existential quasifinite type. We say that $b$ is an *e-isolated tuple* in $\alpha$ if $\mathrm{diag}_{\restriction 1}(\alpha, b)$ is an isolated point of $D_b$. We may say $b$ is e-isolated *by* $\pi(x)$ in $\alpha$.

It is sometimes convenient to use the syntactic counterpart of $D_b$ which we now define. If $p(x)$ is a complete type, we write $p_{\restriction\forall}(x)$, respectively $p_{\restriction\exists}(x)$ for the set of universal, respectively existential, formulas in $p(x)$. We write $p_{\restriction 1}(x)$ for $p_{\restriction L}(x) \cup p_{\restriction\forall}(x) \cup p_{\restriction\exists}(x)$. We say that a type is realized in $Y$ if it is realized in some $(N, \sigma)$ for some $\sigma \in Y$. Let $S_x$ be the set of types of the form $p_{\restriction 1}(x)$, where $p(x)$ is some complete parameter-free type realized in $Y$. On $S_x$ define the topology where the basic open sets are of the form

$$[\,\pi(x)\,]_S \quad = \quad \Big\{q_{\restriction 1}(x)\ :\ \pi(x)\ \subseteq\ q(x)\Big\},$$

where $\pi(x)$ is some existential quasifinite type, and $q(x)$ ranges over the parameter-free types realized in $Y$. When $[\pi(x)]_S$ isolates $p_{\restriction 1}(x)$ in $S_x$, we say that $p(x)$ is *e-isolated by* $\pi(x)$.

**Fact 3.1.** *Let $b$ be a tuple in $N$ and let $p_{\restriction L}(x)$ be the parameter-free type of $b$ in the language $L$. There is a homeomorphism $h : D_b \to [p_{\restriction L}(x)]_S$. For every existential quasifinte type $\pi(x)$ containing $p_{\restriction L}(x)$, the image under $h$ of the set $[\pi(b)]_D$ is the set $[\pi(x)]_S$.*

*Proof.* Let $h$ be the function that takes the universal diagram $\mathrm{diag}_{\restriction 1}(\alpha, b)$ to the universal type $\{\varphi(x)\ :\ \varphi(b) \in \mathrm{diag}_{\restriction 1}(\alpha, b)\}$.

It is clear that $h$ maps $D_b$ injectively to $S_x$. For surjecivity, let $q(x)$ be a complete parameter-free type realized in $Y$, say $(N, \sigma) \models q(a)$ for some $\sigma \in Y$, and suppose that $q_{\restriction 1}(x)$ belongs to $[\pi(x)]_S$. As $p_{\restriction L}(x) \subseteq q(x)$, there is an isomorphism $g : N \to N$

such that $g(a) = b$. Then $q_{\restriction 1}(x)$ is the image of $\text{diag}_{\restriction 1}(\sigma^g, b)$ under $h$. This proves surjectivity. □

From this fact it is clear that $b$ is e-isolated in $\alpha$ if and only if $p(x)$, the parameter-free type of $b$ in $(N, \alpha)$, is e-isolated. The following lemma is also clear.

**Lemma 3.2.** *Let $p(x)$ be a complete parameter-free type realized in $Y$ and let $\pi(x)$ be an existential quasifinite type such that $p_{\restriction L} \subseteq \pi(x) \subseteq p(x)$. Then the following are equivalent*

1. *$p(x)$ is e-isolated by $\pi(x)$;*
2. *$\pi(x) \ \rightarrow \ p_{\restriction 1}(x)$ holds in every $\sigma \in Y$.*

**Definition 3.3.** Let $\alpha \in Y$. We say that $(N, \alpha)$ is an e-atomic model, or, for short, that $\alpha$ is *e-atomic*, if every finite tuple in $N$ is e-isolated. In other words, $(N, \alpha)$ realizes $p_{\restriction 1}(x)$ only if $p(x)$ is e-isolated.

The notion of e-atomic is close to Ivanov's notion of $(A, \exists)$–atomic in [Ivan] but, since the context is different, a circumstantial comparison is not straightforward.

**Remark 3.4.** As remarked in Section 2, when $T$ is $\omega$–categorical, every expansion is smooth. In this case, if the model companion $T_{\text{mc}}$ of $T_0$ exists, the e-atomic expansions are exactly the atomic models of $T_{\text{mc}}$.

**Theorem 3.5.** *Any two e-atomic expansions are conjugate.*

*Proof.* Let $\alpha$ and $\beta$ be e-atomic. We prove the following claim: any finite 1–elementary partial map $f : (N, \alpha) \rightarrow (N, \beta)$ can be extended to an isomorphisms. Since $T_{\text{mc}}$ is assumed to be complete, the empty map between existentially closed models is elementary, so the theorem follows from the claim.

To prove the claim it suffices to show that for any finite tuple $b$ we can extend $f$ to some 1–elementary map defined on $b$. The claim then follows by back and forth. Let $a$ be an enumeration of $\text{dom} f$. The tuple $a\,b$ is e-isolated in $\alpha$, say by some existential quasifinite type $\pi(v, x)$. Let $p(v, x) = \text{tp}(a, b)$. By fattening $\pi$ if necessary, we may assume that it contains $p_{\restriction L}(v, x)$. Since $\beta$ is smooth and $f$ is 1–elementary, the type $\pi(fa, x)$ is realized in $\beta$, say by $c$. By lemma 3.2, $\pi(v, x) \ \rightarrow p_{\restriction 1}(v, x)$ holds both in $\alpha$ and $\beta$, so $f \cup \{\langle b, c\rangle\}$ gives the required extension. □

**Theorem 3.6.** *If an e-atomic expansion exists, then the set of e-atomic expansions is comeagre in $Exp(N, T_0)$.*

*Proof.* We prove that the set of e-atomic expansions is a dense $G_\delta$ subset of $Y$, hence comeagre in $\text{Exp}(N, T_0)$.

To prove density, let $\psi(x)$ be a parameter- and quantifier-free formula. Let $a \in N$ be such that $\psi(a)$ is consistent with $T_0$. We show that $(N, \alpha) \models \psi(a)$ for some e-atomic

$\alpha$. Write $p_{\restriction L}(x)$ for the parameter-free type of $a$ in the signature $L$. Let $\beta$ be any e-atomic expansion and let $c$ be a realization of $p_{\restriction L}(x) \wedge \psi(x)$ in $(N, \beta)$. Let $g$ be an automorphism of $N$ such that $g(c) = a$. Then $\alpha := \beta^g$ is the required expansion. Hence the set of e-atomic expansions is dense.

We now prove that the set of e-atomic expansions is a $G_\delta$ subset of $Y$. Let $b$ be a finite tuple and denote by $X_b$ the set of expansions in $Y$ where $b$ is e-isolated. It suffices to prove that $X_b$ is an open subset of $Y$.

Let $\alpha \in X_b$ and let $[\pi_\alpha(b)]_D$ be the basic open subset of $D_b$ that isolates $\mathrm{diag}_{\restriction 1}(\alpha, b)$. We may assume $\pi_\alpha(b)$ has the form $\exists y\, [\, p_{\alpha\restriction L}(b, y) \wedge \varphi_\alpha(b, y)\,]$. So let $a_\alpha$ be a witness of the existential quantifier. We have that $Y \cap [\varphi_\alpha(a_\alpha, b)]_N \subseteq X_b$. It follows that

$$Y \;\cap \bigcup_{\alpha \in X_b} [\varphi_\alpha(a_\alpha, b)]_N \;\;=\;\; X_b.$$

Hence $X_b$ is an open subset of $Y$. □

In [Tru1], a notion of generic automorphisms is introduced and a number of examples are given of countable, $\omega$–categorical structures that have generic automorphisms. The following definition, which appears in [Ivan], generalizes the notion of generic automorphisms to arbitrary expansions.

**Definition 3.7.** We say that an expansion $\tau$ is *Truss-generic* if $\{\tau^g \;:\; g \in \mathrm{Aut}(N)\}$ is a comeagre subset of $\mathrm{Exp}(N, T_0)$.

**Remark 3.8.** There is at most one comeagre subset of $\mathrm{Exp}(N, T_0)$ of the form $\{\tau^g \;:\; g \in \mathrm{Aut}(N)\}$. This is because any two sets of this form are either equal or disjoint, and two comeagre sets in a Baire space have nonempty intersection.

**Theorem 3.9.** *Let $\alpha$ be any expansion. Then the following are equivalent:*

1. *$\alpha$ is e-atomic;*
2. *$\alpha$ is Truss-generic.*

*Proof.* Let $\alpha$ be e-atomic. By Theorem 3.6, the set $X$ of e-atomic expansions is comeagre. By Corollary 3.5, and because $X$ is closed under conjugacy by elements of $\mathrm{Aut}(N)$, $X$ is of the form $\{\tau^g \;:\; g \in \mathrm{Aut}(N)\}$ for any e-atomic $\tau$. By Remark 3.8, $X$ is exactly the set of Truss-generic expansions.

Vice versa, let $\alpha$ be Truss-generic. As smoothness and existential closure are guaranteed by Fact 2.6, we only need to prove that $\alpha$ omits $p_{\restriction 1}(x)$ for any complete parameter-free type $p(x)$ that is not e-isolated. It suffices to prove that the set of expansions in $Y$ that omit $p_{\restriction 1}(x)$ is dense $G_\delta$ in $Y$, hence comeagre in $\mathrm{Exp}(N, T_0)$. Then some Truss-generic expansion omits it and, as Truss-generic expansions are conjugated, the same holds for $\alpha$.

Denote by $X_b$ the set of expansions in $Y$ that model $\neg p_{\restriction 1}(b)$. The set of expansions in $Y$ that omit $p_{\restriction 1}(x)$ is the intersection of $X_b$ as the tuple $b$ ranges over $N$. So it suffices to show that $X_b$ is open dense in $Y$.

First we prove density. Let $\psi(a,b)$ be a quantifier-free formula where $a$ and $b$ are disjoint tuples. We need to show that there is an expansion in $Y$ that models $\psi(a,b) \wedge \neg p_{\restriction 1}(b)$. Let $q_{\restriction L}(z,x)$ be the parameter-free type of $a,b$ in the language $L$. We claim that $\psi(z,x) \wedge q_{\restriction L}(z,x) \wedge \neg p_{\restriction 1}(x)$ is consistent in $Y$, say it is realized by $a',b'$ in some expansion $\sigma \in Y$. If not, then $\psi(z,x) \wedge q_{\restriction L}(z,x) \rightarrow p_{\restriction 1}(x)$ holds in every expansion in $Y$, which contradicts that $p(x)$ is not e-isolated and proves the claim. There is an automorphism $g : N \rightarrow N$ such that $g(a'\,b') = a\,b$. We conclude that $\psi(a,b) \wedge \neg p_{\restriction 1}(b)$ holds in $(N, \sigma^g)$.

Now we prove that $X_b$ is open in $Y$. Let $\sigma \in X_b$. We shall show that $\sigma$ belongs to a basic open set contained in $X_b$. If $(N,\sigma) \models \neg p_{\restriction\forall}(b)$ the claim is obvious, so suppose that $(N,\sigma) \models \neg\varphi(b)$ for some existential formula $\varphi(x)$. The expansions in $Y$ are existentially closed, hence (see, for instance, Theorem 7.2.4 in [Hodg2]) there is an existential formula $\psi(x)$, consistent in $(N,\sigma)$, such that $\psi(x) \rightarrow \varphi(x)$ holds for every $\tau \in Y$. Then $\sigma \in [\exists x\, \psi(x)]_N \subseteq X_b$ as required. □

**Theorem 3.10.** *The following are equivalent:*

1. *Truss-generic expansions exist;*
2. *for every finite $b$, the isolated points are dense in $D_b$;*
3. *for every finite $x$, the isolated points are dense in $S_x$.*

*Proof.* The equivalence $2 \Leftrightarrow 3$ is clear by Fact 3.1. Since the existence of e-atomic models implies that isolated points are dense in $S_x$, the implication $1 \Rightarrow 3$ follows from Theorem 3.9. To prove the converse we assume 2 and construct a set $\Delta$ which is the quantifier-free diagram of an e-atomic model.

The diagram $\Delta$ is defined by finite approximations. Assume that at stage $i$ we have a finite set $\Delta_i$ of quantifier-free sentences with parameters in $N$ which is consistent with $T_0$. Below we define $\Delta_{i+1}$. The definition uses a fixed arbitrary enumeration of length $\omega$ of all the types of the form $p_{\restriction L}(x) \cup \{\varphi(x)\}$ with finitely many parameters in $N$, where $\varphi(x)$ quantifier-free. This enumeration exists because $T$ is small by assumption.

If $i$ is even, consider the $i/2$-th type in the given enumeration. If this type is consistent with $T_0 \cup \Delta_i$, let $c$ be such that $T_0 \cup p_{\restriction L}(c) \cup \{\varphi(c)\}$ holds for some expansion and define $\Delta_{i+1} := \Delta_i \cup \{\varphi(c)\}$. Otherwise let $\Delta_{i+1} := \Delta_i$. If $i$ is odd, let $b$ a tuple that enumerates all the parameters in $\Delta_i$. Recall that we have assumed 2, so there is an expansion $\alpha$ which models $\Delta_i$ and such that $\mathrm{diag}_{\restriction 1}(\alpha, b)$ is isolated in $D_b$, say by the type $\exists y\, [p_{\restriction L}(b,y) \wedge \varphi(b,y)]$ where $\varphi(b,y)$ is quantifier-free. Let $a$ satisfy $p_{\restriction L}(b,x) \wedge \varphi(b,x)$ and define $\Delta_{i+1} := \Delta_i \cup \{\varphi(b,a)\}$.

Let $(N, \alpha)$ be the model with diagram $\Delta$. We claim that even stages guarantee both smoothness and existential closure. Smoothness is clear. To prove existential closure observe that if $\varphi(x)$ is a quantifier-free formula with parameters in $N$ that has a solution in some extension of $(N, \alpha)$, then in particular it is consistent with $T_0 \cup \Delta_i$ for every $i$, so at some stage $\varphi(c)$ is added to the diagram of $(N, \alpha)$. Odd stages ensure that every type $p_{\restriction 1}(x)$ realized in $(N, \alpha)$ is e-isolated so 1 follows by Theorem 3.9. □

**Example 3.11.** Truss-generic automorphisms of the random graph. Let $L$ be the language of graphs and let $T$ be the theory of the random graph. Let $L_0$ and $T_0$ be as in Example 2.8. It is known [Kik] that $T_0$ has no model companion. The existence of Truss-generic automorphisms of the random graph has been first proved in [Tru1] and extended to generic tuples in [HHLS], essentially using [Hru2]. These proofs use amalgamation properties of finite structures.

In the case of the random graph we can give a precise description of the isolated tuples. The existence of Truss-generic automorphisms of the random graph follows by the proposition below and Theorem 3.10. This proof is by no means shorter than the one in [HHLS], and it still uses [Hru2].

**Proposition 3.12.** Let $T$ be the theory of the random graph and let $N$ be a countable random graph. Let $L_0$ and $T_0$ be as above (i.e. as in Example 2.8). Then for every finite tuple $b$ in $N$, the e-isolated points in $D_b$ are dense.

*Proof.* By the main result in [Hru2], for every finite set $B$ of a random graph $N$ there is a finite set $A$ such that $B \subseteq A \subseteq N$ and every partial isomorphism $g : N \to N$ with $\operatorname{dom} g, \operatorname{rng} g \subseteq B$ has an extension to an automorphism of $A$.

Let $\psi(b)$ be any existential formula consistent with $T_0$. Let $(N, \alpha)$ be a model that realizes $\psi(b)$. We shall show that $[\psi(b)]_D$ contains an isolated point. By the result in [Hru2] mentioned above, there is a model $(N, \sigma)$ which has a finite substructure $(A, \sigma \restriction A)$ that models $\psi(b)$. We may assume that $\sigma$ is rich. Let $\varphi(a, b)$ be the quantifier-free diagram of $A$ in $(N, \sigma)$. We claim that $\exists z\, \varphi(z, b)$ isolates a point of $D_b$, namely $\operatorname{diag}_{\restriction 1}(\sigma, b)$.

To prove the claim, let $\tau \in Y$ model $\exists z\, \varphi(z, b)$ and prove that $(N, \tau) \equiv_{1,b} (N, \sigma)$. As $\varphi(a, b)$ is a diagram of a substructure we can assume $(N, \tau)$ and $(N, \sigma)$ overlap on $A$ so, as they both are existentially closed and can be amalgamated over $A$, they are 1-elementarily equivalent. □

**Example 3.13.** Cycle-free automorphisms of the random graph. Let $L$, $T$, $N$, and $L_0$ be as in Example 3.11. The theory $T_0$ says that $f$ is an automorphism with inverse $f^{-1}$, and moreover for every positive integer $n$ it contains the axiom $\forall x\, f^n x \neq x$. These axioms claim that $f$ has no finite cycles. It is known [KuMac] that $T_0$ has a model-companion. Now we prove that there is no Truss-generic expansion in $\operatorname{Exp}(N, T_0)$.

Suppose for a contradiction that there exists a Truss-generic expansion $(N, \tau)$. Let $b$ be an element of $N$. As $T$ is $\omega$-categorical, existential quasifinite types are equivalent to existential formulas. So, by Theorem 3.10, there is an existential formula $\varphi(b)$ that isolates $\mathrm{diag}_{\restriction 1}(\tau, b)$ in $D_b$. As the symbol $f^{-1}$ can be eliminated at the cost of a few extra existential quantifiers, we can assume that it does not occur in $\varphi(b)$. Let $n$ be a positive integer which is larger than the number of occurrences of the symbol $f$ in $\varphi(b)$. Denote by $f_\tau$ the interpretation of $f$ in $(N, \tau)$. Let $A \subseteq N$ be a finite set containing $b$ and such that the sets $\{c, f_\tau c, \dots, f_\tau^{n-1} c\}$, for $c \in A$, are pairwise disjoint and let $B$ be the union of all these sets. Clearly we can choose $A$ such that $B$ contains witnesses of all the existential quantifiers in $\varphi(b)$. The latter requirement guarantees that if $\alpha$ is an expansion such that $\alpha \restriction B = \tau \restriction B$, then $(N, \alpha) \models \varphi(b)$. Define $d := f_\tau^n b$ and $e := f_\tau d$. Let $e' \in N$ realize the type $\mathrm{tp}_{\restriction L}(e/f_\tau[B])$ and be such that $r(b, e) \not\leftrightarrow r(b, e')$. As $b \notin f_\tau[B]$, the theory of random graph ensures the existence of such an $e'$. Let $g := f_\tau \restriction B \cup \{\langle d, e' \rangle\}$. We claim that $g : N \to N$ is a partial isomorphism. To prove the claim it suffices to check that $r(a, d) \leftrightarrow r(ga, e')$ for every $a \in B$. We know that $r(a, d) \leftrightarrow r(ga, e)$. As $ga \in f_\tau[B]$, by the choice of $e'$ we have $r(ga, e) \leftrightarrow r(ga, e')$. Then $r(a, d) \leftrightarrow r(ga, e')$ follows. Finally, it is easy to see that the homogeneity of $N$ yields an extension of $g$ to a cycle-free automorphism of $N$, hence an expansion $\alpha$. By construction, $\alpha \restriction B = \tau \restriction B$ so, as observed above, $(N, \alpha) \models \varphi(b)$. But $(N, \tau)$ and $(N, \alpha)$ disagree on the truth of $r(b, f^{n+1} b)$. This contradicts that $\varphi(b)$ isolates $\mathrm{diag}_{\restriction 1}(\tau, b)$.

Example 3.13 shows that the existence of a model companion is not sufficient to guarantee the existence of Truss-generic expansions. The following corollary of Theorem 3.10 gives a sufficient condition.

**Corollary 3.14.** *If $T_0$ is small and $T_{\mathrm{mc}}$ exists, then $N$ has a Truss-generic expansion.*

*Proof.* Modulo $T_{\mathrm{mc}}$ every formula is equivalent to an existential (or, equivalently, to a universal) one, then $S_x$ is the set of all complete parameter-free types consistent with $T_{\mathrm{mc}}$. Though the topology on $S_x$ is not the standard one, the usual argument (e.g. Theorem 4.2.11 of [Mark]) suffices to prove that the isolated types are dense. □

**Remark 3.15.** Question 4 in Section 4 of [Tru2] asks what the precise relation is between Lascar-generic automorphisms (*beaux automorphismes*) and Truss generics. Let $T$ be a theory with quantifier elimination in a language $L$. Let $L_0 = L \cup \{f\}$, where $f$ is a unary function symbol. Let $T_0$ be $T$ together with the sentences which say that $f$ is an automorphism. A Lascar-generic automorphism is the interpretation of $f$ in models that have certain properties of universality, homogeneity and saturation. The existence of Lascar generics is equivalent, under technical hypotheses, to the existence of the model companion of $T_0$. We refer to [Lasc2], [ChaPi], and [BaZam] for details. So Corollary 3.14 says that, when $T_0$ is small, the existence of Lascar generics implies the existence of Truss generics. The assumption that $T_0$ is small is necessary by Example 3.13. Then Theorem 3.9 implies that Truss generic automorphisms are the e-atomic models of the theory of Lascar generics. In particular, when $T$ is $\omega$-categorical, the Truss generic automorphisms are the atomic models of the theory of Lascar generics.

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