A NOTE ON THE PRANDTL LAYERS

YAN GUO, TOAN NGUYEN

Abstract. This note concerns a nonlinear ill-posedness of the Prandtl equation and an invalidity of asymptotic boundary-layer expansions of incompressible fluid flows near a solid boundary. Our analysis is built upon recent remarkable linear ill-posedness results established by Gérard-Varet and Dormy [2], and an analysis in Guo and Tice [5]. We show that the asymptotic boundary-layer expansion is not valid for non-monotonic shear layer flows in Sobolev spaces. We also introduce a notion of Weak Lipschitz well-posedness and prove that the nonlinear Prandtl equation is not well-posed in this sense near non-stationary and non-monotonic shear flows. On the other hand, we are able to verify that Oleinik's monotonic solutions are well-posed.

Contents

1. Introduction

One of classical problems in fluid dynamics is the vanishing viscosity limit of Navier-Stokes solutions near a solid boundary. To describe the problem, let us consider the two-dimensional incompressible Navier-Stokes equations:

$$\begin{aligned}\partial_t \begin{pmatrix} u^\nu \\ v^\nu \end{pmatrix} + (u^\nu \partial_x + v^\nu \partial_y) \begin{pmatrix} u^\nu \\ v^\nu \end{pmatrix} + \nabla p^\nu &= \nu \Delta \begin{pmatrix} u^\nu \\ v^\nu \end{pmatrix} \\ \partial_x u^\nu + \partial_y v^\nu &= 0.\end{aligned} \tag{1.1}$$

Here, $(x,y) \in \mathbb{T}\times\mathbb{R}_+$ and $(u^\nu, v^\nu) \in \mathbb{R}\times\mathbb{R}$ are the tangential and normal components of the velocity, respectively, corresponding to the boundary $y = 0$. We impose the no-slip boundary conditions: $(u^\nu, v^\nu)_{|y=0} = 0$. A natural question is how one relates solutions of the Navier-Stokes equations to those of the Euler equations (i.e., equations (1.1) with $\nu = 0$) with boundary condition $v^0|_{y=0} = 0$ in the zero viscosity limit? Formally, one may expect an asymptotic description as follows:

$$\begin{pmatrix} u^\nu \\ v^\nu \end{pmatrix}(t,x,y) = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix}(t,x,y) + \begin{pmatrix} u_p \\ \sqrt{\nu} v_p \end{pmatrix}(t,x,y/\sqrt{\nu}) \tag{1.2}$$

Date: Last updated: October 23, 2010.

This work is grown out of the previous joint work with David Gérard-Varet [3], and the second author greatly thanks him for many fruitful discussions. Y. Guo's research is supported in part by DMS-0530862 and a Chinese NSF grant.

where (u^0, v^0) solves the Euler equation and (u_p, v_p) is the boundary layer correction that describes the transition near the boundary from zero velocity u^ν of the Navier-Stokes flow to the potentially nonzero velocity u^0 of the Euler flow and thus plays a significant role in the thin layer with order $\mathcal{O}(\sqrt{\nu})$. We may also express the pressure p^ν as

$$p^\nu(t,x,y) = p^0(t,x,y) + p_p(t,x,\frac{y}{\sqrt{\nu}}).$$

We then can formally plug these formal Ansatz into (1.1) and derive the boundary layer equations for (u_p, v_p) at the leading order in $\sqrt{\nu}$. For our convenience, we denote $Y = y/\sqrt{\nu}$ and define

$$\begin{aligned} u(t,x,Y) &:= u^0(t,x,0) + u_p(t,x,Y), \\ v(t,x,Y) &:= \partial_y v^0(t,x,0)Y + v_p(t,x,Y). \end{aligned}$$

The boundary layer or Prandtl equation for (u,v) then reads:

$$\left\{ \begin{array}{rcl} \partial_t u + u\partial_x u + v\partial_Y u - \partial_Y^2 u + \partial_x P &=& 0, \quad Y>0, \\ \partial_x u + \partial_Y v &=& 0, \quad Y>0, \\ u|_{t=0} &=& u_0(x,y) \\ u|_{Y=0} = v|_{Y=0} &=& 0, \\ \lim_{Y\to+\infty} u &=& U(t,x), \end{array} \right. \tag{1.3}$$

where $U = u^0(t,x,0)$ and $P = P(t,x)$ are the normal velocity and pressure describing the Euler flow just outside the boundary layer, and satisfy the Bernoulli equation

$$\partial_t U + U\partial_x U + \partial_x P = 0.$$

This formal idea was proposed by Ludwig Prandtl [7] in 1904 to describe the fluid flows near the boundary. Mathematically, we are interested in the following two problems:

- well-posedness of the Prandtl equation (1.3);
- rigorous justification of the asymptotic boundary layer expansion.

Sammartino and Caflisch [8] resolved these issues in an analytic setting where the initial data and the outer Euler flow are assumed to be analytic functions. Oleinik [6] established the existence and uniqueness of the Cauchy problem (1.3) in a monotonic setting where the initial and boundary data are assumed to be monotonic in y along the boundary-layer profile. For further mathematical results, see the review paper [1]. In this paper, we address the above issues in a Sobolev setting. Our work is based on a recent result of Gérard-Varet and Dormy [2] where they established ill-posedness for the Cauchy problem of the linearized Prandtl equation around non-monotonic shear flows.

In what follows, we shall work with the Euler flow which is constant on the boundary, that is, $U \equiv const$. Also, by a shear flow to the Prandtl, we always mean that a special solution to (1.3) has a form of $(u_s, 0)$ with $u_s = u_s(t,Y)$. Thus, u_s solves the heat equation:

$$\left\{ \begin{array}{rcl} \partial_t u_s &=& \partial_Y^2 u_s, \qquad Y>0, \\ u_s|_{t=0} &=& U_s, \end{array} \right. \tag{1.4}$$

with initial shear layer U_s, and with the same boundary conditions at $Y=0$ and $Y=+\infty$ as in (1.3).

We shall work on the standard Sobolev spaces L^2 and H^m, $m \geq 0$, with usual norms:

$$\|u\|_{L^2_{x,Y}} := \Big(\int_{\mathbb{T}\times\mathbb{R}_+} |u|^2 dxdY \Big)^{1/2} \quad \text{and} \quad \|u\|_{H^m_{x,Y}} := \sum_{k=0}^m \sum_{i+j=k} \|\partial_x^i \partial_Y^j u\|_{L^2}.$$

For initial data, we will often take them to be in a weighted H^m Sobolev spaces. For instance, we say $u_0 \in e^{-\alpha Y} H^m_{x,Y}$ if $e^{\alpha Y} u_0 \in H^m_{x,Y}$ and has a finite norm, for some $\alpha > 0$ (see, for example, [2, 3] where this type of weighted spaces is used for initial data). We occasionally drop the subscripts x, Y in $H^m_{x,Y}$ when no confusion is possible, and write H^m_α to refer to the weighted space $e^{-\alpha Y} H^m_{x,Y}$.

To state our results precisely, we introduce the following definition of well-posedness; here, we say that u belongs to $U + \mathcal{X}$, for some functional space, to mean that $u - U \in \mathcal{X}$.

Definition 1.1 (Weak Lipschitz well-posedness)**.** *For a given Euler flow u^0, denote $U(t,x) = u^0(t,x,0)$. We say the Cauchy problem* (1.3) *is locally Weak Lipschitz well-posed if for some integers $m \geq 1$, there exists a $T > 0$, a continuous function $C(\cdot,\cdot)$, $\delta_0 > 0$, and $\alpha > 0$, such that for any initial data u_0^1, u_0^2 in $U + e^{-\alpha Y} H^m_{x,Y}(\mathbb{T}\times\mathbb{R}_+)$ and $\|e^{\alpha Y}(u_0^1 - u_0^2)\|_{H^m_{x,Y}} \leq \delta_0$, there are unique distributional solutions u_1, u_2 of* (1.3) *in $U + L^\infty(]0,T[; H^1_{x,Y}(\mathbb{T}\times\mathbb{R}_+))$ with initial data $u_j|_{t=0} = u_0^j$, $j = 1, 2$, and there holds*

$$\begin{aligned} &\sup_{0\leq t\leq T} \|u_1(t) - u_2(t)\|_{H^1_{x,Y}} \\ &\qquad \leq C(\|e^{\alpha Y}[u_0^1 - U]\|_{H^m_{x,Y}}, \|e^{\alpha Y}[u_0^2 - U]\|_{H^m_{x,Y}}) \|e^{\alpha Y}[u_0^1 - u_0^2]\|_{H^m_{x,Y}}. \end{aligned} \tag{1.5}$$

We note that when we choose $u_2 \equiv 0$ in the above definition, we obtain an estimate for solutions in the $H^1_{x,Y}$ space. We call such a Lipschitz well-posedness Weak because we allow the initial data to be in $H^m_{x,Y}$ for sufficiently large m.

Our first main result then reads

Theorem 1.2 (No Lipschitz continuity of the flow)**.** *The Cauchy problem* (1.3) *is not locally Weak Lipschitz well-posed in the sense of Definition 1.1.*

Our result is an improvement of a recent result obtained by D. Gérard-Varet and the second author [3] without additional sources in the Prandtl equation. In Section 5, we will show that in the monotonic framework of Oleinik (see Assumption (O) in Section 5), the Cauchy problem (1.3) is well-posed in the sense of Definition 1.1. The key idea is to use the Crocco transformation to obtain certain energy estimates for $\partial_x u$. We note that as shown in [2], the ill-posedness in the non-monotonic case is due to high-frequency in x and the lack of control on $\partial_x u$ in the original coordinates in (1.3).

Finally, regarding the validity of the asymptotic boundary layer expansion, we ask whether one can write

$$\begin{pmatrix} u^\nu \\ v^\nu \end{pmatrix}(t,x,y) = \begin{pmatrix} u^0 - u^0|_{y=0} \\ 0 \end{pmatrix}(y) + \begin{pmatrix} u_s \\ 0 \end{pmatrix}(t, \frac{y}{\sqrt{\nu}}) + (\sqrt{\nu})^\gamma \begin{pmatrix} \tilde{u}^\nu \\ \sqrt{\nu}\tilde{v}^\nu \end{pmatrix}(t, x, \frac{y}{\sqrt{\nu}}), \tag{1.6}$$

and

$$p^\nu(t,x,y) = (\sqrt{\nu})^\gamma \tilde{p}^\nu(t, x, \frac{y}{\sqrt{\nu}}),$$

for shear flows u_s and for some $\gamma > 0$, where $(u^0(y), 0)^t$ is the Euler flow. Our second main result asserts that this is false in general, for all $\gamma > 0$.

Theorem 1.3 (No asymptotic expansions). *For arbitrary $\gamma > 0, m \ge 0, T > 0$, and any large n, there exist a shear flow u_{s_n}, which has a non-degenerate critical point, and initial data $u_0^n, v_0^n \in e^{-\alpha Y} H^m_{x,Y}(\mathbb{T}\times\mathbb{R}_+)$ with*

$$\|e^{\alpha Y}(u_0^n, v_0^n)\|_{H^m_{x,Y}} \le 1$$

such that if the expansion (1.6) *with $(\tilde u^\nu, \tilde v^\nu)|_{t=0} = (u_0^n, v_0^n)$ is valid in $L^\infty([0,T]; H^1_{x,Y}(\mathbb{T}\times\mathbb{R}_+))$ and $\tilde p^\nu \in L^\infty([0,\varepsilon_0]; L^2_{x,Y}(\mathbb{T}\times\mathbb{R}_+))$, then there must hold*

$$\sup_{0\le t\le T} \|\tilde u^\nu(t)\|_{L^2_{x,Y}} \ge n.$$

We note that in Grenier's result [4] on invalidity of asymptotic expansions, he allows the initial perturbation data to be arbitrarily small of size ν^n and shows that in a very short time of size $\sqrt{\nu}\log(1/\nu)$, the solution u grows rapidly to $\mathcal{O}(\nu^{1/4})$ in L^∞. Our result is weaker in capturing how badly the solution grows, but strengthens his result in the sense that the expansion is invalid in order $\mathcal{O}(\nu^\gamma)$, for any $\gamma > 0$. Furthermore, the blow-up norm in [4] is H^1 in the original variable y, whereas our result concerns the norm in the stretched variable $Y = y/\sqrt{\nu}$.

2. Linear ill-posedness

In this section, we recall the previous linear ill-posedness results obtained by Gérard-Varet and Dormy [2] that will be used to prove our nonlinear illposedness. For notational simplicity, we define the linearized Prandtl operator $\mathcal{L}_s$ around a shear flow u_s:

$$\mathcal{L}_s u := -\partial_Y^2 u + u_s\partial_x u + v\partial_Y u_s, \qquad v = -\int_0^Y \partial_x u\, dy'.$$

With our notation, the nonlinear Prandtl equation (1.3) in the perturbation variable $\tilde u := u - u_s$ then reads (dropping the titles):

$$\begin{cases} \partial_t u + \mathcal{L}_s u &=& -u\partial_x u - v\partial_Y u, \qquad Y>0,\\ u|_{t=0} &=& u_0, \end{cases} \tag{2.1}$$

with zero boundary conditions at $Y = 0$ and $Y = \infty$.

Removing the nonlinear term in (2.1), we call the resulting equation as the linearized Prandtl equation around the shear flow u_s:

$$\partial_t u + \mathcal{L}_s u \;=\; 0, \qquad u|_{t=0} \;=\; u_0. \tag{2.2}$$

Denote by $T(s,t)$ the linearized solution operator, that is,

$$T(s,t)u_0 := u(t)$$

where $u(t)$ is the solution to the linearized equation with $u|_{t=s} = u_0$. The following ill-posedness result is for the linearized equation (2.2).

Theorem 2.1 ([2]). *There exists an initial shear layer U_s to* (1.4) *which has a non-degenerate critical point such that for all $\varepsilon_0 > 0$ and all $m \ge 0$, there holds*

$$\sup_{0\le s\le t\le\varepsilon_0} \|T(s,t)\|_{\mathcal{L}(H^m_\alpha, L^2)} = +\infty, \tag{2.3}$$

where $\|\cdot\|_{\mathcal{L}(H^m_\alpha, L^2)}$ denotes the standard operator norm in the functional space $\mathcal{L}(H^m_\alpha, L^2)$ consisting of linear operators from the weighted space $H^m_\alpha = e^{-\alpha Y}H^m$ to the usual L^2 space.

Sketch of proof. In fact, the instability estimate (2.3) stated in [2] was from the weighted space H^m_α to another weighted space $H^{m'}_\alpha$. From their construction, (2.3) remains true when the targeting space is not weighted. We thus sketch their proof where it applies to the usual L^2 space as stated. We recall that the main ingredient in the proof is their construction of approximate growing solutions u^ε to (2.2) such that for all small ε, u^ε solves

$$\partial_t u^\varepsilon + \mathcal{L}_s u^\varepsilon = \varepsilon^M r^\varepsilon,$$

for arbitrary large M, where u^ε and r^ε satisfy:

$$c e^{\theta_0 t/\sqrt{\varepsilon}} \le \|u^\varepsilon(t)\|_{L^2} \le C e^{\theta_0 t/\sqrt{\varepsilon}}, \qquad \|e^{\alpha Y} r^\varepsilon(t)\|_{H^m} \le C\varepsilon^{-m} e^{\theta_0 t/\sqrt{\varepsilon}},$$

for all t in $[0,T]$, $m \ge 0$, and for some $\theta_0, c, C > 0$.

The proof is then by contradiction. That is, we assume that $\|T(s,t)\|_{\mathcal{L}(H^m_w, L^2)}$ is bounded for all $0 \le s \le t \le \varepsilon_0$, for some $\varepsilon_0 > 0$ and some $m \ge 0$. We then introduce $u(t) := T(0,t)u^\varepsilon(0)$, and $v = u - u^\varepsilon$, where u^ε is the growing solution defined above. The function v then satisfies

$$\partial_t v + \mathcal{L}_s v = -\varepsilon^M r_\varepsilon, \quad v|_{t=0} = 0, \tag{2.4}$$

and thus obeys the standard Duhamel representation

$$v(t) \;=\; -\varepsilon^M \int_0^t T(s,t) r^\varepsilon(s)\, ds.$$

Thus, thanks to the bound on the $T(s,t)$ and the remainder r^ε, we get that

$$\|v(t)\|_{L^2} \;\le\; C\varepsilon^M \int_0^t \|e^y r^\varepsilon(s)\|_{H^m}(s) ds \;\le\; C\,\varepsilon^{M+\frac{1}{2}-m}\, e^{\frac{\theta_0 t}{\sqrt{\varepsilon}}}.$$

Also, from the definition of $u(t)$, we have

$$\|u(t)\|_{L^2} = \|T(0,t)u^\varepsilon(0)\|_{L^2} \;\le\; C\|e^{\alpha Y}\, u^\varepsilon(0)\|_{H^m} \;\le\; C\,\varepsilon^{-m}.$$

Combining these estimates together with the lower bound on $u^\varepsilon(t)$, we deduce

$$C\,\varepsilon^{-m} \;\ge\; \|u(t)\|_{L^2} \;\ge\; \|u^\varepsilon(t)\|_{L^2} - \|v(t)\|_{L^2} \;\ge\; \Big(c \,-\, C\,\varepsilon^{M+\frac{1}{2}-m}\Big)\, e^{\frac{\theta_0 t}{\sqrt{\varepsilon}}}.$$

This then yields a contradiction for small enough ε, M large, and $t = K|\ln\varepsilon|\sqrt{\varepsilon}$ with a sufficiently large K. The theorem is therefore proved. □

Next, we also recall the following uniqueness result for the linearized equation.

Proposition 2.2. *([3]) Let $u_s = u_s(t,y)$ be a smooth shear flow satisfying*

$$\sup_{t\ge 0}\Big(\sup_{y\ge 0}|u_s| + \int_0^\infty y|\partial_y u_s|^2 dy\Big) < +\infty.$$

Let $u \in L^\infty(]0,T[; L^2(\mathbb{T}\times\mathbb{R}_+))$ and $\partial_y u \in L^2(0,T\times\mathbb{T}\times\mathbb{R}_+)$ be a solution to the linearized equation of (2.2) around the shear flow, with $u|_{t=0} = 0$. Then, $u \equiv 0$.

Proof. For sake of completeness, we recall here the proof in [3]. Let $w \in L^\infty(]0,T[; L^2(\mathbb{T}\times\mathbb{R}_+))$ and $\partial_y w \in L^2(0,T\times\mathbb{T}\times\mathbb{R}_+)$ be a solution to the linearized equation of (2.2) around the shear flow, with $w|_{t=0} = 0$. Let us define $\hat{w}_k(t,y)$, $k \in \mathbb{Z}$, the Fourier transform of $w(t,x,y)$ in x variable. We observe that for each k, $\hat{w}_k$ solves

$$\left\{\begin{array}{rcl} \partial_t \hat{w}_k + iku_s\hat{w}_k - ik\partial_y u_s \int_0^y \hat{w}_k(y')dy' - \partial_y^2 \hat{w}_k &=& 0\\ \hat{w}_k(t,0) &=& 0\\ \hat{w}_k(0,y) &=& 0. \end{array}\right. \tag{2.5}$$

Taking the standard inner product of the equation (2.5) against the complex conjugate of $\hat w_k$ and using the standard Cauchy–Schwartz inequality to the term $\int_0^y \hat w_k dy'$, we obtain

$$\begin{aligned}\frac12\partial_t\|\hat w_k\|^2_{L^2(\mathbb{R}_+)} + \|\partial_y \hat w_k\|^2_{L^2(\mathbb{R}_+)} &\le |k|\int_0^\infty |u_s||\hat w_k|^2 dy + |k|\int_0^\infty |\partial_y u_s|y^{1/2}|\hat w_k|\|\hat w_k\|_{L^2(\mathbb{R}_+)}dy\\ &\le |k|\Big(\sup_{t,y}|u_s| + \int_0^\infty y|\partial_y u_s|^2 dy\Big)\|\hat w_k\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}$$

Applying the Gronwall lemma into the last inequality yields

$$\|\hat w_k(t)\|_{L^2(\mathbb{R}_+)} \le Ce^{C|k|t}\|\hat w_k(0)\|_{L^2(\mathbb{R}_+)},$$

for some constant C. Thus, $\hat w_k(t)\equiv 0$ for each $k\in\mathbb{Z}$ since $\hat w_k(0)\equiv 0$. That is, $w\equiv 0$, and the proposition is proved. □

3. No asymptotic boundary layer expansions

In this section, we will disprove the nonlinear asymptotic boundary-layer expansion. Our proof is based on the linear ill-posedness result, Theorem 2.1. Let u_s be the shear flow in Theorem 2.1 such that (2.3) holds. Thus, we have that for a fixed $\varepsilon_0 > 0, m\ge 0$, and any large n, there are s_n, t_n with $0\le s_n\le t_n\le\varepsilon_0$ and a sequence of u_0^n such that

$$\|e^{\alpha Y}u_0^n\|_{H^{m+1}} = 1 \qquad\text{and}\qquad \|u_L^n(t_n)\|_{L^2}\ge 2n \tag{3.1}$$

with $u_L^n(t)$ being the solution to the linearized equation (2.2) with $u_L^n(s_n) = u_0^n$.

Now, let $u_{s'}$ be some shear flow (later on, we choose it as a translation of the above u_s). We are then interested in validity of the first order expansion (as compared to (1.2)):

$$\begin{pmatrix}u^\nu\\ v^\nu\end{pmatrix}(t,x,y) = \begin{pmatrix}u^0 - u^0|_{y=0}\\ 0\end{pmatrix}(y) + \begin{pmatrix}u_{s'}\\ 0\end{pmatrix}(t,\frac{y}{\sqrt\nu}) + (\sqrt\nu)^\gamma\begin{pmatrix}\tilde u^\nu\\ \sqrt\nu\tilde v^\nu\end{pmatrix}(t,x,\frac{y}{\sqrt\nu}), \qquad \gamma>0, \tag{3.2}$$

and

$$p^\nu(t,x,y) = (\sqrt\nu)^\gamma\tilde p^\nu(t,x,\frac{y}{\sqrt\nu}),\qquad \gamma>0,$$

where we will take the initial data for such a expansion to be

$$(\tilde u_0^{\nu,n},\tilde v_0^{\nu,n}) := (u_0^n, v_0^n),\qquad\text{with}\quad v_0^n := -\int_0^y \partial_x u_0^n dy'. \tag{3.3}$$

We note that since u_0^n is normalized, $(\tilde u_0^{\nu,n},\tilde v_0^{\nu,n})$ belongs to $e^{-\alpha Y}H^m$ with a finite norm of size independent of n.

We now prove Theorem 1.3 by contradiction. That is, we assume that expansion (3.2) is valid in the Sobolev spaces. That is, for any initial data $\tilde u_0^\nu, \tilde v_0^\nu \in H^m(\mathbb{T}\times\mathbb{R}_+)$, $m\ge 0$, there is a $\varepsilon_0>0$ such that there holds the expansion for $t\in[0,\varepsilon_0]$ with $\tilde u^\nu,\tilde v^\nu\in L^\infty([0,\varepsilon_0];H^1(\mathbb{T}\times\mathbb{R}_+))$, $(\tilde u^\nu,\tilde v^\nu)|_{t=0} = (\tilde u_0^\nu,\tilde v_0^\nu)$, and $\tilde p^\nu\in L^\infty([0,\varepsilon_0];L^2(\mathbb{T}\times\mathbb{R}_+))$. We let $(\tilde u,\tilde v)$ and $\tilde p$ be the weak limits of $(\tilde u^\nu,\tilde v^\nu)$ and $\tilde p^\nu$ in $L^\infty([0,\varepsilon_0];H^1(\mathbb{T}\times\mathbb{R}_+))$ and in $L^\infty([0,\varepsilon_0];L^2(\mathbb{T}\times\mathbb{R}_+))$, respectively, as $\nu\to 0$.

Hence, plugging these expansions into (1.1), we obtain

$$\begin{aligned}\partial_t\tilde u^\nu + (u^0 - u^0|_{y=0} + u_s)\partial_x\tilde u^\nu + \tilde v^\nu(\sqrt\nu\partial_y u^0 + \partial_Y u_s) + \partial_x\tilde p^\nu - \partial_Y^2\tilde u^\nu&\\ = -(\sqrt\nu)^\gamma(\tilde u^\nu\partial_x\tilde u^\nu + \tilde v^\nu\partial_Y\tilde u^\nu) + \nu\partial_x^2\tilde u^\nu + \nu\partial_y^2 u^0&\end{aligned}$$

and

$$\nu\partial_t\tilde v^\nu + \nu(u^0 - u^0|_{y=0} + u_s + (\sqrt{\nu})^\gamma \tilde u^\nu)\partial_x \tilde v^\nu + (\sqrt{\nu})^{\gamma+2}\tilde v^\nu \partial_Y \tilde v^\nu + \partial_Y \tilde p^\nu = \nu^2\partial_x^2\tilde v^\nu + \nu\partial_Y^2\tilde v^\nu$$

We take $\nu\to 0$ in these expressions. Since $(\tilde u^\nu,\tilde v^\nu)(t)$ converges to $(\tilde u,\tilde v)$ weakly in H^1, the nonlinear terms $(\tilde u^\nu\partial_x + \tilde v^\nu\partial_Y)\tilde u^\nu$ and $(\tilde u^\nu\partial_x + \tilde v^\nu\partial_Y)\tilde v^\nu$ have their weak limits in L^1, and thus disappear in the limiting equations due to the factor of $(\sqrt{\nu})^\gamma$. Similar treatments hold for the linear terms. Note that $(u^0 - u^0|_{y=0})(y) = y\partial_y u^0 = \sqrt{\nu}Y\partial_y u^0$ also vanishes in the limit. We thus obtain the following equations for the limits in the sense of distribution:

$$\begin{aligned}\partial_t\tilde u + u_s\partial_x\tilde u + \tilde v\partial_Y u_s - \partial_Y^2\tilde u + \partial_x\tilde p &= 0,\\ \partial_Y\tilde p &= 0\end{aligned} \tag{3.4}$$

and the divergence-free condition for $(\tilde u,\tilde v)$. From the second equation, $\tilde p = \tilde p(t,x)$. Now, setting $Y=+\infty$ in (3.4) and noting that $(\tilde u,\tilde v)$ belong to the H^1 Sobolev space and u_s has a finite limit as $Y\to+\infty$, we must get $\partial_x\tilde p\equiv 0$ in the distributional sense. That is, the next order in the asymptotic expansion solves the linearized Prandtl equation:

$$\partial_t\tilde u + \mathcal{L}_s\tilde u = 0, \qquad \tilde u|_{t=0} = u_0, \tag{3.5}$$

with zero boundary conditions at $Y=0$ and $Y=+\infty$, for arbitrary shear flow $u_s = u_s(t,Y)$.

Now, for n and s_n being fixed as in (3.1), we consider the expansion (3.2) for $u_{s_n} = u_s(t+s_n)$ and initial data $(\tilde u_0^{\nu,n},\tilde v_0^{\nu,n})$ defined as in (3.3). Let $(\tilde u^{\nu,n},\tilde v^{\nu,n})$ be the corresponding solution in the expansion in $L^\infty([0,\varepsilon_0];H^1(\mathbb{T}\times\mathbb{R}_+))$ whose existence is guaranteed by the contradiction assumption. Let $(\tilde u^n,\tilde v^n)$ be their limiting solutions when $\nu\to0$. As shown above, we then obtain the linearized Prandtl equation for $\tilde u^n$ with initial data u_0^n:

$$\partial_t\tilde u^n + \mathcal{L}_{s_n}\tilde u^n = 0, \qquad \tilde u^n|_{t=0} = u_0^n.$$

Thus, if we define $u^n(t) := \tilde u^n(t-s_n)$, the above equation immediately yields

$$\partial_t u^n + \mathcal{L}_s u^n = 0, \qquad u^n|_{t=s_n} = u_0^n,$$

which, by uniqueness of the linear flow, yields $u^n\equiv u_L^n$ on $[s_n,T]$ and

$$\|\tilde u^n(t_n-s_n)\|_{L^2} = \|u^n(t_n)\|_{L^2}\ge 2n.$$

This implies that for small ν, $\|\tilde u^{n,\nu}(t_n-s_n)\|_{L^2}\ge n$. The proof of Theorem 1.3 is complete.

4. Nonlinear ill-posedness

Again, using the previous linear results, Theorem 2.1, we can prove Theorem 1.2 for the nonlinear equation (1.3). We proceed by contradiction. That is, we assume that the Cauchy problem (1.3) is (H^m,H^1) locally Lipschitz well-posed for some $m\ge0$ in the sense of Definition 1.1. Let C,δ_0,T be the constants given in the definition. Let u_s be the fixed shear flow in Theorem 2.1 such that (2.3) holds. By definition, (2.3) yields that for fixed $\varepsilon_0>0$ and any large n, there are s_n,t_n with $0\le s_n\le t_n\le\varepsilon_0$ and a sequence of u_0^n such that

$$\|e^{\alpha Y}u_0^n\|_{H^m} = 1 \qquad\text{and}\qquad \|u_L^n(t_n)\|_{L^2}\ge n \tag{4.1}$$

with $u_L^n(t)$ being the solution to the linearized equation (2.2) with $u_L^n(s_n) = u_0^n$. We now fix n large.

Next, define $v_0^{\delta,n} := u_s(s_n) + \delta u_0^n$, with δ a small parameter less than δ_0. Let $v^{\delta,n}$ be the solution to the nonlinear equation (1.3) with $v^{\delta,n}|_{t=0} = v_0^{\delta,n}$. By the Lipschitz well-posedness applied to two solutions $v^{\delta,n}$ and the shear flow $u_{s_n}(t) := u_s(t+s_n)$, we then obtain

$$\operatorname{ess}\sup_{t\in[0,T]} \|v^{\delta,n}(t) - u_s(t+s_n)\|_{H^1} \le C\,\delta\|e^{\alpha Y}u_0^n\|_{H^m} = C\,\delta,$$

for the constant C as in the well-posedness definition, which is therefore independent of n. In other words, the sequence $u^{\delta,n} := \frac{1}{\delta}(v^{\delta,n} - u_{s_n})$ is bounded in $L^\infty(0,T;H^1(\mathbb{T}\times\mathbb{R}_+))$ uniformly with respect to δ, and moreover it solves

$$\partial_t u^{\delta,n} + \mathcal{L}_{s_n}u^{\delta,n} = \delta N(u^{\delta,n}), \qquad u^{\delta,n}(0) = u_0^n, \tag{4.2}$$

noting that $\mathcal{L}_{s_n}$ is the operator linearized around the shear profile u_{s_n} and N is the nonlinear term: $N(u^{\delta,n}) := -u^{\delta,n}\partial_x u^{\delta,n} - v^{\delta,n}\partial_Y u^{\delta,n}$. From the uniform bound on $u^{\delta,n}$, we deduce that, up to a subsequence,

$$u^{\delta,n} \to u^n \quad L^\infty(0,T;H^1(\mathbb{T}\times\mathbb{R}_+)) \text{ weak}^* \text{ as } \delta\to 0.$$

We shall show that u^n solves the linearized equation (2.2) in the sense of distribution. To see this, we only need to check with the nonlinear term. First, on any compact set K of $\mathbb{R}^+$, we obtain by applying the standard Cauchy inequality and using the divergence-free condition:

$$|v^{\delta,n}| \le \int_0^Y |\partial_x u^{\delta,n}|dY' \le CY^{1/2}\Big(\int_{\mathbb{R}^+}|\partial_x u^{\delta,n}|^2 dY\Big)^{1/2},$$

and

$$\begin{aligned}\int_{\mathbb{T}\times K}|u^{\delta,n}v^{\delta,n}|dY\,dx &\le C_K\int_{\mathbb{T}}\int_K |u^{\delta,n}|\Big(\int_{\mathbb{R}^+}|\partial_x u^{\delta,n}|^2 dY\Big)^{1/2}dY\,dx\\ &\le C_K\Big(\int_{\mathbb{T}}\int_K |u^{\delta,n}|^2 dY\,dx\Big)^{1/2}\Big(\int_{\mathbb{T}}\int_{\mathbb{R}^+}|\partial_x u^{\delta,n}|^2 dY\,dx\Big)^{1/2}\\ &\le C_K\|u^{\delta,n}\|_{H^1}^2,\end{aligned}$$

for some constant C_K depending on K. Now, from the divergence-free condition, we can rewrite $N(u^{\delta,n})$ as

$$N(u^{\delta,n}) = -\partial_x(u^{\delta,n})^2 - \partial_Y(u^{\delta,n}v^{\delta,n})$$

we have, for any smooth function ϕ that is compactly supported in K,

$$\begin{aligned}\delta\Big|\int_{\mathbb{T}\times\mathbb{R}_+}N(u^{\delta,n})\phi\,dx\,dy\Big| &\le C_{K,\phi}\delta\int_{\mathbb{T}\times K}\Big(|u^{\delta,n}|^2 + |u^{\delta,n}v^{\delta,n}|\Big)dx\,dY\\ &\le C_{K,\phi}\delta\|u^{\delta,n}\|_{H^1}^2 \longrightarrow 0,\end{aligned}$$

as $\delta\to 0$, thanks to the uniform bound on $u^{\delta,n}$ in H^1. Here, $C_{K,\phi}$ is some constant that depends on K and $W^{1,\infty}$ norm of ϕ. Thus, the nonlinearity $\delta N(u^{\delta,n})$ converges to zero in the above sense of distribution. This shows that by taking the limits of equation (4.2), u^n solves

$$\partial_t u^n + \mathcal{L}_{s_n}u^n = 0, \quad u^n|_{t=0} = u_0^n.$$

By shifting the time t to $t - s_n$, re-labeling $\tilde u^n(t) := u^n(t-s_n)$, and noting that by definition $\mathcal{L}_{s_n}(t) = \mathcal{L}_s(t+s_n)$, one has

$$\partial_t\tilde u^n + \mathcal{L}_s\tilde u^n = 0, \quad \tilde u^n|_{t=s_n} = u_0^n,$$

that is, $\tilde{u}^n$ solves the linearized equation (2.2) around the shear flow u_s. By uniqueness of the linear flow (recalled in Proposition 2.2), $\tilde{u}^n \equiv u_L^n$ on $[s_n, T]$. This therefore leads to a contradiction due to (4.1) and the fact that the bound for $u^{\delta,n}$ yields a uniform bound for u^n and thus for $\tilde{u}^n$:

$$n \le \|u_L^n(t_n)\|_{L^2} = \|\tilde{u}^n(t_n)\|_{L^2} \le \sup_{t\in[s_n,T]} \|\tilde{u}^n(t)\|_{H^1} \le C,$$

for arbitrarily large n. This completes the proof of Theorem 1.2.

5. Well-posedness of the Oleinik's solutions

In this section, we check that the Oleinik solutions to the Prandtl equation (1.3) are well-posed in the sense of Definition 1.1. Here, since now we only deal with the Prandtl equation, we shall write (x, y) to refer (x, Y) in (1.3), and use both ∂ and subscripts whenever it is convenient to denote corresponding derivatives. To fit into the monotonic framework studied by Oleinik, we make the following assumption on the initial data and outer Euler flow:

(O) Assume that $U(t, x)$ is a smooth positive function and $\partial_x U, \partial_t U/U$ are bounded; the initial data $u_0(x, y)$ is an increasing function in y with $u_0(x, 0) = 0$ and $u_0(x, y) \to U(0, x)$ as $y \to \infty$, and furthermore, for some positive constants θ_0, C_0,

$$\theta_0 \le \frac{\partial_y u_0(x,y)}{U(0,x) - u_0(x,y)} \le C_0. \tag{5.1}$$

We also assume that all functions $\partial_y u_0, \partial_x u_0, \partial_x\partial_y u_0$ are bounded, and so are the ratios $\partial_y^2 u_0/\partial_y u_0$ and $\partial_y^3 u_0 \partial_y u_0/\partial_y^2 u_0$.

We now apply the Crocco change of variables:

$$(t, x, y) \mapsto (t, x, \eta), \qquad \text{with} \quad \eta := \frac{u(t,x,y)}{U(t,x)},$$

and the Crocco unknown function:

$$w(t, x, \eta) := \frac{\partial_y u(t,x,y)}{U(t,x)}.$$

The Prandtl equation (1.3) then yields

$$\left\{ \begin{array}{rcl} \partial_t w + \eta U \partial_x w - A\partial_\eta w - Bw &=& w^2 \partial_\eta^2 w, \qquad 0 < \eta < 1, x \in \mathbb{T} \\ (w\partial_\eta w + \partial_x U + \partial_t U/U)|_{\eta=0} &=& 0, \\ w|_{\eta=1} &=& 0, \end{array} \right. \tag{5.2}$$

with initial conditions: $w|_{t=0} = w_0 = \partial_y u_0/U$. Here,

$$A := (\eta^2 - 1)\partial_x U + (\eta - 1)\frac{\partial_t U}{U}, \qquad B := -\eta \partial_x U - \frac{\partial_t U}{U}.$$

To see how the boundary conditions are imposed, one notes that $\eta = 0$ and $\eta = 1$ correspond to the values at $y = 0$ and $y = +\infty$, respectively. At $y = +\infty$, it is clear that $w = \partial_y u = 0$ since u approaches to $U(t, x)$ as $y \to +\infty$, while by using the imposed conditions on u and v at $y = 0$, we obtain from the equation (1.3) that

$$0 = \partial_y^2 u - \partial_x P = \partial_y w + \partial_x U + \partial_t U/U = w\partial_\eta w + \partial_x U + \partial_t U/U.$$

Theorem 5.1. *([6]) Assume* (O). *Then there exists a $T > 0$ such that the problems* (5.2) *and* (1.3) *have a unique solution w and u on their respective domains, and there hold*

$$\theta_1(1-\eta) \le w(t,x,\eta) \le \theta_2(1-\eta), \qquad |\partial_x w(t,x,\eta)|\,,\ |\partial_t w(t,x,\eta)| \le \theta_2(1-\eta) \tag{5.3}$$

for all $(t,x,\eta) \in [0,T]\times\mathbb{T}\times(0,1)$, and

$$\theta_1 \le \frac{\partial_y u(t,x,y)}{U(t,x)-u(t,x,y)} \le \theta_2, \qquad e^{-\theta_2 y} \le 1-\frac{u(t,x,y)}{U(t,x)} \le e^{-\theta_1 y}, \tag{5.4}$$

for all $(t,x,y) \in [0,T]\times\mathbb{T}\times\mathbb{R}_+$, for some positive constants θ_1,θ_2. In addition, weak derivatives $\partial_t u, \partial_x u, \partial_y\partial_x u, \partial_y^2 u, \partial_y^3 u$ are bounded functions in $[0,T]\times\mathbb{T}\times\mathbb{R}_+$.

Proof. In fact, the authors in [6, Section 4.1, Chapter 4] established the theorem in the case $x \in [0,X]$ with zero boundary conditions at $x = 0$. Their analysis is based on the line method to discretize the t and x variables and to solve a set of second order differential equations in variable η. It is straightforward to check that these lines of analysis work as well in the periodic case $x \in \mathbb{T}$ with minor changes in the choice of boundary conditions. We thus omit to repeat the proof here. □

Using the estimates in Theorem 5.1, we are able to prove that

Theorem 5.2. *The Cauchy problem* (1.3) *under the assumption* (O) *is well-posed in the sense of Definition 1.1, with some constant α and some continuous function $C(\cdot,\cdot)$ appeared in the Weak Lipschitz estimate* (1.5) *that depend on θ_0, C_0 in our assumption* (O).

In the proof, we need the following lemma.

Lemma 5.3. *Under the same assumptions as in Theorem 5.1, we obtain*

$$I(t) \le C I(0), \qquad 0 \le t \le T, \tag{5.5}$$

with

$$I(t) := \int_{\mathbb{T}\times[0,1]} \Big[\frac{|w_{1x}-w_{2x}|^2}{(1-\eta)^\beta} + \frac{|w_1-w_2|^2}{(1-\eta)^\beta}\Big](t,x,\eta)\,dx\,d\eta\Big], \qquad \forall 0 \le \beta < 3,$$

for arbitrary two solutions w_1, w_2 to (5.2).

Proof of Lemma 5.3. We consider w_1, w_2 being solutions to (5.2). We first note that $I(t)$ is well-defined for $\beta < 3$ by the bounds in Theorem 5.1 that $|w_j| \le C(1-\eta)$ and $|w_{jx}| \le C(1-\eta)$. Let us introduce $\phi = w_1 - w_2$. Then, ϕ solves

$$\begin{cases} \phi_t + \eta U\phi_x - A\phi_\eta - B\phi - (w_1+w_2)\partial_\eta^2 w_2 \phi = w_1^2\partial_\eta^2\phi, & 0<\eta<1, x\in\mathbb{T} \\ (w_1\phi_\eta + w_{2\eta}\phi)|_{\eta=0} = 0, \\ \phi|_{\eta=1} = 0, \end{cases}$$

for A, B being defined as in (5.2). In particular, we have $|A| \le C(1-\eta)$ and $|B| \le C$. Multiplying the equation by $e^{-k\eta}\phi/(1-\eta)^\beta$ and integrating it over $\mathbb{T}\times(0,1)$, we easily obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}\times(0,1)} \frac{e^{-k\eta}|\phi|^2}{(1-\eta)^\beta}dx\,d\eta = -\int_{\mathbb{T}\times(0,1)} \Big[\eta U\phi_x - A\phi_\eta - B\phi - (w_1+w_2)\partial_\eta^2 w_2\phi - w_1^2\phi_{\eta\eta}\Big]\frac{e^{-k\eta}\phi}{(1-\eta)^\beta}dx\,d\eta.$$

We treat each term on the right-hand side. Using the bounds on A, B and on $w_j, \partial_\eta^2 w_j$, it is easy to see that

$$\begin{aligned}\Big|\int_{\mathbb{T}\times(0,1)} \Big[\eta U\phi_x - A\phi_\eta - B\phi + (w_1+w_2)\partial_\eta^2 w_2\phi\Big]\frac{e^{-k\eta}\phi}{(1-\eta)^\beta}dxd\eta\Big| \\ \le \epsilon\int_{\mathbb{T}\times(0,1)}\frac{e^{-k\eta}A^2}{(1-\eta)^\beta}|\phi_\eta|^2dxd\eta + C_\varepsilon\int_{\mathbb{T}\times(0,1)}\frac{e^{-k\eta}|\phi|^2}{(1-\eta)^\beta}dxd\eta,\end{aligned}$$

for arbitrary small ε. For the last term, integration by parts yields

$$\begin{aligned}\int_{\mathbb{T}\times(0,1)}\frac{e^{-k\eta}w_1^2}{(1-\eta)^\beta}\phi_{\eta\eta}\phi dxd\eta = &-\int_{\mathbb{T}\times(0,1)}\frac{e^{-k\eta}w_1^2}{(1-\eta)^\beta}|\phi_\eta|^2dxd\eta \\ &-\int_{\mathbb{T}\times(0,1)}\partial_\eta\Big(\frac{e^{-k\eta}w_1^2}{(1-\eta)^\beta}\Big)\phi_\eta\phi dxd\eta - \int_{\mathbb{T}\times\{\eta=0\}}\frac{e^{-k\eta}w_1^2}{(1-\eta)^\beta}\phi_\eta\phi dx.\end{aligned}$$

Again, by integration by parts, we have

$$-\int_{\mathbb{T}\times(0,1)}\partial_\eta\Big(\frac{e^{-k\eta}w_1^2}{(1-\eta)^\beta}\Big)\phi_\eta\phi dxd\eta = \frac12\int_{\mathbb{T}\times(0,1)}\partial_\eta^2\Big(\frac{e^{-k\eta}w_1^2}{(1-\eta)^\beta}\Big)|\phi|^2dxd\eta + \frac12\int_{\mathbb{T}\times\{\eta=0\}}\partial_\eta\Big(\frac{e^{-k\eta}w_1^2}{(1-\eta)^\beta}\Big)|\phi|^2dx.$$

Thanks to the bounds $|w_j|\le C(1-\eta)$, we have

$$\partial_\eta^2\Big(\frac{e^{-k\eta}w_1^2}{(1-\eta)^\beta}\Big)\le C\frac{e^{-k\eta}}{(1-\eta)^\beta}.$$

Collecting all boundary terms, we need to estimate

$$\frac12\int_{\mathbb{T}\times\{\eta=0\}}\Big[-k\frac{e^{-k\eta}w_1^2}{(1-\eta)^\beta}|\phi|^2 + \partial_\eta\Big(\frac{w_1^2}{(1-\eta)^\beta}\Big)e^{-k\eta}|\phi|^2 - \frac{e^{-k\eta}w_1^2}{(1-\eta)^\beta}\phi_\eta\phi\Big]dx.$$

Note that at $\eta=0$, $w_1\neq 0$ and $w_1\phi_\eta = -w_{2\eta}\phi$. Thus, by taking k sufficiently large in the above expression, we can bound it by

$$-\frac{k}{4}\int_{\mathbb{T}\times\{\eta=0\}}w_1^2|\phi|^2dx.$$

Combining the above estimates and choosing ε sufficiently small, with noting that $|A|\le C(1-\eta)\le Cw_1$, we thus obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}\times(0,1)}\frac{e^{-k\eta}|\phi|^2}{(1-\eta)^\beta}dxd\eta + \int_{\mathbb{T}\times(0,1)}\frac{e^{-k\eta}w_1^2}{(1-\eta)^\beta}|\phi_\eta|^2dxd\eta \\ + \int_{\mathbb{T}\times\{\eta=0\}}w_1^2|\phi|^2dx \;\le\; C\int_{\mathbb{T}\times(0,1)}\frac{e^{-k\eta}|\phi|^2}{(1-\eta)^\beta}dxd\eta.\end{aligned}\tag{5.6}$$

To obtain estimates for ϕ_x, we take x-derivative of the equation for ϕ and integrate the resulting equation over $\mathbb{T}\times(0,1)$ against $e^{-k\eta}\phi_x/(1-\eta)^\beta$. We arrive at

(5.7)
$$\begin{aligned}
&\frac12\frac{d}{dt}\int_{\mathbb{T}\times(0,1)}\frac{e^{-k\eta}|\phi_x|^2}{(1-\eta)^\beta}dxd\eta\\
&\qquad = -\int_{\mathbb{T}\times(0,1)}\Big[\eta U\phi_{xx}+\eta U_x\phi_x - A_x\phi_\eta - A\phi_{x\eta}-B_x\phi - B\phi_x\\
&\qquad\quad -((w_1+w_2)\partial_\eta^2 w_2)_x\phi - (w_1+w_2)\partial_\eta^2 w_2\phi_x - w_1^2\phi_{x\eta\eta} - 2w_1w_{1x}\phi_{\eta\eta}\Big]\frac{e^{-k\eta}\phi_x}{(1-\eta)^\beta}dxd\eta.
\end{aligned}$$

Similarly as in deriving the estimate (5.6), integration by parts and the bounds on A, B, w_j easily yields

$$\begin{aligned}
&\int_{\mathbb{T}\times(0,1)}\Big[w_1^2\phi_{x\eta\eta}+2w_1w_{1x}\phi_{\eta\eta}\Big]\frac{e^{-k\eta}\phi_x}{(1-\eta)^\beta}dxd\eta\\
&\qquad\le -\frac12\int_{\mathbb{T}\times(0,1)}\frac{e^{-k\eta}w_1^2}{(1-\eta)^\beta}|\phi_{x\eta}|^2dxd\eta + C\int_{\mathbb{T}\times(0,1)}\frac{|\phi_x|^2+w_1^2|\phi_\eta|^2}{(1-\eta)^\beta}e^{-k\eta}dxd\eta\\
&\qquad\quad -\int_{\mathbb{T}\times(0,1)}\partial_\eta\Big[\frac{e^{-k\eta}\phi_x}{(1-\eta)^\beta}\Big]\phi_{x\eta}\phi_x dxd\eta + \int_{\mathbb{T}\times\{\eta=0\}}\Big[w_1^2\phi_{x\eta}+2w_1w_{1x}\phi_\eta\Big]\phi_x dx.
\end{aligned}\tag{5.8}$$

Here, we note that there is a crucial factor of w_1^2 in front of the term $|\phi_\eta|^2$ thanks to the bounds: $w_j\sim(1-\eta)$ and $|w_{jx}|\le C(1-\eta)$. Again, applying integration by parts to the third term on the right-hand side yields

$$\begin{aligned}
&-\int_{\mathbb{T}\times(0,1)}\partial_\eta\Big[\frac{e^{-k\eta}w_1^2}{(1-\eta)^\beta}\Big]\phi_{x\eta}\phi_x dxd\eta\\
&\qquad = \frac12\int_{\mathbb{T}\times(0,1)}\partial_\eta^2\Big[\frac{e^{-k\eta}w_1^2}{(1-\eta)^\beta}\Big]|\phi_x|^2dxd\eta + \int_{\mathbb{T}\times\{\eta=0\}}\partial_\eta\Big[\frac{e^{-k\eta}w_1^2}{(1-\eta)^\beta}\Big]|\phi_x|^2dx,
\end{aligned}$$

where the last boundary term is clearly bounded by

$$-\frac{k}{2}\int_{\mathbb{T}\times\{\eta=0\}}w_1^2|\phi_x|^2dx.$$

We now estimate the boundary term in (5.8). We recall that at the boundary $\eta=0$, we have $w_1\phi_\eta = -w_{2\eta}\phi$. Thus,

$$w_1^2\phi_{x\eta} = w_1(-w_{2\eta}\phi_x - w_{2x\eta}\phi - w_{1x}\phi_\eta) = -w_1(w_{2\eta}\phi_x + w_{2x\eta}\phi) + w_{1x}w_{2\eta}\phi.$$

That is, the normal derivative ϕ_η on the boundary can always be eliminated to yield

$$\int_{\mathbb{T}\times\{\eta=0\}}\Big[w_1^2\phi_{x\eta}+2w_1w_{1x}\phi_\eta\Big]\phi_x dx \le C\int_{\mathbb{T}\times\{\eta=0\}}(|\phi|^2+|\phi_x|^2)dx.$$

The remaining terms on the right-hand side of (5.7) are again easily bounded by

$$C\int_{\mathbb{T}\times(0,1)}e^{-k\eta}\frac{|\phi|^2+|\phi_x|^2}{(1-\eta)^\beta}dxd\eta.$$

Putting these estimates into (5.7), we have obtained

(5.9)
$$\frac{d}{dt}\int_{\mathbb{T}\times(0,1)}\frac{e^{-k\eta}|\phi_x|^2}{(1-\eta)^\beta}dxd\eta \;\le\; C\int_{\mathbb{T}\times(0,1)}\frac{|\phi|^2+|\phi_x|^2+|w_1|^2|\phi_\eta|^2}{(1-\eta)^\beta}dxd\eta + C\int_{\mathbb{T}\times\{\eta=0\}}|\phi|^2dx.$$

Adding together this inequality with a large constant M times the inequality (5.6), we can get rid of the boundary term and the term involving $|\phi_\eta|^2$ on the right-hand side of (5.9) and thus obtain

$$\frac{d}{dt}\int_{\mathbb{T}\times(0,1)} e^{-k\eta}\frac{M|\phi|^2+|\phi_x|^2}{(1-\eta)^\beta}dxd\eta \;\le\; C(M)\int_{\mathbb{T}\times(0,1)} e^{-k\eta}\frac{M|\phi|^2+|\phi_x|^2}{(1-\eta)^\beta}dxd\eta. \tag{5.10}$$

The claimed estimate (5.5) thus immediately follows from (5.10) by the standard Gronwall inequality, and this completes the proof of Lemma 5.3. □

We are now ready to give

Proof of Theorem 5.2. We only need to check the Lipschitz estimate (1.5). Let $U(t,x)$ be a fixed Euler flow, and take $u_{01}(x,y)$ and $u_{02}(x,y)$ be arbitrary smooth functions satisfying the assumption (O). Let u_1,u_2 be solutions to (1.3) and w_1,w_2 the corresponding solutions to (5.2) constructed by Theorem 5.1. Set $z=u_1-u_2$ and $h=v_1-v_2$ with v_j being determined through the divergence-free condition with u_j. Then, z and h solve

$$\partial_t z + u_1\partial_x z + z\partial_x u_2 + v_1\partial_y z + h\partial_y u_2 = \partial_y^2 z, \qquad h = -\int_0^y \partial_x z dy', \tag{5.11}$$

with $z|_{y=0} = z|_{y=+\infty} = 0$.

Multiplying the equation for z by $e^{-kt}z$ for some large k, taking integration over $\mathbb{T}\times\mathbb{R}_+$, and applying integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}\times\mathbb{R}_+}|z|^2dxdy + \int_{\mathbb{T}\times\mathbb{R}_+}\Big[(k+\partial_x u_2)|z|^2 + \partial_y u_2 hz + |\partial_y z|^2\Big]dxdy \;=\; 0. \tag{5.12}$$

By the definition of h, we can estimate

$$\begin{aligned}\Big|\int_{\mathbb{T}\times\mathbb{R}_+}\partial_y u_2 hzdxdy\Big| &= \Big|\int_{\mathbb{T}\times\mathbb{R}_+}\partial_y(u_2-U)z\Big(\int_0^y\partial_x zdy'\Big)dxdy\Big| \\ &\le \sup_{t,x}\Big(\int_{\mathbb{R}_+}y^{1/2}\partial_y(u_2-U)dy\Big)\|z\|\|\partial_x z\|\end{aligned}$$

for some $\alpha<1/2$, where $\|\cdot\|$ denotes the standard L^2 norm on $\mathbb{T}\times\mathbb{R}_+$. Thanks to bounds (5.4), u_2 converges exponentially to U as $y\to\infty$ and thus the integral $\int_{\mathbb{R}_+}y^{1/2}\partial_y(u_2-U)dy$ is finite. In addition, since the derivatives $\partial_x u_j,\partial_y u_j$ are bounded, by taking k sufficiently large, the identity (5.12) yields

$$\frac{d}{dt}\int_{\mathbb{T}\times\mathbb{R}_+}|z|^2dxdy + \int_{\mathbb{T}\times\mathbb{R}_+}\Big[|z|^2+|z_y|^2\Big]dxdy \;\le\; C\|z_x\|^2. \tag{5.13}$$

We will next derive estimates for z_y. For this, we take derivative with respect to y to the equation for z and multiply the resulting equation by $\partial_y z$. With noting that $z|_{y=0} = 0$ and $z_{yy}|_{y=0} = 0$ (obtained by setting $y = 0$ in (5.11)), easy computations yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}\times\mathbb{R}_+}|z_y|^2dxdy + \int_{\mathbb{T}\times\mathbb{R}_+}|z_{yy}|^2dxdy + \int_{\mathbb{T}\times\mathbb{R}_+}\Big[\frac{1}{2}u_1\partial_x|z_y|^2\\ + (v_{1y}+u_{2x})|z_y|^2 + u_{1y}z_xz_y + u_{2y}h_yz_y + u_{2xy}zz_y + v_1\frac{1}{2}\partial_y|z_y|^2 + u_{2yy}hz_y\Big]dxdy \;=\; 0.\end{aligned}$$

Again, by using the boundedness of u_{jx}, u_{jxy}, u_{jyy}, the divergence-free condition $h_y = -z_x$, and similar estimates on the term involving h as above, we easily get

$$\frac{d}{dt}\int_{\mathbb{T}\times\mathbb{R}_+}|z_y|^2dxdy \;\le\; C\Big(\|z\|^2 + \|z_y\|^2 + \|z_x\|^2\Big). \tag{5.14}$$

We note that by using the fact that the derivatives u_{jx}, u_{jxy}, u_{jyy} are not only bounded, but also decay exponentially in y, similar estimates as done above also yield

$$\frac{d}{dt}\int_{\mathbb{T}\times\mathbb{R}_+}y^n|z_y|^2dxdy \;\le\; C\Big(\|z\|^2 + \|z_y\|^2 + \|z_x\|^2\Big), \qquad \forall n \ge 0. \tag{5.15}$$

Finally, we may wish to give similar estimates for z_x. That is, taking x-derivative to the equation for z, testing the resulting equation by z_x, and using the boundary condition $z_x|_{y=0} = 0$, one may get

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{T}\times\mathbb{R}_+}|z_x|^2dxdy + \int_{\mathbb{T}\times\mathbb{R}_+}|z_{xy}|^2dxdy\\ &\quad + \int_{\mathbb{T}\times\mathbb{R}_+}\Big[(u_{1x}+u_{2x})|z_x|^2 + u_{2xx}zz_x + v_{1x}z_xz_y + u_{2y}h_xz_x + u_{2xy}hz_x\Big]dxdy \;=\; 0.\end{aligned} \tag{5.16}$$

However, it is not at all immediate to estimate the term $u_{2y}h_xz_x$ in the above identity to yield a similar bound as in (5.14) since h has the same order as z_x by its definition (see (5.11)).

Therefore, we shall derive estimates for z_x through the equation (5.2) and the estimates on w obtained in Lemma 5.3. First, we recall that u is defined through w by the relation (see, for example, [6, Eq. (4.1.52)]):

$$y = \int_0^{u(t,x,y)/U(t,x)}\frac{1}{w(t,x,\eta')}d\eta'.$$

Differentiating this identity with respect to x, we immediately obtain[1]

$$u_x = u\frac{U_x}{U} + wU\int_0^{u/U}\frac{w_x}{w^2}(t,x,\eta')d\eta', \tag{5.17}$$

for u and w being solutions to (1.3) and (5.2). We apply this expression to u_1, w_1 and u_2, w_2, respectively and derive an estimate for $z_x = u_{1x} - u_{2x}$. In regions where $u_1 \ge u_2$, it will appear to be convenient to estimate z_x as follows:

$$\begin{aligned}|z_x| \le& C\Big[|z| + |w_1(t,x,u_1/U) - w_2(t,x,u_2/U)|\int_0^{u_2/U}\Big|\frac{w_{2x}}{w_2^2}\Big|(t,x,\eta')d\eta'\\ &+ |w_1|\Big|\int_{u_1/U}^{u_2/U}\frac{w_{1x}}{w_1^2}(t,x,\eta')d\eta'\Big| + |w_1|\int_0^{u_2/U}\Big|\frac{w_{1x}}{w_1^2} - \frac{w_{2x}}{w_2^2}\Big|(t,x,\eta')d\eta'\Big].\end{aligned} \tag{5.18}$$

[1]There is an unfortunate typo in [6, Eq. (4.1.53)] where the integral in (5.17) was $\int_0^{u/U}\frac{w_x}{w}(t,x,\eta')d\eta'$.

Whereas in regions where $u_1 \le u_2$ we estimate

$$\begin{aligned}|z_x| \le & C\Big[|z| + |w_1(t,x,u_1/U) - w_2(t,x,u_2/U)| \int_0^{u_1/U} \Big|\frac{w_{1x}}{w_1^2}\Big|(t,x,\eta')d\eta' \\ & + |w_2|\Big|\int_{u_1/U}^{u_2/U} \frac{w_{2x}}{w_2^2}(t,x,\eta')d\eta'\Big| + |w_2| \int_0^{u_1/U} \Big|\frac{w_{1x}}{w_1^2} - \frac{w_{2x}}{w_2^2}\Big|(t,x,\eta')d\eta'\Big].\end{aligned} \tag{5.19}$$

From the definition $w_j(t,x,u_j/U) = \partial_y u_j(t,x,y)$, we have $|w_1(t,x,u_1/U) - w_2(t,x,u_2/U)| = |z_y|$. Also, note that $|w_{jx}/w_j|$ is uniformly bounded. We have

$$\int_0^{u_j/U} \Big|\frac{w_{jx}}{w_j^2}\Big|(t,x,\eta')d\eta' \ \le\ C\int_0^{u_j/U} \Big|\frac{1}{w_j}\Big|(t,x,\eta')d\eta' = Cy,$$

and

$$|w_j|\Big|\int_{u_1/U}^{u_2/U} \frac{w_{jx}}{w_j^2}(t,x,\eta')d\eta'\Big| \le C|w_j|\Big|\int_{u_1/U}^{u_2/U} \frac{1}{1-\eta'}d\eta'\Big| \le C\Big(\frac{|w_j|}{1-u_1/U} + \frac{|w_j|}{1-u_2/U}\Big)|u_1 - u_2|.$$

Now, if $u_1 \ge u_2$, we use the estimate (5.18) and the fact that $|w_j| \le C(1 - u_j/U)$. We thus obtain

$$\frac{|w_1|}{1-u_1/U} + \frac{|w_1|}{1-u_2/U} \le 2\frac{|w_1|}{1-u_1/U} \le C.$$

Similarly, if $u_1 \le u_2$, we use (5.19) and replace w_1 by w_2 in the above inequality, leading to the similar uniform bound. This explains our choice of expressions in (5.18)-(5.19). By combining these estimates, the second and third terms in (5.18) when $u_1 \ge u_2$ and in (5.19) when $u_1 \le u_2$ are bounded by

$$C(|z| + y|z_y|).$$

Finally, we give estimates for the last term in inequalities (5.18) and (5.19). Using the estimates on w, w_x, we have

$$\Big|\frac{w_{1x}}{w_1^2} - \frac{w_{2x}}{w_2^2}\Big| \le C\frac{|w_{1x} - w_{2x}|}{(1-\eta')^2} + C\frac{|w_1 - w_2|}{(1-\eta')^2}, \qquad \forall \eta' \in (0,1),$$

which together with the standard Hölder inequality implies that

$$\begin{aligned}&\int_{\mathbb{T}\times\mathbb{R}_+} |w_j|^2\Big|\int_0^{u_k/U} \Big(\frac{w_{1x}}{w_1^2} - \frac{w_{2x}}{w_2^2}\Big)(t,x,\eta')d\eta'\Big|^2 dxdy \\ &\le C\sup_{t,x}\int_{\mathbb{R}_+} |\partial_y u_j|^2(1-\frac{u_k}{U})^{\beta-2}dy \int_{\mathbb{T}\times[0,1]} \Big[\frac{|w_{1x}-w_{2x}|^2}{(1-\eta')^\beta} + \frac{|w_1-w_2|^2}{(1-\eta')^\beta}\Big]dxd\eta' \\ &\le C\sup_{t,x}\int_{\mathbb{R}_+} e^{-2\theta_1 y}e^{(3-\beta)\theta_2 y}dy \int_{\mathbb{T}\times[0,1]} \Big[\frac{|w_{1x}-w_{2x}|^2}{(1-\eta')^\beta} + \frac{|w_1-w_2|^2}{(1-\eta')^\beta}\Big]dxd\eta' \\ &\le C\int_{\mathbb{T}\times[0,1]} \Big[\frac{|w_{1x}-w_{2x}|^2}{(1-\eta')^\beta} + \frac{|w_1-w_2|^2}{(1-\eta')^\beta}\Big]dxd\eta',\end{aligned}$$

for some $\beta < 3$ satisfying $(3-\beta)\theta_2 \le \theta_1$.

Thus, we have obtained

$$\|z_x\|_{L^2}^2 \le C\Big[\|z\|_{L^2}^2 + \|yz_y\|_{L^2}^2 + \int_{\mathbb{T}\times[0,1]} \Big[\frac{|w_{1x}-w_{2x}|^2}{(1-\eta)^\beta} + \frac{|w_1-w_2|^2}{(1-\eta)^\beta}\Big]dxd\eta\Big], \tag{5.20}$$

for some $\beta < 3$. Now, applying Lemma 5.3 into (5.20), we then have the following estimate:

$$\|z_x\|^2_{L^2} \le C\Big[\|z\|^2_{L^2} + \|yz_y\|^2_{L^2} + \int_{\mathbb{T}\times[0,1]} \Big[\frac{|w_{1x}-w_{2x}|^2}{(1-\eta)^\beta} + \frac{|w_1-w_2|^2}{(1-\eta)^\beta}\Big](0,x,\eta)dxd\eta\Big]. \tag{5.21}$$

Combining this with estimates (5.13), (5.14), and (5.15) and applying the standard Gronwall's inequality, we easily obtain

$$\|z\|^2_{H^1}(t) \le C(T)\Big[\|z_0\|^2_{H^1} + \|yz_{0y}\|^2_{L^2} + \int_{\mathbb{T}\times[0,1]} \Big[\frac{|w_{1x}-w_{2x}|^2}{(1-\eta)^\beta} + \frac{|w_1-w_2|^2}{(1-\eta)^\beta}\Big](0,x,\eta)dxd\eta\Big], \tag{5.22}$$

where we have denoted $z_0 = u_{01} - u_{02}$.

Note that $\|yz_{0y}\|^2_{L^2} \le \|e^y z_{0y}\|^2_{L^2}$. It thus remains to express the last estimate in terms of initial data u_{01} and u_{02}. We note that for $\eta = u_1(0,x,y)/U(t,x)$,

$$\begin{aligned}|w_1-w_2|(0,x,\eta) &\le |w_1(0,x,u_1/U) - w_2(0,x,u_2/U)| + |w_2(0,x,u_1/U) - w_2(0,x,u_2/U)| \\ &\le |\partial_y(u_1-u_2)(0,x,y)| + |\partial_\eta w_2||u_1-u_2|(0,x,y).\end{aligned}$$

In addition, for $\eta = u_1(0,x,y)/U(t,x)$, assumptions on initial data (see (O)) gives $(1-\eta)^{-1} \le Ce^{\theta_2 y}$ and $|\eta_y| = |\partial_y u_{01}/U| \le C(1-u_{01}/U)$. Thus, we can make change of variable η back to y and estimate

$$\begin{aligned}\int_{\mathbb{T}\times[0,1]} \frac{|w_1-w_2|^2}{(1-\eta)^\beta}(0,x,\eta)dxd\eta \; &\le \; C\int_{\mathbb{T}\times\mathbb{R}_+} e^{(\beta-1)\theta_2 y}(|\partial_y(u_{01}-u_{02})|^2 + |u_{01}-u_{02}|^2)(x,y)\; dxdy \\ &\le \; C\|e^{(\beta-1)\theta_2 y/2}(u_{01}-u_{02})\|^2_{H^1}.\end{aligned}$$

Similarly, we have

$$|w_{1x}-w_{2x}|(0,x,\eta) \; \le \; |\partial_x\partial_y(u_{01}-u_{02})|(x,y) + C|\partial_x(u_{01}-u_{02})|(x,y),$$

and thus

$$\int_{\mathbb{T}\times[0,1]} \frac{|w_{1x}-w_{2x}|^2}{(1-\eta)^\beta}(0,x,\eta)dxd\eta \; \le \; C\|e^{(\beta-1)\theta_2 y/2}(u_{01}-u_{02})\|^2_{H^2}.$$

Putting these into (5.22), we have obtained

$$\|(u_1-u_2)(t)\|^2_{H^1} \; \le \; C\|e^{\alpha y}(u_{01}-u_{02})\|^2_{H^2}, \tag{5.23}$$

for $\alpha = (\beta-1)\theta_2/2$. Theorem 5.2 thus follows. □

References

[1] E, W. Boundary layer theory and the zero-viscosity limit of the Navier-Stokes equation. *Acta Math. Sin. (Engl. Ser.) 16*, 2 (2000), 207–218.

[2] D. Gérard-Varet and E. Dormy, *On the ill-posedness of the Prandtl equation*, J. Amer. Math. Soc. 23 (2010), no. 2, 591–609.

[3] D. Gérard-Varet and T. Nguyen, *Remarks on the ill-posedness of the Prandtl equation*, preprint 2010. arXiv:1008.0532v1.

[4] Grenier, E. On the nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math. 53*, 9 (2000), 1067–1091.

[5] Y. Guo and I. Tice, *Compressible, inviscid Rayleigh-Taylor instability*, Indiana University Mathematics Journal, to appear. arXiv:0911.4098v1.

[6] O. A. Oleinik and V. N. Samokhin, *Mathematical models in boundary layer theory,* vol. 15 of Applied Mathematics and Mathematical Computation. Chapman & Hall/CRC, Boca Raton, FL, 1999.

[7] Prandtl, L. Uber flüssigkeits-bewegung bei sehr kleiner reibung. In *Actes du 3me Congrés international dse Mathématiciens, Heidelberg*. Teubner, Leipzig, 1904, pp. 484–491.
[8] M. Sammartino and R. Caflisch, *Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I-II* Comm. Math. Phys. 192 (1998), no. 2, 433–461.

Division of Applied Mathematics, Brown University, Providence, RI, USA
E-mail address: Yan_Guo@Brown.edu; Toan_Nguyen@Brown.edu