A PBW basis criterion for pointed Hopf algebras*

Michael Helbig†

November 2, 2010

Abstract

We give a necessary and sufficient PBW basis criterion for Hopf algebras generated by skew-primitive elements and abelian group of group-like elements with action given via characters. This class of pointed Hopf algebras has shown great importance in the classification theory and can be seen as generalized quantum groups.
We apply the criterion to classical examples and liftings of Nichols algebras which were determined in [9].

Key Words: Hopf algebras, Nichols algebras, lifting, PBW basis, Gröbner basis

Introduction

In the famous Poincaré-Birkhoff-Witt theorem for universal enveloping algebras of finite-dimensional Lie algebras a class of new bases appeared. Since then many PBW theorems for more general situations were discovered. We want to name those for quantum groups: Lusztig's axiomatic approach [13, 14] and Ringel's approach via Hall algebras [17]. Let us also mention the work of Berger [4], Rosso [18], and Yamane [19].

Our starting point of view is the following: Part of the classification program of finite-dimensional pointed Hopf algebras with the lifting method of Andruskiewitsch and Schneider [1] is the knowledge of the dimension resp. a basis of the deformations of a Nichols algebra (the so-called *liftings*). Another aspect is to find the redundant relations in the ideal. These liftings are among the class we consider here. We want to present a necessary and sufficient PBW basis criterion for Hopf algebras generated by skew-primitive elements and abelian group of group-like elements with action given via characters. This class contains all quantum groups, Nichols algebras and their liftings and it is conjectured that any finite-dimensional pointed Hopf algebra over the complex numbers is of that form.

The very general and for us important work is [11], where a PBW theorem for the here considered class of Hopf algebras is formulated: Kharchenko shows in [11, Thm. 2]

*This work is part of the author's PhD thesis [8] written under the supervision of Prof. H.-J. Schneider.
†eMail: michael@helbig123.de

these Hopf algebras have a PBW basis in special q-commutators, namely the *hard super letters* coming from the theory of Lyndon words, see Section 3. However, the definition of *hard* is not constructive (see also [7, 6] for the word problem for Lie algebras) and in view of treating concrete examples there is a lack of deciding whether a given set of iterated q-commutators establishes a PBW basis.

On the other hand the diamond lemma [5] (see also Section 6, Theorem 6.1) is a very general method to check whether an associative algebra given in terms of generators and relations has a certain basis, or equivalently the relations form a Gröbner basis. As mentioned before, we construct such a Gröbner basis for a character Hopf algebra in Theorem 3.1 and give a necessary and sufficient criterion for a set of super letters being a PBW basis, see Theorem 4.2. The PBW Criterion 4.2 is formulated in the languague of q-commutators. This seems to be the natural setting, since the criterion involves only q-commutator identities of Proposition 1.2; as a side effect we find redundant relations.

The main idea is to combine the diamond lemma with the combinatorial theory of Lyndon words resp. super letters and the q-commutator calculus of Section 1. In order to apply the diamond lemma we give a general construction to identify a smash product with a quotient of a free algebra, see Proposition 5.5 in Section 5.

Further the PBW Criterion 4.2 is a generalization of [4] and [3, Sect. 4] in the following sense: In [4] a condition involving the q-Jacoby identity for the generators x_i occurs (it is called "q-Jacobi sum"). However, this condition can be formulated more generally for iterated q-commutators (not only for x_i), so also higher than quadratic relations can be considered. The intention of [4] was a q-generalization of the classical PBW theorem, so powers of q-commutators are not covered at all and also his algebras do not contain a group algebra. On the other hand, [3, Sect. 4] deals with powers of q-commutators (root vector relations) and also involves the group algebra. But here it is assumed that the powers of the commutators lie in the group algebra and fulfill a certain centrality condition. As mentioned above these assumptions are in general not preserved; in the PBW Criterion 4.2 the centrality condition is replaced by a more general condition involving the restricted q-Leibniz formula of Proposition 1.2.

This work is organized as follows: In Section 1 we develop a general calculus for q-commutators in an arbitrary algebra, which is needed throughout the thesis; new formulas for q-commutators are found in Proposition 1.2. We recall in Section 2 the theory of Lyndon words, super letters and super words. We show that the set of all super words can be seen indeed as a set of words, i.e., as a free monoid. In Section 3 we recall the result of [10] about a structural description of the here considered Hopf algebras, in terms of generators and relations. With this result we are able to formulate in Section 4 the main result of this work, namely the PBW basis criterion. Sections 5 to 7 are dedicated to the proof of the criterion. Finally in Sections 8 and 9 we apply the PBW Criterion 4.2 to classical examples and the liftings of Nichols algebras obtained in [9].

1 q-commutator calculus

In this section let A denote an arbitrary algebra over a field $\Bbbk$ of characteristic $\operatorname{char}\Bbbk = p \geq 0$. The main result of this section is Proposition 1.2, which states important q-commutator formulas in an arbitrary algebra.

1.1 q-calculus

For every $q \in \Bbbk$ we define for $n \in \mathbb{N}$ and $0 \leq i \leq n$ the *q-numbers* $(n)_q := 1 + q + q^2 + \ldots + q^{n-1}$, the *$q$-factorials* $(n)_q! := (1)_q(2)_q \ldots (n)_q$, and the *$q$-binomial coefficients* $\binom{n}{i}_q := \frac{(n)_q!}{(n-i)_q!(i)_q!}$. Note that the latter right-handside is well-defined since it is a polynomial over $\mathbb{Z}$ evaluated in q. We denote the *multiplicative order* of any $q \in \Bbbk^\times$ by $\operatorname{ord} q$. If $q \in \Bbbk^\times$ and $n > 1$, then

$$\binom{n}{i}_q = 0 \text{ for all } 1 \leq i \leq n-1 \Longleftrightarrow \begin{cases} \operatorname{ord} q = n, & \text{if } \operatorname{char}\Bbbk = 0 \\ p^k \operatorname{ord} q = n \text{ with } k \geq 0, & \text{if } \operatorname{char}\Bbbk = p > 0, \end{cases} \tag{1.1}$$

see [15, Cor. 2]. Moreover for $1 \leq i \leq n$ there are the *q-Pascal identities*

$$q^i \binom{n}{i}_q + \binom{n}{i-1}_q = \binom{n}{i}_q + q^{n+1-i} \binom{n}{i-1}_q = \binom{n+1}{i}_q, \tag{1.2}$$

and the *q-binomial theorem*: For $x, y \in A$ and $q \in \Bbbk^\times$ with $yx = qxy$ we have

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i}_q x^i y^{n-i}. \tag{1.3}$$

Note that for $q = 1$ these are the usual notions.

1.2 q-commutators

For all $a, b \in A$ and $q \in \Bbbk$ we define the *q-commutator*

$$[a, b]_q := ab - qba.$$

The q-commutator is bilinear. If $q = 1$ we get the classical commutator of an algebra. If A is graded and a, b are homogeneous elements, then there is a natural choice for the q. We are interested in the following special case:

Example 1.1. Let $\theta \geq 1$, $X = \{x_1, \ldots, x_\theta\}$, $\langle X \rangle$ the free monoid and $A = \Bbbk\langle X \rangle$ the free $\Bbbk$-algebra. For an abelian group Γ let $\widehat{\Gamma}$ be the character group, $g_1, \ldots, g_\theta \in \Gamma$ and $\chi_1, \ldots, \chi_\theta \in \widehat{\Gamma}$. If we define the two monoid maps

$$\deg_\Gamma : \langle X \rangle \to \Gamma,\ \deg_\Gamma(x_i) := g_i \quad \text{and} \quad \deg_{\widehat{\Gamma}} : \langle X \rangle \to \widehat{\Gamma},\ \deg_{\widehat{\Gamma}}(x_i) := \chi_i,$$

for all $1 \leq i \leq \theta$, then $\Bbbk\langle X\rangle$ is Γ- and $\widehat{\Gamma}$-graded.

Let $a \in \Bbbk\langle X\rangle$ be Γ-homogeneous and $b \in \Bbbk\langle X\rangle$ be $\widehat{\Gamma}$-homogeneous. We set

$$g_a := \deg_\Gamma(a), \quad \chi_b := \deg_{\widehat{\Gamma}}(b), \quad \text{and} \quad q_{a,b} := \chi_b(g_a).$$

Further we define $\Bbbk$-linearly on $\Bbbk\langle X\rangle$ the q-commutator

$$[a,b] := [a,b]_{q_{a,b}}. \tag{1.4}$$

Note that $q_{a,b}$ is a bicharacter on the homogeneous elements and depends only on the values

$$q_{ij} := \chi_j(g_i) \text{ with } 1 \leq i,j \leq \theta.$$

For example $[x_1,x_2] = x_1x_2 - \chi_2(g_1)x_2x_1 = x_1x_2 - q_{12}x_2x_1$. Further if a,b are $\mathbb{Z}^\theta$-homogeneous they are both Γ- and $\widehat{\Gamma}$-homogeneous. In this case we can build iterated q-commutators, like $\big[x_1,[x_1,x_2]\big] = x_1[x_1,x_2] - \chi_1\chi_2(g_1)[x_1,x_2]x_1 = x_1[x_1,x_2] - q_{11}q_{12}[x_1,x_2]x_1$.

Later we will deal with algebras which still are $\widehat{\Gamma}$-graded, but not Γ-graded such that Eq. (1.4) is not well-defined. However, the q-commutator calculus, which we next want to develop, will be a major tool for our calculations such that we need the general definition with the q as an index.

Proposition 1.2. *For all $a,b,c,a_i,b_i \in A$, $q,q',q'',q_i,\zeta \in \Bbbk$, $1 \leq i \leq n$ and $r \geq 1$ we have:*

(1) q-derivation properties:

$$[a,bc]_{qq'} = [a,b]_q c + qb[a,c]_{q'}, \qquad [ab,c]_{qq'} = a[b,c]_{q'} + q'[a,c]_q b,$$

$$[a,b_1\dots b_n]_{q_1\dots q_n} = \sum_{i=1}^n q_1\dots q_{i-1} b_1\dots b_{i-1}[a,b_i]_{q_i} b_{i+1}\dots b_n,$$

$$[a_1\dots a_n,b]_{q_1\dots q_n} = \sum_{i=1}^n q_{i+1}\dots q_n a_1\dots a_{i-1}[a_i,b]_{q_i} a_{i+1}\dots a_n.$$

(2) q-Jacobi identity:

$$\big[[a,b]_{q'},c\big]_{q''q} = \big[a,[b,c]_q\big]_{q'q''} - q'b[a,c]_{q''} + q[a,c]_{q''}b.$$

(3) q-Leibniz formulas:

$$[a,b^r]_{q^r} = \sum_{i=0}^{r-1} q^i \binom{r}{i}_\zeta b^i \big[\dots\big[[a,\underbrace{b]_q,b\big]_{q\zeta}\dots,b}_{r-i}\big]_{q\zeta^{r-i-1}},$$

$$[a^r,b]_{q^r} = \sum_{i=0}^{r-1} q^i \binom{r}{i}_\zeta \big[\underbrace{a,\dots\big[a,[a}_{r-i},b]_q\big]_{q\zeta}\dots\big]_{q\zeta^{r-i-1}} a^i.$$

(4) restricted q-Leibniz formulas: *If* $\operatorname{char} k = 0$ *and* $\operatorname{ord}\zeta = r$, *or* $\operatorname{char} \Bbbk = p > 0$ *and* $p^k \operatorname{ord}\zeta = r$, *then*

$$[a, b^r]_{q^r} = \big[\dots \big[\underbrace{[a, b]_q, b\big]_{q\zeta} \dots, b}_{r}\big]_{q\zeta^{r-1}},$$

$$[a^r, b]_{q^r} = \big[\underbrace{a, \dots \big[a, [a}_{r}, b]_q\big]_{q\zeta} \dots\big]_{q\zeta^{r-1}}.$$

Proof. (1) The first part is a direct calculation, e.g.

$$[a, bc]_{qq'} = abc - qq'bca = abc - qbac + qbac - qq'bca = [a, b]_q c + qb[a, c]_{q'}.$$

The second part follows by induction.
(2) Using the $\Bbbk$-linearity and (1) we get the result immediately.
(3) By induction on r: $r = 1$ is obvious, so let $r \geq 1$. Using (1) we get

$$[a, b^{r+1}]_{q^{r+1}} = [a, b^r b]_{q^r q} = [a, b^r]_{q^r} b + q^r b^r [a, b]_q.$$

By induction assumption $[a, b^r]_{q^r} b = \sum_{i=0}^{r-1} q^i \binom{r}{i}_\zeta b^i \big[\dots \big[\underbrace{[a, b]_q, b\big]_{q\zeta} \dots, b}_{r-i}\big]_{q\zeta^{r-i-1}} b$, where

$$b^i \big[\dots \big[\underbrace{[a, b]_q, b\big]_{q\zeta} \dots, b}_{r-i}\big]_{q\zeta^{r-i-1}} b =$$

$$b^i \big[\dots \big[\underbrace{[a, b]_q, b\big]_{q\zeta} \dots, b}_{r+1-i}\big]_{q\zeta^{r-i}} + q\zeta^{r-i} b^{i+1} \big[\dots \big[\underbrace{[a, b]_q, b\big]_{q\zeta} \dots, b}_{r-i}\big]_{q\zeta^{r-i-1}}.$$

In total we get

$$[a, b^{r+1}]_{q^{r+1}} = \sum_{i=0}^{r} q^i \binom{r}{i}_\zeta b^i \big[\dots \big[\underbrace{[a, b]_q, b\big]_{q\zeta} \dots, b}_{r+1-i}\big]_{q\zeta^{r-i}}$$
$$+ \sum_{i=0}^{r-1} q^{i+1} \binom{r}{i}_\zeta \zeta^{r-i} b^{i+1} \big[\dots \big[\underbrace{[a, b]_q, b\big]_{q\zeta} \dots, b}_{r-i}\big]_{q\zeta^{r-i-1}}.$$

Shifting the index of the second sum and using Eq. (1.2) for ζ we get the formula. The second formula is proven in the same way. (4) Follows from (3) and Eq. (1.1). □

2 Lyndon words and q-commutators

In this section we recall the theory of Lyndon words [12, 16] as far as we are concerned and then introduce the notion of super letters and super words [11].

2.1 Words and the lexicographical order

Let $\theta \geq 1$, $X = \{x_1, x_2, \ldots, x_\theta\}$ be a finite totally ordered set by $x_1 < x_2 < \ldots < x_\theta$, and $\langle X \rangle$ the free monoid; we think of X as an alphabet and of $\langle X \rangle$ as the words in that alphabet including the empty word 1. For a word $u = x_{i_1} \ldots x_{i_n} \in \langle X \rangle$ we define $\ell(u) := n$ and call it the *length* of u.

The *lexicographical order* $\leq$ on $\langle X \rangle$ is defined for $u, v \in \langle X \rangle$ by $u < v$ if and only if either v begins with u, i.e., $v = uv'$ for some $v' \in \langle X \rangle \backslash \{1\}$, or if there are $w, u', v' \in \langle X \rangle$, $x_i, x_j \in X$ such that $u = wx_iu'$, $v = wx_jv'$ and $i < j$. E.g., $x_1 < x_1x_2 < x_2$.

2.2 Lyndon words and the Shirshov decomposition

A word $u \in \langle X \rangle$ is called a *Lyndon word* if $u \neq 1$ and u is smaller than any of its proper endings, i.e., for all $v, w \in \langle X \rangle \backslash \{1\}$ such that $u = vw$ we have $u < w$. We denote by

$$\mathcal{L} := \{u \in \langle X \rangle \mid u \text{ is a Lyndon word}\}$$

the set of all Lyndon words. For example $X \subset \mathcal{L}$, but $x_i^n \notin \mathcal{L}$ for all $1 \leq i \leq \theta$ and $n \geq 2$. Also x_1x_2, $x_1x_1x_2$, $x_1x_2x_2$, $x_1x_1x_2x_1x_2 \in \mathcal{L}$.

For any $u \in \langle X \rangle \backslash X$ we call the decomposition $u = vw$ with $v, w \in \langle X \rangle \backslash \{1\}$ such that w is the minimal (with respect to the lexicographical order) ending the *Shirshov decomposition* of the word u. We will write in this case

$$\mathrm{Sh}(u) = (v|w).$$

E.g., $\mathrm{Sh}(x_1x_2) = (x_1|x_2)$, $\mathrm{Sh}(x_1x_1x_2x_1x_2) = (x_1x_1x_2|x_1x_2)$, $\mathrm{Sh}(x_1x_1x_2) \neq (x_1x_1|x_2)$. If $u \in \mathcal{L} \backslash X$, this is equivalent to w is the longest proper ending of u such that $w \in \mathcal{L}$.

Definition 2.1. We call a subset $L \subset \mathcal{L}$ *Shirshov closed* if $X \subset L$, and for all $u \in L$ with $\mathrm{Sh}(u) = (v|w)$ also $v, w \in L$.

For example $\mathcal{L}$ is Shirshov closed, and if $X = \{x_1, x_2\}$, then $\{x_1, x_1x_1x_2, x_2\}$ is not Shirshov closed, whereas $\{x_1, x_1x_2, x_1x_1x_2, x_2\}$ is.

2.3 Super letters and super words

Let the free algebra $\Bbbk\langle X \rangle$ be graded as in Section 1.1. For any $u \in \mathcal{L}$ we define recursively on $\ell(u)$ the map

$$[\,.\,] : \mathcal{L} \to \Bbbk\langle X \rangle, \quad u \mapsto [u]. \tag{2.1}$$

If $\ell(u) = 1$, then set $[x_i] := x_i$ for all $1 \leq i \leq \theta$. Else if $\ell(u) > 1$ and $\mathrm{Sh}(u) = (v|w)$ we define $[u] := \big[[v], [w]\big]$. This map is well-defined since inductively all $[u]$ are $\mathbb{Z}^\theta$-homogeneous such that we can build iterated q-commutators; see Section 1.1. The elements

$[u] \in \Bbbk\langle X\rangle$ with $u \in \mathcal{L}$ are called *super letters*. E.g. $[x_1x_1x_2x_1x_2] = \big[[x_1x_1x_2],[x_1x_2]\big] = \big[[x_1,[x_1,x_2]],[x_1,x_2]\big]$. If $L \subset \mathcal{L}$ is Shirshov closed then the subset of $\Bbbk\langle X\rangle$

$$[L] := \big\{[u] \,\big|\, u \in L\big\}$$

is a set of iterated q-commutators. Further $[\mathcal{L}] = \big\{[u] \,\big|\, u \in \mathcal{L}\big\}$ is the set of all super letters and the map $[\,.\,] : \mathcal{L} \to [\mathcal{L}]$ is a bijection, which follows from [10, Lem. 2.5]. Hence we can define an order $\leq$ of the super letters $[\mathcal{L}]$ by

$$[u] < [v] :\Leftrightarrow u < v,$$

thus $[\mathcal{L}]$ is a new alphabet containing the original alphabet X; so the name "letter" makes sense. Consequently, products of super letters are called *super words*. We denote

$$[\mathcal{L}]^{(\mathbb{N})} := \big\{[u_1]\dots[u_n] \,\big|\, n \in \mathbb{N},\, u_i \in \mathcal{L}\big\}$$

the subset of $\Bbbk\langle X\rangle$ of all super words. Any super word has a unique factorization in super letters [10, Prop. 2.6], hence we can define the lexicographical order on $[\mathcal{L}]^{(\mathbb{N})}$, as defined above on regular words. We denote it also by $\leq$.

2.4 A well-founded ordering of super words

The *length* of a super word $U = [u_1][u_2]\dots[u_n] \in [L]^{(\mathbb{N})}$ is defined as $\ell(U) := \ell(u_1u_2\dots u_n)$.

Definition 2.2. For $U, V \in [\mathcal{L}]^{(\mathbb{N})}$ we define $U \prec V$ by

- $\ell(U) < \ell(V)$, or
- $\ell(U) = \ell(V)$ and $U > V$ lexicographically in $[\mathcal{L}]^{(\mathbb{N})}$.

This defines a total ordering of $[\mathcal{L}]^{(\mathbb{N})}$ with minimal element 1. As X is assumed to be finite, there are only finitely many super letters of a given length. Hence every nonempty subset of $[\mathcal{L}]^{(\mathbb{N})}$ has a minimal element, or equivalently, $\preceq$ fulfills the descending chain condition: $\preceq$ is *well-founded*. This makes way for inductive proofs on $\preceq$.

2.5 The free monoid $\langle X_L\rangle$

Let $L \subset \mathcal{L}$. We want to stress the two different aspects of a super letter $[u] \in [L]$:

- On the one hand it is by definition a polynomial $[u] \in \Bbbk\langle X\rangle$.
- On the other hand, as we have seen, it is a letter in the alphabet $[L]$.

To distinguish between these two point of views we define for the latter aspect a new alphabet corresponding to the set of super letters $[L]$: To be technically correct we regard the free monoid $\langle 1,\dots,\theta\rangle$ of the ciphers $\{1,\dots,\theta\}$ (telephone numbers), together with the trivial bijective monoid map $\nu : \langle x_1,\dots,x_\theta\rangle \to \langle 1,\dots,\theta\rangle,\ x_i \mapsto i$ for all $1 \leq i \leq \theta$. Hence we can transfer the lexicographical order to $\langle 1,\dots,\theta\rangle$. The image $\nu(\mathcal{L})\subset\langle 1,\dots,\theta\rangle$ can be seen as the set of "Lyndon telephone numbers". We define the set

$$X_L := \{x_u \mid u\in\nu(L)\}.$$

Note that if $X\subset L$ (e.g. $L\subset\mathcal{L}$ is Shirshov closed), then $X\subset X_L$. E.g., if $X=\{x_1,x_2\}\subset L=\{x_1,x_1x_2,x_2\}$ then $\nu(L)=\{1,12,2\}$ and $X\subset X_L=\{x_1,x_{12},x_2\}$.

Notation 2.3. From now on we will not distinguish between L and $\nu(L)$ and write for example x_u instead of $x_{\nu(u)}$ for $u\in L$. In this manner we will also write $g_{\nu(u)},\chi_{\nu(u)}$ equivalently for g_u,χ_u if $u\in L$, as defined in Example 1.1. E.g. $g_{112}=g_{x_1x_1x_2}=g_{x_1}g_{x_1}g_{x_2}=g_1g_1g_2$, $\chi_{112}=\chi_{x_1x_1x_2}=\chi_{x_1}\chi_{x_1}\chi_{x_2}=\chi_1\chi_1\chi_2$.

As seen in [10, Prop. 2.6] we have the bijection of super words and the free monoid $\langle X_L\rangle$

$$\rho:[L]^{(\mathbb{N})}\to\langle X_L\rangle,\quad \rho\big([u_1]\dots[u_n]\big):=x_{u_1}\dots x_{u_n}. \tag{2.2}$$

E.g., $[x_1x_2x_2][x_1x_2]\overset{\rho}{\mapsto}x_{122}x_{12}$. Hence we can transfer all orderings to $\langle X_L\rangle$: For all $U,V\in\langle X_L\rangle$ we set

$$\ell(U):=\ell(\rho^{-1}(U)),\quad U<V:\Leftrightarrow\rho^{-1}(U)<\rho^{-1}(V),\quad U\prec V:\Leftrightarrow\rho^{-1}(U)\prec\rho^{-1}(V).$$

3 A class of pointed Hopf algebras

In this chapter we deal with the class of pointed Hopf algebras for which we give the PBW basis criterion. Let us recall the notions and results of [11, Sect. 3]: A Hopf algebra A is called a *character Hopf algebra* if it is generated as an algebra by elements $a_1,\dots,a_\theta$ and an abelian group $G(A)=\Gamma$ of all group-like elements such that for all $1\le i\le\theta$ there are $g_i\in\Gamma$ and $\chi_i\in\widehat{\Gamma}$ with

$$\Delta(a_i)=a_i\otimes 1+g_i\otimes a_i\qquad\text{and}\qquad ga_i=\chi_i(g)a_ig.$$

As mentioned in the introduction this covers a wide class of examples of Hopf algebras.

Theorem 3.1. [10, Thm. 3.4] *If A is a character Hopf algebra, then*

$$A\cong(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])/I,$$

where the smash product $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ and the ideal I are constructed in the following way:

3.1 The smash product $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$

Let $\Bbbk\langle X\rangle$ be Γ- and $\widehat{\Gamma}$-graded as in Section 1.1, and $\Bbbk[\Gamma]$ be endowed with the usual bialgebra structure $\Delta(g)=g\otimes g$ and $\varepsilon(g)=1$ for all $g\in\Gamma$. Then we define

$$g\cdot x_i:=\chi_i(g)x_i,\quad\text{for all } 1\le i\le\theta.$$

In this case, $\Bbbk\langle X\rangle$ is a $\Bbbk[\Gamma]$-module algebra and we calculate $gx_i = \chi_i(g)x_ig$, $gh = hg = \varepsilon(g)hg$ in $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$. Further $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ is a Hopf algebra with structure determined for all $1 \le i \le \theta$ and $g \in \Gamma$ by

$$\Delta(x_i) := x_i \otimes 1 + g_i \otimes x_i \qquad \text{and} \qquad \Delta(g) := g \otimes g.$$

3.2 Ideals associated to Shirshov closed sets

In this subsection we fix a Shirshov closed $L \subset \mathcal{L}$. We want to introduce the following notation for an $a \in \Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ and $W \in [\mathcal{L}]^{(\mathbb{N})}$: We will write $a \prec_L W$ (resp. $a \preceq_L W$), if a is a linear combination of

- $U \in [L]^{(\mathbb{N})}$ with $\ell(U) = \ell(W)$, $U > W$ (resp. $U \ge W$), and
- Vg with $V \in [L]^{(\mathbb{N})}$, $g \in \Gamma$, $\ell(V) < \ell(W)$.

Furthermore, we set for each $u \in L$ either $N_u := \infty$ or $N_u := \operatorname{ord} q_{u,u}$ (resp. $N_u := p^k \operatorname{ord} q_{u,u}$ with $k \ge 0$ if $\operatorname{char}\Bbbk = p > 0$) and we want to distinguish the following two sets of words depending on L:

$$C(L) := \big\{ w \in \langle X\rangle \backslash L \mid \exists u, v \in L : w = uv,\ u < v,\ \text{and } \operatorname{Sh}(w) = (u|v) \big\},$$
$$D(L) := \big\{ u \in L \mid N_u < \infty \big\}.$$

Note that $C(L) \subset \mathcal{L}$ and $D(L) \subset L \subset \mathcal{L}$ are sets of Lyndon words. For example, if $L = \{x_1, x_1x_1x_2, x_1x_2, x_2\}$, then $C(L) = \{x_1x_1x_1x_2,\ x_1x_1x_2x_1x_2,\ x_1x_2x_2\}$.

Moreover, let $c_w \in (\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi_w}$ for all $w \in C(L)$ such that $c_w \prec_L [w]$; and let $d_u \in (\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi_u^{N_u}}$ for all $u \in D(L)$ such that $d_u \prec_L [u]^{N_u}$. Then let I be the $\widehat{\Gamma}$-homogeneous ideal of $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ generated by the following elements:

$$[w] - c_w \qquad \text{for all } w \in C(L), \tag{3.1}$$
$$[u]^{N_u} - d_u \qquad \text{for all } u \in D(L). \tag{3.2}$$

4 A PBW basis criterion

In this section we want to state a PBW basis criterion which is applicable for any character Hopf algebra. Suppose we have a smash product $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ together with an ideal I as in Sections 3.1 and 3.2.

At first we need to define several algebraic objects for the formulation of the PBW Criterion 4.2. The main idea is not to work in the free algebra $\Bbbk\langle X\rangle$ but in the free algebra $\Bbbk\langle X_L\rangle$ where $\langle X_L\rangle$ is the free monoid of Section 2.5.

4.1 The free algebra $\Bbbk\langle X_L\rangle$ and $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$

In Section 2.5 we associated to a super letter $[u]\in[L]$ a new variable $x_u\in X_L$, where X_L contains X. Hence the free algebra $\Bbbk\langle X_L\rangle$ also contains $\Bbbk\langle X\rangle$. We define the action of Γ on $\Bbbk\langle X_L\rangle$ and q-commutators by

$$\begin{aligned} g\cdot x_u &:= \chi_u(g)x_u && \text{for all } g\in\Gamma, u\in L,\\ [x_u,x_v] &:= x_ux_v - q_{u,v}x_vx_u && \text{for all } u,v\in L. \end{aligned}$$

In this way $\Bbbk\langle X_L\rangle$ becomes a $\Bbbk[\Gamma]$-module algebra and $gx_u=\chi_u(g)x_ug$ in the smash product $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$.

4.2 The subspace $I_{\prec U}\subset\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$

Via ρ of Eq. (2.2) we now define certain elements of $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$: For all $w\in C(L)$ resp. $u\in D(L)$ we write $c_w=\sum\alpha U+\sum\beta Vg\prec_L[w]$ resp. $d_u=\sum\alpha'U'+\sum\beta'V'g'\prec_L[u]^{N_u}$, with $\alpha,\alpha',\beta,\beta'\in\Bbbk$ and $U,U',V,V'\in[L]^{(\mathbb{N})}$ (such decompositions may not be unique; we just fix one). Then we define in $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$

$$c_w^\rho:=\sum\alpha\rho(U)+\sum\beta\rho(V)g \quad\text{resp.}\quad d_u^\rho:=\sum\alpha'\rho(U')+\sum\beta'\rho(V')g'.$$

For all $u,v\in L$ with $u<v$ we define elements $c^\rho_{(u|v)}\in\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$: If $w=uv$ and $\mathrm{Sh}(w)=(u|v)$ we set

$$c^\rho_{(u|v)}:=\begin{cases} x_w, & \text{if } w\in L,\\ c^\rho_w, & \text{if } w\notin L.\end{cases}$$

Else if $\mathrm{Sh}(w)\neq(u|v)$ let $\mathrm{Sh}(u)=(u_1|u_2)$. Then we define inductively on the length of $\ell(u)$

$$c^\rho_{(u|v)}:=\partial^\rho_{u_1}(c^\rho_{(u_2|v)})+q_{u_2,v}c^\rho_{(u_1|v)}x_{u_2}-q_{u_1,u_2}x_{u_2}c^\rho_{(u_1|v)}, \tag{4.1}$$

where $\partial^\rho_{u_1}$ is defined $\Bbbk$-linearly by

$$\begin{aligned} \partial^\rho_{u_1}(x_{l_1}\dots x_{l_n}) &:= c^\rho_{(u_1|l_1)}x_{l_2}\dots x_{l_n}+\sum_{i=2}^n q_{u_1,l_1\dots l_{i-1}}x_{l_1}\dots x_{l_{i-1}}\big[x_{u_1},x_{l_i}\big]x_{l_{i+1}}\dots x_{l_n},\\ \partial^\rho_{u_1}(\rho(V)g) &:= \big[x_{u_1},\rho(V)\big]_{q_{u_1,u_2v}\chi_{u_1}(g)}g. \end{aligned}$$

For any $U\in\langle X_L\rangle$ let $I_{\prec U}$ denote the subspace of $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$ spanned by the elements

$$\begin{aligned} &Vg\big([x_u,x_v]-c^\rho_{(u|v)}\big)Wh && \text{for all } u,v\in L, u<v,\\ &V'g'\big(x_u^{N_u}-d_u^\rho\big)W'h' && \text{for all } u\in L, N_u<\infty \end{aligned}$$

with $V,V',W,W'\in\langle X_L\rangle$, $g,g',h,h'\in\Gamma$ such that

$$Vx_ux_vW\prec U \qquad\text{and}\qquad V'x_u^{N_u}W'\prec U.$$

Finally we want to define the following elements of $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$ for $u,v,w\in L$, $u<v<w$, resp. $u\in L$, $N_u<\infty$, $u\le v$, resp. $v<u$:

$$\begin{aligned}
J(u<v<w) &:= [c^\rho_{(u|v)}, x_w]_{q_{uv,w}} - [x_u, c^\rho_{(v|w)}]_{q_{u,vw}} \\
&\qquad + q_{u,v}x_v[x_u,x_w] - q_{v,w}[x_u,x_w]x_v,\\
L(u,u<v) &:= \big[\underbrace{x_u,\dots[x_u}_{N_u-1}, c^\rho_{(u|v)}]_{q_{u,u}q_{u,v}}\dots\big]_{q_{u,u}^{N_u-1}q_{u,v}} - [d^\rho_u, x_v]_{q_{u,v}^{N_u}},\\
L(u,u\le v) &:= \begin{cases} L(u,u<v), & \text{if } u<v,\\ L(u) := -[d^\rho_u, x_u]_1, & \text{if } u=v,\end{cases}\\
L(u,v<u) &:= \big[\dots[c^\rho_{(v|u)}, \underbrace{x_u]_{q_{v,u}q_{u,u}}\dots, x_u}_{N_u-1}\big]_{q_{v,u}q_{u,u}^{N_u-1}} - [x_v, d^\rho_u]_{q_{v,u}^{N_u}}.
\end{aligned}$$

Remark 4.1. Note that

$$J(u<v<w)\in\big([x_u,x_v]-c^\rho_{(u|v)}, [x_v,x_w]-c^\rho_{(v|w)}\big)$$

by the q-Jacobi identity of Proposition 1.2, and

$$L(u,u\le v)\in\big([x_u,x_v]-c^\rho_{(u|v)},\ x_u^{N_u}-d^\rho_u\big),\quad L(u,v<u)\in\big([x_v,x_u]-c^\rho_{(v|u)},\ x_u^{N_u}-d^\rho_u\big)$$

by the restricted q-Leibniz formula of Proposition 1.2.

4.3 The PBW criterion

Theorem 4.2. *Let $L\subset\mathcal{L}$ be Shirshov closed and I be an ideal of $\Bbbk\langle X\rangle\#\Bbbk[\Gamma]$ as in Section 3.2. Then the following assertions are equivalent:*

(1) *The residue classes of $[u_1]^{r_1}[u_2]^{r_2}\dots[u_t]^{r_t}g$ with $t\in\mathbb{N}$, $u_i\in L$, $u_1>\dots>u_t$, $0<r_i<N_{u_i}$, $g\in\Gamma$, form a $\Bbbk$-basis of the quotient algebra $(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])/I$.*

(2) *The algebra $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$ respects the following conditions:*
   (a) q-Jacobi condition: *$\forall\ u,v,w\in L$, $u<v<w$:*

$$J(u<v<w)\in I_{\prec x_ux_vx_w}.$$

   (b) restricted q-Leibniz conditions: *$\forall\ u,v\in L$ with $N_u<\infty$, $u\le v$ resp. $v<u$:*
      (i) *$L(u,u\le v)\in I_{\prec x_u^{N_u}x_v}$, resp.*
      (ii) *$L(u,v<u)\in I_{\prec x_vx_u^{N_u}}$,*

(2') *The algebra $\Bbbk\langle X_L\rangle\#\Bbbk[\Gamma]$ respects the following conditions:*
   (a) *Condition* (2a) *only for $uv\notin L$ or $\mathrm{Sh}(uv)\neq(u|v)$.*
   (b) (i) *Condition* (2bi) *only for $u=v$ and $u<v$ where $v\neq uv'$ for all $v'\in L$.*
      (ii) *Condition* (2bii) *only for $v<u$ where $v\neq v'u$ for all $v'\in L$.*

We need to formulate several statements over the next sections. Afterwards the proof of Theorem 4.2 will be carried out in Section 7.

5 $(\Bbbk\langle X\rangle\#H)/I$ as a quotient of a free algebra

In order to make the diamond lemma applicable for $(\Bbbk\langle X\rangle\#H)/I$, also not just for the regular letters X but for some super letters $[L]$, we will define a quotient of a certain free algebra, which is the special case of the following general construction:

In this section let X, S be arbitrary sets such that $X \subset S$, and H be a bialgebra with $\Bbbk$-basis G. Then

$$\Bbbk\langle X\rangle \subset \Bbbk\langle S\rangle \quad \text{and} \quad H = \operatorname{span}_\Bbbk G \subset \Bbbk\langle G\rangle,$$

if we view the set G as variables. Further we set $\langle S, G\rangle := \langle S \cup G\rangle$ where we may assume that the union is disjoint. By omitting $\otimes$

$$\Bbbk\langle X\rangle \otimes H = \operatorname{span}_\Bbbk\{ug \,|\, u \in \langle X\rangle, g \in G\} \subset \Bbbk\langle S, G\rangle$$

Now let $\Bbbk\langle X\rangle$ be a H-module algebra. Next we define the ideals corresponding to the extension of the variable set X to S, and to the smash product structure and the multiplication of H, and study their properties afterwards.

Definition 5.1. (1) Let A be an algebra, $B \subset A$ a subset. Then let $(B)_A$ denote the ideal generated by the set B.

(2) Let $f_s \in \Bbbk\langle X\rangle$ for all $s \in S$. Further let $1_H \in G$ and $f_{gh} := gh \in H = \operatorname{span}_\Bbbk G$ for all $g, h \in G$. We then define the ideals

$$\begin{aligned} I_S &:= (s - f_s \mid s \in S)_{\Bbbk\langle S,G\rangle}, \\ I_G &:= \big(gs - (g_{(1)} \cdot f_s)g_{(2)},\ gh - f_{gh},\ 1_H - 1 \mid g, h \in G, s \in S\big)_{\Bbbk\langle S,G\rangle}, \end{aligned}$$

where 1 is the empty word in $\Bbbk\langle S, G\rangle$.

Remark 5.2. We may assume that $1_H \in G$, if $H \neq 0$: Suppose $1_H \notin G$ and write 1_H as a linear combination of G. Suppose all coefficients are 0, then $1_H = 0_H$ hence $H = 0$; a contradiction. So there is a g with non-zero coefficient and we can exchange this g with 1_H.

Example 5.3. Let $H = \Bbbk[\Gamma]$ be the group algebra with the usual bialgebra structure $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. Here $G = \Gamma$, $f_{gh} \in \Gamma$ is just the product in the group, and

$$I_\Gamma = \big(gs - (g \cdot f_s)g,\ gh - f_{gh},\ 1_\Gamma - 1 \mid g, h \in \Gamma,\ s \in S\big).$$

Lemma 5.4. *For any $g \in \Gamma$ we have*

$$g(\Bbbk\langle S, G\rangle) \subset \operatorname{span}_\Bbbk\{ug \,|\, u \in \langle X\rangle, g \in G\} + I_G.$$

Proof. Let $a_1 \dots a_n \in \langle S, G\rangle$. We proceed by induction on n. If $n = 1$ then either $a_1 \in S$ or $a_1 \in G$. Then either $ga_1 \in (g_{(1)} \cdot f_{a_1})g_{(2)} + I_G \subset \operatorname{span}_\Bbbk\{ug \,|\, u \in \langle X\rangle, g \in G\} + I_G$ or $ga_1 \in f_{ga_1} + I_G \subset \operatorname{span}_\Bbbk\{ug \,|\, u \in \langle X\rangle, g \in G\} + I_G$. If $n > 1$, then let us consider $ga_1a_2 \dots a_n$. Again either $a_1 \in S$ or $a_1 \in G$ and we argue for ga_1 as in the induction basis; then by using the induction hypothesis we achieve the desired form. $\square$

Proposition 5.5. *Assume the above situation. Then*

$$\Bbbk\langle X\rangle\#H \;\cong\; \Bbbk\langle S,G\rangle/(I_S+I_G),$$

and for any ideal I of $\Bbbk\langle X\rangle\#H$ also I_S+I_G+I is an ideal of $\Bbbk\langle S,G\rangle$ such that

$$(\Bbbk\langle X\rangle\#H)/I \;\cong\; \Bbbk\langle S,G\rangle/(I_S+I_G+I).$$

Further we have the following special cases:

$$H\cong\Bbbk: \qquad \Bbbk\langle X\rangle \;\cong\; \Bbbk\langle S\rangle/I_S, \qquad \Bbbk\langle X\rangle/I \;\cong\; \Bbbk\langle S\rangle/(I_S+I). \tag{5.1}$$

$$S=X: \qquad \Bbbk\langle X\rangle\#H \;\cong\; \Bbbk\langle X,G\rangle/I_G, \qquad (\Bbbk\langle X\rangle\#H)/I \;\cong\; \Bbbk\langle X,G\rangle/(I_G+I). \tag{5.2}$$

Proof. (1) The algebra map

$$\Bbbk\langle S,G\rangle \to \Bbbk\langle X\rangle\#H, \quad s\mapsto f_s\#1_H, \quad g\mapsto 1_{\Bbbk\langle X\rangle}\#g$$

is surjective and contains I_S+I_G in its kernel; this is a direct calculation using the definitions. Hence we have a surjective algebra map on the quotient

$$\Bbbk\langle S,G\rangle/(I_S+I_G) \longrightarrow \Bbbk\langle X\rangle\#H. \tag{5.3}$$

In order to see that this map is bijective, we verify that a basis is mapped to a basis.

(a) The residue classes of the elements of $\{ug \,|\, u\in\langle X\rangle, g\in G\}$ $\Bbbk$-generate $\Bbbk\langle S,G\rangle/(I_S+I_G)$: Let $A\in\langle S,G\rangle$. Then either $A\in\langle S\rangle$ or it contains an element of G. In the first case $A\in\Bbbk\langle X\rangle+I_S$ by definition of I_S, and then $A\in\Bbbk\langle X\rangle 1_H+I_S+I_G$ since $1_H-1\in I_\Gamma$. In the other case let $A=A_1gA_2$ with $A_1\in\langle S\rangle$, $g\in G$, $A_2\in\langle S,G\rangle$. We argue for A_1 like before, and $gA_2\in\operatorname{span}_\Bbbk\{ug \,|\, u\in\langle X\rangle, g\in G\}+I_G$ by Lemma 5.4.

(b) The residue classes of $\{ug \,|\, u\in\langle X\rangle, g\in G\}$ are mapped by Eq. (5.3) to the $\Bbbk$-basis $\langle X\rangle\#G$ of the right-hand side. Hence the residue classes are linearly independent, thus form a basis of $\Bbbk\langle S,G\rangle/(I_S+I_G)$.

(2) $I_S+I_\Gamma+I$ is an ideal: Let $A\in\langle S,G\rangle$ and $a\in I\subset\operatorname{span}_\Bbbk\{ug \,|\, u\in\langle X\rangle, g\in G\}$. Then by (1a) above $A\in\operatorname{span}_\Bbbk\{ug \,|\, u\in\langle X\rangle, g\in G\}+I_S+I_G$, and since I is an ideal of $\Bbbk\langle X\rangle\#H$, we have $Aa, aA\in I_S+I_G+I$ by the isomorphism Eq. (5.3).

Using the isomorphism theorem and part (1) we get

$$\Bbbk\langle S,G\rangle/(I_S+I_G+I) \;\cong\; \big(\Bbbk\langle S,G\rangle/(I_S+I_G)\big)\big/\big((I_S+I_G+I)/(I_S+I_G)\big) \;\cong\; (k\langle X\rangle\#H)/I,$$

where the last $\cong$ holds since $(I_S+I_G+I)/(I_S+I_G)$ is mapped to I by the isomorphism Eq. (5.3).

(3) The special cases follow from the facts that $I_S=0$ if $S=X$, and if $H\cong\Bbbk$ then $G=\{1_H\}$. Hence $I_G=(1_H-1)$ and $\Bbbk\langle X\rangle\cong\Bbbk\langle X\rangle\#\Bbbk\cong\Bbbk\langle S,\{1_H\}\rangle/(I_S+(1_H-1))\cong \Bbbk\langle S\rangle/I_S$. □

We now return to the situation of Section 3, and rewrite Proposition 5.5 for the case $S=X_L$ and $H=\Bbbk[\Gamma]$:

Corollary 5.6. *Let $L \subset \mathcal{L}$ be Shirshov closed and*

$$I_L := \big(x_u - [x_v, x_w] \mid u \in L,\ \mathrm{Sh}(u) = (v|w)\big)_{\Bbbk\langle X_L,\Gamma\rangle}$$
$$I'_\Gamma := \big(gx_u - \chi_u(g)x_u g,\ gh - f_{gh},\ 1_\Gamma - 1 \mid g,h \in \Gamma,\ u \in L\big)_{\Bbbk\langle X_L,\Gamma\rangle}.$$

Then for any ideal I of $\Bbbk\langle X\rangle \# \Bbbk[\Gamma]$ also $I_L + I'_\Gamma + I$ is an ideal of $\Bbbk\langle X_L, \Gamma\rangle$ such that

$$(\Bbbk\langle X\rangle \# \Bbbk[\Gamma])/I \cong \Bbbk\langle X_L, \Gamma\rangle/(I_L + I'_\Gamma + I).$$

Further we have the analog special cases of Proposition 5.5.

Proof. We apply Proposition 5.5 to the case $S = X_L$, $H = \Bbbk[\Gamma]$, $f_{x_u} = [u]$ for all $u \in L$. Then $I_{X_L} = \big(x_u - [u] \mid u \in L\big)_{\Bbbk\langle X_L,\Gamma\rangle}$ and I_Γ is as in Example 5.3. We are left to prove $I_L + I'_\Gamma + I = I_{X_L} + I_\Gamma + I$, which follows from the Lemma below. □

Lemma 5.7. *We have*

(1) *$[u] \in x_u + I_L$ for all $u \in L$; hence $I_{X_L} = I_L$.*

(2) *$I_\Gamma \subset I'_\Gamma + I_L$*

Proof. (2) follows from (1), which we prove by induction on $\ell(u)$: For $\ell(u) = 1$ there is nothing to show. Let $\ell(u) > 1$ and $\mathrm{Sh}(u) = (v|w)$. Then by the induction assumption we have

$$\begin{aligned}[u] &= [v][w] - q_{v,w}[w][v] \in (x_v + I_L)(x_w + I_L) - q_{vw}(x_w + I_L)(x_v + I_L)\\ &\subset [x_v, x_w] + I_L = x_u - (\underbrace{x_u - [x_v, x_w]}_{\in I_L}) + I_L = x_u + I_L.\end{aligned}$$

□

Example 5.8. Let $X = \{x_1, x_2\} \subset L = \{x_1, x_1x_2, x_2\}$. Then $I_L = \big(x_{12} - [x_1, x_2]\big)$ and by Corollary 5.6 $\Bbbk\langle x_1, x_2\rangle \cong \Bbbk\big\langle x_1, x_{12}, x_2 \bigm| x_{12} = [x_1, x_2]\big\rangle$, and

$$\begin{aligned}\Bbbk\langle x_1, x_2\rangle \# \Bbbk[\Gamma] \cong \Bbbk\langle x_1, x_{12}, x_2, \Gamma \mid x_{12} &= [x_1, x_2],\\ gx_u &= \chi_u(g)x_u g,\ gh = f_{gh},\ 1_\Gamma - 1; \forall u \in L, g, h \in \Gamma\rangle.\end{aligned}$$

6 Bergman's diamond lemma

Following Bergman [5], let Y be a set, $\Bbbk\langle Y\rangle$ the free $\Bbbk$-algebra and Σ an index set. We fix a subset $\mathcal{R} = \{(W_\sigma, f_\sigma) \mid \sigma \in \Sigma\} \subset \langle Y\rangle \times \Bbbk\langle Y\rangle$, and define the ideal

$$I_{\mathcal{R}} := (W_\sigma - f_\sigma \mid \sigma \in \Sigma)_{\Bbbk\langle Y\rangle}.$$

An *overlap of* $\mathcal{R}$ is a triple (A, B, C) such that there are $\sigma, \tau \in \Sigma$ and $A, B, C \in \langle Y\rangle\backslash\{1\}$ with $W_\sigma = AB$ and $W_\tau = BC$. In the same way an *inclusion of* $\mathcal{R}$ is a triple (A, B, C) such that there are $\sigma \neq \tau \in \Sigma$ and $A, B, C \in \langle Y\rangle$ with $W_\sigma = B$ and $W_\tau = ABC$.

Let $\preceq_\diamond$ be a *with* $\mathcal{R}$ *compatible well-founded monoid partial ordering* of the free monoid $\langle Y\rangle$, i.e.:

- $(\langle Y\rangle, \preceq_\diamond)$ is a partial ordered set.
- $B \prec_\diamond B' \Rightarrow ABC \prec_\diamond AB'C$ for all $A, B, B', C \in \langle Y\rangle$.
- Each non-empty subset of $\langle Y\rangle$ has a minimal element w.r.t. $\preceq_\diamond$.
- f_σ is a linear combination of monomials $\prec_\diamond W_\sigma$ for all $\sigma \in \Sigma$; in this case we write $f_\sigma \prec_\diamond W_\sigma$.

For any $A \in \langle Y\rangle$ let $I_{\prec_\diamond A}$ denote the subspace of $\Bbbk\langle Y\rangle$ spanned by all elements $B(W_\sigma - f_\sigma)C$ with $B, C \in \langle Y\rangle$ such that $BW_\sigma C \prec_\diamond A$. The next theorem is a short version of the diamond lemma:

Theorem 6.1. [5, Thm 1.2] *Let $\mathcal{R} = \{(W_\sigma, f_\sigma) \mid \sigma \in \Sigma\} \subset \langle Y\rangle \times \Bbbk\langle Y\rangle$ and $\preceq_\diamond$ be a with $\mathcal{R}$ compatible well-founded monoid partial ordering on $\langle Y\rangle$. Then the following conditions are equivalent:*

(1) (a) *$f_\sigma C - Af_\tau \in I_{\prec_\diamond ABC}$ for all overlaps (A, B, C).*
(b) *$Af_\sigma C - f_\tau \in I_{\prec_\diamond ABC}$ for all inclusions (A, B, C).*

(2) *The residue classes of the elements of $\langle Y\rangle$ which do not contain any W_σ with $\sigma \in \Sigma$ as a subword form a $\Bbbk$-basis of $\Bbbk\langle Y\rangle/I_{\mathcal{R}}$.*

We now define the ordering for our situation, where $L \subset \mathcal{L}$ is Shirshov closed and $Y = X_L \cup \Gamma$: Let $\pi_L : \langle X_L, \Gamma\rangle \to \langle X_L\rangle$ be the monoid map with $x_u \mapsto x_u$ and $g \mapsto 1$ for all $u \in L$, $g \in \Gamma$ (π_L deletes all g in a word of $\langle X_L, \Gamma\rangle$).

Moreover, for a $A \in \langle X_L, \Gamma\rangle$ let $n_\Gamma(A)$ denote the number of letters $g \in \Gamma$ in the word A and $t(A)$ the $n_\Gamma(A)$-tuple of non-negative integers

$$\begin{aligned}&(\text{number of letters after the last } g \in \Gamma \text{ in } A, \dots, \\ &\qquad\qquad \dots, \text{number of letters after the first } g \in \Gamma \text{ in } A) \in \mathbb{N}^{n_\Gamma(A)}.\end{aligned}$$

Definition 6.2. For $A, B \in \langle X_L, \Gamma\rangle$ we define $A \prec_\diamond B$ by

- $\pi_L(A) \prec \pi_L(B)$, or
- $\pi_L(A) = \pi_L(B)$ and $n_\Gamma(A) < n_\Gamma(B)$, or
- $\pi_L(A) = \pi_L(B)$, $n_\Gamma(A) = n_\Gamma(B)$ and $t(A) < t(B)$ under the lexicographical order of $\mathbb{N}^{n_\Gamma(A)}$, i.e., $t(A) \neq t(B)$, and the first non-zero term of $t(B) - t(A)$ is positive.

$\preceq_\diamond$ is a well-founded monoid partial ordering of $\langle X_L, \Gamma\rangle$, which is straightforward to verify, and will be compatible with the later regarded $\mathcal{R}$.

Note that we have the following correspondence between $\prec$ of Section 2.4 and $\prec_\diamond$, which follows from the definitions: For any $U, V \in [L]^{(\mathbb{N})}$, $g, h \in \Gamma$ we have $\rho(U)g, \rho(V)h \in \langle X_L\rangle\Gamma$ and

$$U \prec V \iff \rho(U)g \prec_\diamond \rho(V)h. \tag{6.1}$$

7 Proof of Theorem 4.2

Again suppose the assumptions of Theorem 4.2. By Corollary 5.6

$$(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])/I \cong \Bbbk\langle X_L,\Gamma\rangle/(I_L+I'_\Gamma+I),$$

thus $(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])/I$ has the basis $[u_1]^{r_1}[u_2]^{r_2}\dots[u_t]^{r_t}g$ if and only if $\Bbbk\langle X_L,\Gamma\rangle/(I_L+I'_\Gamma+I)$ has the basis $x_{u_1}^{r_1}x_{u_2}^{r_2}\dots x_{u_t}^{r_t}g$ ($t\in\mathbb{N}$, $u_i\in L$, $u_1>\dots>u_t$, $0<r_i<N_u$, $g\in\Gamma$). The latter we can reformulate equivalently in terms of the Diamond Lemma 6.1:

• We define $\mathcal{R}$ as the set of the elements

$$(1_\Gamma,\ 1), \tag{7.1}$$
$$(gh,\ f_{gh}),\ \text{for all } g,h\in\Gamma, \tag{7.2}$$
$$\big(gx_u,\ \chi_u(g)x_ug\big),\ \text{for all } g\in\Gamma, u\in L, \tag{7.3}$$
$$\big(x_ux_v,\ c^\rho_{(u|v)}+q_{u,v}x_vx_u\big),\ \text{for all } u,v\in L \text{ with } u<v, \tag{7.4}$$
$$\big(x_u^{N_u},\ d^\rho_u\big),\ \text{for all } u\in L \text{ with } N_u<\infty, \tag{7.5}$$

where we again see $c^\rho_{(u|v)},\ d^\rho_u\in\Bbbk\langle X_L\rangle\otimes\Bbbk[\Gamma]\subset\operatorname{span}_\Bbbk\{Ug\mid U\in\langle X_L\rangle, g\in\Gamma\}\subset\Bbbk\langle X_L,\Gamma\rangle$. Then the residue classes of $c^\rho_{(u|v)}, d^\rho_u$ modulo $I_L+I'_\Gamma$ correspond to $c_{(u|v)}$ and d_u by the isomorphism of Corollary 5.6, and we have $I_\mathcal{R}=I_L+I'_\Gamma+I$.

• Note that $\prec_\diamond$ is compatible with $\mathcal{R}$: In Eq. (7.1) resp. (7.2) we have $1\prec_\diamond 1_\Gamma$ resp. $f_{gh}\prec_\diamond gh$ since $n_\Gamma(1)=0<1=n_\Gamma(1_\Gamma)$ resp. $n_\Gamma(f_{gh})=1<2=n_\Gamma(gh)$ ($f_{gh}\in\Gamma$). Eq. (7.3): $t(x_ug)=(0)<(1)=t(gx_u)$, hence $x_ug\prec_\diamond gx_u$. Moreover, by [10, Lem. 3.6] we have $c^\rho_{(u|v)}+q_{u,v}x_vx_u\prec_\diamond x_ux_v$, and $d^\rho_u\prec_\diamond x_u^{N_u}$ by assumption.

• By the Diamond Lemma 6.1 we have to consider all possible overlaps and inclusions of $\mathcal{R}$. The only inclusions happen with Eq. (7.1), namely $(1,1_\Gamma,h)$, $(g,1_\Gamma,1)$, $(1,1_\Gamma,x_u)$. But they all fulfill the condition (1b) of the Diamond Lemma 6.1: for example $h-f_{1_\Gamma h}=h-h=0\in I_{\prec_\diamond 1_\Gamma h}$, and $x_u-\chi_u(1_\Gamma)x_u1_\Gamma=x_u(1_\Gamma-1)\in I_{\prec_\diamond 1_\Gamma x_u}$.

So we are left to check the conditon (1a) for all overlaps: (g,h,k) with $g,h,k\in\Gamma$ fulfills it by the associativity of Γ; for (g,h,x_u) we have

$$f_{gh}x_u-\chi_u(h)gx_uh=\chi_u(gh)x_uf_{gh}-\chi_u(h)\chi_u(g)x_ugh=0,$$

calculating modulo $I_{\prec_\diamond ghx_u}$ and using $\chi_u(f_{gh})=\chi_u(gh)$ since $f_{gh}\in\Gamma$. The next overlap is (g,x_u,x_v) where $u<v$: Calculating modulo $I_{\prec_\diamond gx_ux_v}$ we get

$$\chi_u(g)x_ugx_v-g\big(c^\rho_{(u|v)}+q_{u,v}x_vx_u\big)=\chi_u(g)\chi_v(g)x_ux_vg-$$
$$\chi_{uv}(g)\big(c^\rho_{(u|v)}+q_{u,v}x_vx_u\big)g=\chi_{uv}(g)\big(x_ux_v-\big(c^\rho_{(u|v)}+q_{u,v}x_vx_u\big)\big)g=0,$$

since $c_{(u|v)}\in(\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi_{uv}}$ and $x_ux_vg\prec_\diamond gx_ux_v$. For the overlap $(g,x_u,x_u^{N_u-1})$ we obtain modulo $I_{\prec_\diamond gx_u^{N_u}}$

$$\chi_u(g)x_ugx_u^{N_u-1}-gd^\rho_u=\chi_u(g)^{N_u}\big(x_u^{N_u}-d^\rho_u\big)g=0,$$

because $d_u \in (\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi_u^{N_u}}$ and $x_u^{N_u}\vartheta_g \prec_\diamond \vartheta_g x_u^{N_u}$. The remaining overlaps are those with Eqs. (7.4) and (7.5); for these we formulate the following three Lemmata which are equivalent to (2) of the Theorem 4.2:

Lemma 7.1. *The overlap (x_u, x_v, x_w), $u < v < w$, fulfills condition 6.1(1a), i.e., $a := \big(c^\rho_{(u|v)} + q_{u,v}x_vx_u\big)x_w - x_u\big(c^\rho_{(v|w)} + q_{v,w}x_wx_v\big) \in I_{\prec_\diamond x_ux_vx_w}$, if and only if $J(u < v < w) \in I_{\prec_\diamond x_ux_vx_w}$.*

Proof. We calculate in $\Bbbk\langle X_L, \Gamma\rangle$

$$\begin{aligned} J(u < v < w) &= c^\rho_{(u|v)}x_w - q_{uv,w}x_wc^\rho_{(u|v)} - \big(x_uc^\rho_{(v|w)} - q_{u,vw}c^\rho_{(v|w)}x_u\big) \\ &\quad + q_{u,v}x_v\big(x_ux_w - q_{u,w}x_wx_u\big) - q_{v,w}\big(x_ux_w - q_{u,w}x_wx_u\big)x_v, \\ a &= c^\rho_{(u|v)}x_w + q_{u,v}x_vx_ux_w - x_uc^\rho_{(v|w)} - q_{v,w}x_ux_wx_v, \end{aligned}$$

and show that the difference is zero modulo $I_{\prec_\diamond x_ux_vx_w}$:

$$\begin{aligned} J(u < v < w) - a &= q_{uv,w}x_w\big(x_ux_v - c^\rho_{(u|v)}\big) + q_{u,vw}\big(c^\rho_{(v|w)} - x_vx_w\big)x_u \\ &= q_{uv,w}x_w\big(q_{u,v}x_vx_u\big) - q_{u,vw}\big(q_{v,w}x_wx_v\big)x_u = 0. \end{aligned}$$

since $x_wx_ux_v, x_vx_wx_u \prec_\diamond x_ux_vx_w$. $\square$

Lemma 7.2. *The overlaps $\big(x_u^{N_u-1}, x_u, x_v\big)$ resp. $\big(x_u, x_v, x_v^{N_v-1}\big)$ fulfill condition 6.1(1a), i.e., $d^\rho_ux_v - x_u^{N_u-1}\big(c^\rho_{(u|v)} + q_{u,v}x_vx_u\big) \in I_{\prec_\diamond x_u^{N_u}x_v}$ resp. $\big(c^\rho_{(u|v)} + q_{uv}x_vx_u\big)x_v^{N_v-1} - x_ud^\rho_v \in I_{\prec_\diamond x_ux_v^{N_v}}$ if and only if $L(u, u < v) \in I_{\prec_\diamond x_u^{N_u}x_v}$ resp. $L(u, u > v) \in I_{\prec_\diamond x_vx_u^{N_u}}$.*

Proof. We prove it for $\big(x_u^{N_u-1}, x_u, x_v\big)$; the other overlap is proved analogously. We set $r := N_u - 1$, then $\operatorname{ord} q_{u,u} = r + 1$. Using the q-Leibniz formula of Proposition 1.2 we get

$$\begin{aligned} &x_u^r\big(c^\rho_{(u|v)} + q_{u,v}x_vx_u\big) - d^\rho_ux_v = \\ &= \big[x_u^r, c^\rho_{(u|v)}\big]_{q_{u,u}^rq_{u,v}} + q_{u,u}^rq_{u,v}c^\rho_{(u|v)}x_u^r \\ &\qquad\qquad + q_{u,v}\big[x_u^r, x_v\big]_{q_{u,v}^r}x_u + q_{u,v}^{r+1}x_vx_u^{r+1} - d^\rho_ux_v \\ &= \sum_{i=0}^r q_{u,u}^iq_{u,v}^i\binom{r}{i}_{q_{u,u}}\big[\underbrace{x_u, \dots [x_u}_{r-i}, c^\rho_{(u|v)}]_{q_{u,u}q_{u,v}} \dots\big]_{q_{u,u}^{r-i}q_{u,v}}x_u^i \\ &\quad + \sum_{i=0}^{r-1} q_{u,v}^{i+1}\binom{r}{i}_{q_{u,u}}\big[\underbrace{x_u, \dots [x_u}_{r-i}, x_v]_{q_{u,v}} \dots\big]_{q_{u,u}^{r-i-1}q_{u,v}}x_u^{i+1} + q_{u,v}^{r+1}x_vx_u^{r+1} - d^\rho_ux_v. \end{aligned}$$

Because of $x_u^{r-i}x_vx_u^{i+1} \prec_\diamond x_u^{r+1}x_v$ for all $0 \le i \le r$, this is modulo $I_{\prec_\diamond x_u^{r+1}x_v}$ equal to

$$\begin{aligned} &\sum_{i=0}^r q_{u,u}^iq_{u,v}^i\binom{r}{i}_{q_{u,u}}\big[\underbrace{x_u, \dots [x_u}_{r-i}, c^\rho_{(u|v)}]_{q_{u,u}q_{u,v}} \dots\big]_{q_{u,u}^{r-i}q_{u,v}}x_u^i \\ &\quad + \sum_{i=0}^{r-1} q_{u,v}^{i+1}\binom{r}{i}_{q_{u,u}}\big[\underbrace{x_u, \dots [x_u}_{r-i-1}, c^\rho_{(u|v)}]_{q_{u,u}q_{u,v}} \dots\big]_{q_{u,u}^{r-i-1}q_{u,v}}x_u^{i+1} - \big[d^\rho_u, x_v\big]_{q_{u,v}^{r+1}}. \end{aligned}$$

Now shifting the index of the second sum, we obtain

$$\big[\underbrace{x_u, \dots [x_u}_{r}, c^{\rho}_{(u|v)}]_{q_{u,u}q_{u,v}} \dots\big]_{q^r_{u,u}q_{u,v}} - \big[d^{\rho}_u, x_v\big]_{q^{r+1}_{u,v}}$$
$$+ \sum_{i=1}^{r} q^i_{u,v}\Big(q^i_{u,u}\tbinom{r}{i}_{q_{u,u}} + \tbinom{r}{i-1}_{q_{u,u}}\Big)\big[\underbrace{x_u, \dots [x_u}_{r-i}, c^{\rho}_{(u|v)}]_{q_{u,u}q_{u,v}} \dots\big]_{q^{r-i}_{u,u}q_{u,v}} x^i_u.$$

Finally we obtain the claim, since $q^i_{u,u}\binom{r}{i}_{q_{u,u}} + \binom{r}{i-1}_{q_{u,u}} = \binom{r+1}{i}_{q_{u,u}} = 0$ for all $1 \le i \le r$, by Eq. (1.2) and $\operatorname{ord} q_{u,u} = r + 1$. □

Lemma 7.3. *The overlaps* $\big(x^{N_u-i}_u, x^i_u, x^{N_u-i}_u\big)$ *fulfill condition 6.1(1a) for all* $1 \le i < N_u$, *if and only if the overlap* $\big(x^{N_u-1}_u, x_u, x^{N_u-1}_u\big)$ *fulfills condition 6.1(1a), if and only if* $L(u) \in I_{\prec \diamond x^{N_u+1}_u}$.

Proof. This is evident. □

• We are left to prove the equivalence of (2) to its weaker version (2') of Theorem 4.2: For (2'a) we show that if $uv \in L$ and $\mathrm{Sh}(uv) = (u|v)$, then conditon (2a) is already fulfilled: By definition $c^{\rho}_{(u|v)} = x_{uv}$ and

$$\big[c^{\rho}_{(u|v)}, x_w\big]_{q_{uv,w}} = \big[x_{uv}, x_w\big] = c^{\rho}_{(uv|w)}$$

modulo $I_{\prec x_u x_v x_w}$. Now certainly $\mathrm{Sh}(uvw) \ne (uv|w)$, thus

$$c^{\rho}_{(uv|w)} = \partial^{\rho}_u(c^{\rho}_{(v|w)}) + q_{v,w} c^{\rho}_{(u|w)} x_v - q_{u,v} x_v c^{\rho}_{(u|w)}$$

by Eq. (4.1). Hence in this case the q-Jacobi condition is fulfilled by the q-derivation formula of Proposition 1.2.

For (2'b) of Theorem 4.2 it is enough to show the following: Let condition (2bi) hold for $u = v$, i.e., $[x_u, d^{\rho}_u]_1 \in I_{\prec x^{N_u+1}_u}$. Then, if condition (2bi) holds for some $u < v$ with $N_u < \infty$, then (2bi) also holds for $u < uv$ (whenever $uv \in L$). Analogously, if (2bii) holds for $v < u$ with $N_u < \infty$, then also (2bii) holds for $vu < u$ (whenever $vu \in L$).

Note that if $u < v$, then $uv < v$: Either v does not begin with u, then $uv < v$; or let $v = uw$ for some $w \in \langle X \rangle$. Then $u < v = uw < w$ since $v \in \mathcal{L}$. Hence $uv = uuw < uw = v$.

We will prove the first part (2'bi), (2'bii) is the same argument. But before we formulate the following

Lemma 7.4. *Let* $a \in \Bbbk\langle X_L\rangle \# \Bbbk[\Gamma]$, $A, W \in \langle X_L\rangle$ *such that* $a \preceq_L A \prec W$. *Then* $a \in I_{\prec W}$ *if and only if* $a \in I_{\preceq A}$.

Proof. Clearly $I_{\preceq A} \subset I_{\prec W}$, since $A \prec W$. So denote by $\{(W_\sigma, f_\sigma) \mid \sigma \in \Sigma\}$ the set of Eqs. (7.4) and (7.5) with $f_\sigma \prec_L W_\sigma$, and let $a \in I_{\prec W}$, i.e., a is a linear combination of $Ug(W_\sigma - f_\sigma)Vh$ with $U, V \in \langle X_L\rangle$ such that $UW_\sigma V \prec W$. Denote by E the $\prec$-biggest word of all $UW_\sigma V$ with non-zero coefficient. $E \succ A$ contradicts the assumption $a \preceq_L A \prec W$. Hence $E \preceq A$ and therefore $f \in I_{\preceq A}$. □

Suppose (2bi) for $u<v$ with $N_u<\infty$ and $uv\in L$, i.e.,

$$\begin{aligned}
&\big[\underbrace{x_u,\dots[x_u}_{N_u-1},x_{uv}]_{q_{u,u}q_{u,v}}\dots\big]_{q_{u,u}^{N_u-1}q_{u,v}}-[d_u^\rho,x_v]_{q_{u,v}^{N_u}}\in I_{\prec x_u^{N_u}x_v}\\
\Leftrightarrow&\big[\underbrace{x_u,\dots[x_u}_{N_u-2},c^\rho_{(u|uv)}]_{q_{u,u}^2q_{u,v}}\dots\big]_{q_{u,u}^{N_u-1}q_{u,v}}-[d_u^\rho,x_v]_{q_{u,v}^{N_u}}\in I_{\preceq x_u^{N_u-1}x_wUx_v},
\end{aligned}$$

for some $w\in L$ with $w>u$ and $U\in\langle X_L\rangle$ such that $\ell(U)+\ell(w)=\ell(u)$. Here we used the relation $[x_u,x_{uv}]_{q_{u,uv}}-c^\rho_{(u|uv)}$, and Lemma 7.4 since the above polynomial is $\preceq x_u^{N_u-1}x_wUx_v$ (by assumption $c_{(u|uv)}\preceq_L[uuv]$, $d_u\prec_L[u]^{N_u}$). Hence the condition (2bi) for $u<uv$ reads

$$\begin{aligned}
&\big[\underbrace{x_u,\dots[x_u}_{N_u-1},c^\rho_{(u|uv)}]_{q_{u,u}^2q_{u,v}}\dots\big]_{q_{u,u}^{N_u}q_{u,v}}-[d_u^\rho,x_{uv}]_{q_{u,u}^{N_u}q_{u,v}^{N_u}}\in I_{\prec x_u^{N_u}x_{uv}}\\
\Leftrightarrow&\big[x_u,[d_u^\rho,x_v]_{q_{u,v}^{N_u}}\big]_{q_{u,u}^{N_u}q_{u,v}}-[d_u^\rho,x_{uv}]_{q_{u,u}^{N_u}q_{u,v}^{N_u}}\in I_{\prec x_u^{N_u}x_{uv}},
\end{aligned}$$

since $x_uI_{\preceq x_u^{N_u-1}x_wUx_v},I_{\preceq x_u^{N_u-1}x_wUx_v}x_u\subset I_{\prec x_u^{N_u}x_{uv}}$ ($w>u$ and w cannot begin with u since $\ell(w)\leq\ell(u)$, hence $w>uv$. By the q-Jacobi identity

$$\begin{aligned}
\big[x_u,[d_u^\rho,x_v]_{q_{u,v}^{N_u}}\big]_{q_{u,u}^{N_u}q_{u,v}}&=\big[[x_u,d_u^\rho]_{q_{u,u}^{N_u}},x_v\big]_{q_{u,v}^{N_u+1}}+q_{u,u}^{N_u}d_u^\rho[x_u,x_v]-q_{u,v}^{N_u}[x_u,x_v]d_u^\rho\\
&=\big[[x_u,d_u^\rho]_1,x_v\big]_{q_{u,v}^{N_u+1}}+[d_u^\rho,x_{uv}]_{q_{u,v}^{N_u}}=[d_u^\rho,x_{uv}]_{q_{u,v}^{N_u}}.
\end{aligned}$$

For the last two "=" we used $q_{u,u}^{N_u}=1$, the relation $[x_u,x_v]-x_{uv}$ and $[x_u,d_u^\rho]_1\in I_{\prec x_u^{N_u+1}}$ (We can use this condition: Note that $[x_u,d_u^\rho]_1\preceq x_u^{N_u}x_{w'}U'$ for some $w'\in L$, $w'>u$, $U'\in\langle X_L\rangle$, $\ell(U')+\ell(w')=\ell(u)$, hence $[x_u,d_u^\rho]_1\in I_{\preceq x_u^{N_u}x_{w'}U'}$ by Lemma 7.4. Therefore $x_vI_{\preceq x_u^{N_u}x_{w'}U'},I_{\preceq x_u^{N_u}x_{w'}U'}x_v\subset I_{\prec x_u^{N_u}x_{uv}}$, like before).

8 PBW basis in rank one

We want to apply the PBW basis criterion to Hopf algebras of rank one and two for some fixed $L\subset\mathcal{L}$. Especially we want to treat liftings of Nichols algebras. Therefore we define the following scalars which will guarantee a $\widehat{\Gamma}$-graduation:

Definition 8.1. Let $L\subset\mathcal{L}$. Then we define coefficients $\mu_u\in\Bbbk$ for all $u\in D(L)$, and $\lambda_w\in\Bbbk$ for all $w\in C(L)$ by

$$\mu_u=0,\text{ if }g_u^{N_u}=1\text{ or }\chi_u^{N_u}\neq\varepsilon,\qquad\lambda_w=0,\text{ if }g_w=1\text{ or }\chi_w\neq\varepsilon,$$

and otherwise they can be chosen arbitrarily.

In this section let V be a 1-dimensional vector space with basis x_1 and $\mathrm{ord}q_{11}=N\leq\infty$. Since $T(V)\cong\Bbbk[x_1]$ we have $\mathcal{L}=\{x_1\}$. We give the condition when $(T(V)\#\Bbbk[\Gamma])/\big(x_1^N-d_1\big)$ has the PBW basis $\{x_1\}$. By the PBW Criterion 4.2 the only condition in $\Bbbk[x_1]\#\Bbbk[\Gamma]$ is

$$[d_1^\rho,x_1]_1\in I_{\prec x_1^{N+1}}.$$

Examples 8.2. Let $\operatorname{char}\Bbbk = 0$ and $q \in \Bbbk^\times$ with $\operatorname{ord} q = N \geq 2$.

1. *Nichols algebra* A_1. $T(V)/\big(x_1^N\big)$ has basis $\{x_1^r \,|\, 0 \leq r < N\}$.

2. *Taft Hopf algebra.* Let $\mathbb{Z}/(N) = \langle g_1 \rangle$ and $\chi_1(g_1) := q$. The set $\{x_1^r g \mid 0 \leq r < N, g \in \mathbb{Z}/(N)\}$ is a basis of $T(q) \cong \big(\Bbbk[x_1]\#\Bbbk[\mathbb{Z}/(N)]\big)/(x_1^N)$.

3. *Radford Hopf algebra.* Let $\mathbb{Z}/(N^2) = \langle g_1 \rangle$ and $\chi_1(g_1) := q$. The set $\{x_1^r g \mid 0 \leq r < N, g \in \mathbb{Z}/(N^2)\}$ is a basis of $r(q) \cong (\Bbbk[x_1]\#\Bbbk[\mathbb{Z}/(N^2)])/(x_1^N - (1 - g_1^N))$.

4. *Liftings* A_1. The set $\{x_1^r g \,|\, 0 \leq r < N, g \in \Gamma\}$ is a basis of $(T(V)\#\Bbbk[\Gamma])/\big(x_1^N - \mu_1(1 - g_1^N)\big)$,

Proof. (1) and (2) clearly fulfill the only condition above, since $d_1 = 0$.
(3) is a special case of (4): It is $d_1 \in (\Bbbk\langle X\rangle\#\Bbbk[\Gamma])^{\chi_1^N}$ by Definition 8.1 of μ_1. Further

$$\big[\mu_1(1 - g_1^N), x_1\big]_1 = \mu_1\big[1, x_1\big]_1 - \mu_1\big[g_1^N, x_1\big]_1 = -\mu_1(q_{11}^N - 1)x_1 g_1^N = 0,$$

since $\operatorname{ord} q_{11} = N$. □

9 PBW basis in rank two and redundant relations

Let V be a 2-dimensional vector space with basis x_1, x_2, hence $T(V) \cong \Bbbk\langle x_1, x_2\rangle$. In this chapter we apply the PBW Criterion 4.2 to verify for certain $L \subset \mathcal{L}$ that the algebra

$$(T(V)\#\Bbbk[\Gamma])/I,$$

with I as in Section 3.2, has the PBW basis $[L]$. In particular, we examine the Nichols algebras and their liftings of [9]. Moreover, we will see how to find the redundant relations, and in addition, we will treat some classical examples.

9.1 PBW basis for $L = \{x_1 < x_2\}$

This is the easiest case and covers the Cartan Type $A_1 \times A_1$, as well as many other examples. We are interested when $[L]$ builds up a PBW Basis of

$$(T(V)\#\Bbbk[\Gamma])/\big([x_1x_2] - c_{12},\ x_1^{N_1} - d_1,\ x_2^{N_2} - d_2\big),$$

with $N_1 = \operatorname{ord} q_{11}, N_2 = \operatorname{ord} q_{22} \in \{2, 3, \ldots, \infty\}$. If $N_1 = N_2 = \infty$, then by the PBW Criterion 4.2 there is no condition in $\Bbbk\langle x_1, x_2\rangle\#\Bbbk[\Gamma]$ such that we can choose c_{12} arbitrarily with $c_{12} \prec_L [x_1x_2]$ and $\deg_{\widehat{\Gamma}}(c_{12}) = \chi_1\chi_2$:

Examples 9.1.

1. *Quantum plane.* The set $\{x_2^{r_2} x_1^{r_1} \mid r_2, r_1 \geq 0\}$ is a basis of $Q(q_{12}) \cong T(V)/([x_1x_2])$.

2. *Weyl algebra.* If $q_{12}=1$, then $\{x_2^{r_2}x_1^{r_1} \mid r_2, r_1 \geq 0\}$ is a basis of $W \cong T(V)/([x_1x_2]-1)$.

If $\mathrm{ord} q_{11} = N_1 < \infty$ or $\mathrm{ord} q_{22} = N_2 < \infty$, then by the PBW Criterion 4.2 we have to check

$$\left[d_1^\rho, x_1\right]_1 \in I_{\prec x_1^{N_1+1}}, \quad \text{or} \quad \left[d_2^\rho, x_2\right]_1 \in I_{\prec x_2^{N_2+1}}, \text{ and} \tag{9.1}$$

$$[\underbrace{x_1, \dots [x_1}_{N_1-1}, c_{12}^\rho]_{q_{11}q_{12}} \dots]_{q_{11}^{N_1-1}q_{12}} - \left[d_1^\rho, x_2\right]_{q_{12}^{N_1}} \in I_{\prec x_1^{N_1}x_2}, \text{ or} \tag{9.2}$$

$$[\dots [c_{12}^\rho, \underbrace{x_2]_{q_{12}q_{22}} \dots, x_2}_{N_2-1}]_{q_{12}q_{22}^{N_2-1}} - \left[x_1, d_2^\rho\right]_{q_{12}^{N_2}} \in I_{\prec x_1 x_2^{N_2}}. \tag{9.3}$$

Examples 9.2. Let $\lambda_{12}, \mu_1, \mu_2 \in \Bbbk$ as in Definition 8.1.

1. *Nichols algebra $A_1 \times A_1$.* Let $q_{12}q_{21}=1$, then $\{x_2^{r_2}x_1^{r_1} \mid 0 \leq r_i < N_i\}$ is a basis of

$$T(V)/\big([x_1x_2],\, x_1^{N_1},\, x_2^{N_2}\big).$$

2. *Liftings $A_1 \times A_1$.* Let $q_{12}q_{21}=1$, then $\{x_2^{r_2}x_1^{r_1}g \mid 0 \leq r_i < N_i, g \in \Gamma\}$ is a basis of

$$(T(V)\#\Bbbk[\Gamma])/\big([x_1x_2]-\lambda_{12}(1-g_{12}),\ x_1^{N_1}-\mu_1(1-g_1^{N_1}),\ x_2^{N_2}-\mu_2(1-g_2^{N_2})\big).$$

3. *Book Hopf algebra.* Let $q \in \Bbbk^\times$ with $\mathrm{ord} q = N > 2$, $\mathbb{Z}/(N) = \langle g_1 \rangle$, $g := g_2 := g_2$, and $\chi_1(g_i) := q^{-1}$, $\chi_2(g_i) := q$ for $i=1,2$. Then $\{x_2^{r_2}x_1^{r_1}g \mid 0 \leq r_i < N, g \in \Gamma\}$ is a basis of $h(1,q) \cong \big(\Bbbk\langle x_1, x_2\rangle \# \Bbbk[\mathbb{Z}/(N)]\big)/\big([x_1x_2],\ x_1^N,\ x_2^N\big)$.

4. *Frobenius-Lusztig kernel.* Let $q \in \Bbbk^\times$ with $\mathrm{ord} q = N > 2$, $\mathbb{Z}/(N) = \langle g_1 \rangle$, $g := g_2 := g_1$, and $\chi_1(g_i) := q^{-2}$, $\chi_2(g_i) := q^2$ for $i=1,2$. Then $\{x_2^{r_2}x_1^{r_1}g \mid 0 \leq r_i < N, g \in \Gamma\}$ is a basis of $u_q(\mathfrak{sl}_2) \cong \big(\Bbbk\langle x_1, x_2\rangle \# \Bbbk[\mathbb{Z}/(N)]\big)/\big([x_1x_2]-(1-g^2),\ x_1^N,\ x_2^N\big)$.

Proof. In (1) it is $d_1 = d_2 = c_{12} = 0$. (3) and (4) are special cases of (2): By definition of $\lambda_{12}, \mu_1, \mu_2$ the elements have the required $\widehat{\Gamma}$-degree. As in Example 9.1 we show conditions Eq. (9.1). Eq. (9.2): We have $\chi_1\chi_2 = \varepsilon$ if $\lambda_{12} \neq 0$, hence $q_{11}q_{12} = 1$ and then $q_{11} = q_{11}q_{12}q_{21} = q_{21}$, since $q_{12}q_{21} = 1$. Using these equations we calculate

$$[\underbrace{x_1, \dots [x_1}_{N_1-1}, \lambda_{12}(1-g_1g_2)]_{q_{11}q_{12}} \dots]_{q_{11}^{N_1-1}q_{12}} = -\lambda_{12}(1-q_{11}^2)\dots(1-q_{11}^{N_1})x_1^{N_1-1}g_1g_2 = 0.$$

Further $\chi_i^{N_i} = \varepsilon$ if $\mu_1 \neq 0$, thus $q_{21}^{N_1} = 1$; by taking $q_{12}q_{21} = 1$ to the N_1-th power, we deduce $q_{12}^{N_1} = 1$. Then $\left[\mu_1(1-g_1^{N_1}), x_2\right]_{q_{12}^{N_1}} = \mu_1(1-q_{12}^{N_1})x_2 = 0$. The remaining condition Eq. (9.3) works in a similar way. $\square$

9.2 PBW basis for $L = \{x_1 < x_1x_2 < x_2\}$

We now examine the case when $[L]$ is a PBW Basis of $(T(V)\#\Bbbk[\Gamma])/I$, where I is generated by the following elements

$$\begin{aligned} &[x_1x_1x_2] - c_{112}, && x_1^{N_1} - d_1, \\ &[x_1x_2x_2] - c_{122}, && [x_1x_2]^{N_{12}} - d_{12}, \\ & && x_2^{N_2} - d_2, \end{aligned}$$

with $\mathrm{ord} q_{11} = N_1$, $\mathrm{ord} q_{12,12} = N_{12}$, $\mathrm{ord} q_{22} = N_2 \in \{2, 3, \dots, \infty\}$. We have in $\Bbbk\langle x_1, x_{12}, x_2\rangle\#\Bbbk[\Gamma]$ the elements

$$c^\rho_{(1|12)} = c^\rho_{112}, \quad c^\rho_{(1|2)} = x_{12}, \quad c^\rho_{(12|2)} = c^\rho_{122}.$$

At first we want to study the conditions in general. By Theorem 4.2(2') we have to check the following: The only Jacobi condition is for $1 < 12 < 2$, namely

$$\big[c^\rho_{112}, x_2\big]_{q_{112,2}} - \big[x_1, c^\rho_{122}\big]_{q_{1,122}} + (q_{1,12} - q_{12,2})x_{12}^2 \in I_{\prec x_1x_{12}x_2}. \tag{9.4}$$

There are the following restricted q-Leibniz conditions: If $N_1 < \infty$, then we have to check Eqs. (9.1) and (9.2) for $1 < 2$; note that we can omit the restricted Leibniz condition for $1 < 12$ in (2') of Theorem 4.2. In the same way if $N_2 < \infty$, then there are the conditions Eqs. (9.1) and (9.3) for $1 < 2$; we can omit the condition for $12 < 2$. Further Eq. (9.2) resp. (9.3) is equivalent to

$$[\underbrace{x_1, \dots [x_1}_{N_1-2}, c^\rho_{112}]_{q_{11}^2 q_{12}} \dots]_{q_{11}^{N_1-1} q_{12}} - [d^\rho_1, x_2]_{q_{12}^{N_1}} \in I_{\prec x_1^{N_1} x_2}, \tag{9.5}$$

$$[\dots [c^\rho_{122}, \underbrace{x_2]_{q_{12}q_{22}^2} \dots, x_2}_{N_2-2}]_{q_{12}q_{22}^{N_2-1}} - [x_1, d^\rho_2]_{q_{12}^{N_2}} \in I_{\prec x_1 x_2^{N_2}}. \tag{9.6}$$

In the case $N_1 = 2$ resp. $N_2 = 2$ then condition Eq. (9.5) resp. (9.6) is

$$c^\rho_{112} - [d^\rho_1, x_2]_{q_{12}^2} \in I_{\prec x_1^2 x_2} \quad \text{resp.} \quad c^\rho_{122} - [x_1, d^\rho_2]_{q_{12}^2} \in I_{\prec x_1 x_2^2}.$$

Here we see with Corollary 5.6 that by the restricted q-Leibniz formula $[x_1x_1x_2] - c_{112} \in (x_1^2 - d_1)$ resp. $[x_1x_2x_2] - c_{122} \in (x_2^2 - d_2)$, hence these two relations are redundant. Suppose $[d_1, x_2]_{q_{12}^2} \prec_L [x_1x_1x_2]$ resp. $[x_1, d_2]_{q_{12}^2} \prec_L [x_1x_2x_2]$. Thus if we define

$$c^\rho_{112} := [d^\rho_1, x_2]_{q_{12}^2} \quad \text{resp.} \quad c^\rho_{122} := [x_1, d^\rho_2]_{q_{12}^2}, \tag{9.7}$$

then condition Eq. (9.5) resp. (9.6) is fulfilled.

Finally, if $N_{12} < \infty$, then there are the conditions

$$\begin{aligned} &\big[d^\rho_{12}, x_{12}\big]_1 \in I_{\prec x_{12}^{N_{12}+1}}, \\ &[\dots [c^\rho_{112}, \underbrace{x_{12}]_{q_{1,12}q_{12,12}} \dots, x_{12}}_{N_{12}-1}]_{q_{1,12}q_{12,12}^{N_{12}-1}} - [x_1, d^\rho_{12}]_{q_{1,12}^{N_{12}}} \in I_{\prec x_1 x_{12}^{N_{12}}}, \\ &[\underbrace{x_{12}, \dots [x_{12}}_{N_{12}-1}, c^\rho_{122}]_{q_{12,12}q_{12,2}} \dots]_{q_{12,12}^{N_{12}-1} q_{12,2}} - [d^\rho_{12}, x_2]_{q_{12,2}^{N_{12}}} \in I_{\prec x_{12}^{N_{12}} x_2}. \end{aligned} \tag{9.8}$$

Now we want to take a closer look at Eq. (9.4). Essentially, there are two cases: If $q_{11} = q_{22}$ we set $q := q_{112,2} = q_{1,122}$ and then Eq. (9.4) reads

$$\left[c^\rho_{112}, x_2\right]_q - \left[x_1, c^\rho_{122}\right]_q \in I_{\prec x_1 x_{12} x_2}. \tag{9.9}$$

Else if $q_{11} \neq q_{22}$. Suppose $N_{12} = \mathrm{ord} q_{12,12} = 2$, then we define

$$d_{12} := -(q_{1,12} - q_{12,2})^{-1}\big(\left[c_{112}, x_2\right]_{q_{1,2}q_{12,2}} - \left[x_1, c_{122}\right]_{q_{1,122}}\big).$$

It is $[x_1x_2]^2 - d_{12} \in \big([x_1x_1x_2] - c_{112},\, [x_1x_2x_2] - c_{122}\big)$ by the q-Jacobi identity, see Eq. (9.4) and Corollary 5.6, i.e., this relation is redundant. Further $d_{12} \in (\Bbbk\langle X\rangle \# \Bbbk[\Gamma])^{\chi^2_{12}}$. Let us assume that $d_{12} \prec_L [x_1x_2]^2$, e.g., c_{122}, c_{112} are linear combinations of monomials of length < 3. Then for

$$d^\rho_{12} := -(q_{1,12} - q_{12,2})^{-1}\big(\left[c^\rho_{112}, x_2\right]_{q_{1,2}q_{12,2}} - \left[x_1, c^\rho_{122}\right]_{q_{1,122}}\big) \tag{9.10}$$

condition Eq. (9.4) is fulfilled.

As a demonstration we want to proof that the Hopf algebras coming from liftings of a Nichols algebra with Cartan matrix A_2 [9, Thm. 5.9], admit a PBW basis $[L]$ (this is already known for liftings of Nichols algebras of Cartan type A_2 [2], but not for non-Cartan type):

Proposition 9.3 (Liftings A_2)**.** *Consider the Hopf algebras $(\Bbbk\langle x_1, x_2\rangle \# \Bbbk[\Gamma])/I$ where I depends upon (q_{ij}) as follows:*

(1) *Cartan type A_2: $q_{12}q_{21} = q_{11}^{-1} = q_{22}^{-1}$.*
 (a) *If $q_{11} = -1$, then let I be generated by*

$$x_1^2 - \mu_1(1 - g_1^2), \qquad [x_1x_2]^2 - 4\mu_1 q_{21} x_2^2 - \mu_{12}(1 - g_{12}^2), \qquad x_2^2 - \mu_2(1 - g_2^2).$$

 (b) *If $\mathrm{ord} q_{11} = 3$, then let I be generated by*

$$\begin{aligned}
&[x_1x_1x_2] - \lambda_{112}(1 - g_{112}), \quad [x_1x_2x_2] - \lambda_{122}(1 - g_{122}),\\
&x_1^3 - \mu_1(1 - g_1^3),\\
&[x_1x_2]^3 + (1 - q_{11})q_{11}\lambda_{112}[x_1x_2x_2]\\
&\qquad - \mu_1(1 - q_{11})^3 x_2^3 - \mu_{12}(1 - g_{12}^3),\\
&x_2^3 - \mu_2(1 - g_2^3).
\end{aligned}$$

 (c) *If $N := \mathrm{ord} q_{11} \geq 4$, then then let I be generated by, see* [2]*,*

$$\begin{aligned}
&[x_1x_1x_2], \quad [x_1x_2x_2],\\
&x_1^N - \mu_1(1 - g_1^N),\\
&[x_1x_2]^N - \mu_1(q_{11} - 1)^N q_{21}^{\frac{N(N-1)}{2}} x_2^N - \mu_{12}(1 - g_{12}^N),\\
&x_2^N - \mu_2(1 - g_2^N).
\end{aligned}$$

(2) *Let* $q_{12}q_{21} = q_{11}^{-1}$, $q_{22} = -1$.
(a) *If* $4 \neq N := \mathrm{ord} q_{11} \geq 3$, *then let* I *be generated by*

$$[x_1 x_1 x_2], \qquad x_1^N - \mu_1(1 - g_1^N), \qquad x_2^2 - \mu_2(1 - g_2^2).$$

(b) *If* $\mathrm{ord} q_{11} = 4$, *then let* I *be generated by*

$$[x_1 x_1 x_2] - \lambda_{112}(1 - g_{112}), \qquad x_1^4 - \mu_1(1 - g_1^4), \qquad x_2^2 - \mu_2(1 - g_2^2).$$

(3) *Let* $q_{11} = -1$, $q_{12}q_{21} = q_{22}^{-1}$.
(a) *If* $4 \neq N := \mathrm{ord} q_{22} \geq 3$, *then let* I *be generated by*

$$[x_1 x_2 x_2], \qquad x_1^2 - \mu_1(1 - g_1^2), \qquad x_2^N - \mu_2(1 - g_2^N).$$

(b) *If* $\mathrm{ord} q_{22} = 4$, *then let* I *be generated by*

$$[x_1 x_2 x_2] - \lambda_{122}(1 - g_{122}), \qquad x_1^2 - \mu_1(1 - g_1^2), \qquad x_2^4 - \mu_2(1 - g_2^4).$$

(4) *Let* $q_{11} = q_{22} = -1$ *and* $N := \mathrm{ord} q_{12}q_{21} \geq 3$.
(a) *If* $q_{12} \neq \pm 1$, *then let* I *be generated by*

$$x_1^2 - \mu_1(1 - g_1^2), \qquad [x_1 x_2]^N - \mu_{12}(1 - g_{12}^N), \qquad x_2^2.$$

(b) *If* $q_{12} = \pm 1$, *then let* I *be generated by*

$$x_1^2, \qquad [x_1 x_2]^N - \mu_{12}(1 - g_{12}^N), \qquad x_2^2 - \mu_2(1 - g_2^2).$$

All of these Hopf algebras have basis $\{x_2^{r_2}[x_1x_2]^{r_{12}}x_1^{r_1} g \mid 0 \leq r_u < N_u$ *for all* $u \in L$, $g \in \Gamma\}$.

Proof. Note that all defined ideals are $\widehat{\Gamma}$-homogeneous by the definition of the coefficients. The conditions Eq. (9.1) are exactly as in Example 9.1.

(1a) We have $N_1 = N_2 = N_{12} = 2$. Since $d_1^\rho = \mu_1(1 - g_1^2)$ we have by the argument preceding Eq. (9.7), that necessarily

$$c_{112} = [\mu_1(1 - g_1^2), x_2]_{q_{12}^2} \quad \text{and} \quad c_{122} = [x_1, \mu_2(1 - g_2^2)]_{q_{12}^2}$$

and the conditions Eqs. (9.5) and (9.6) are fulfilled. Note that $c_{112} = \mu_1(1 - q_{12}^2)x_2 = 0$: either $\mu_1 = 0$ or else $\mu_1 \neq 0$, but then $\chi_1^2 = \varepsilon$ and $q_{21}^2 = 1$. By squaring the assumption $q_{12}q_{21} = -1$, we obtain $q_{12}^2 = 1$. In the same way $c_{122} = 0$.

Then the conditions Eq. (9.8) are

$$\begin{aligned} \left[4\mu_1 q_{21} x_2^2 + \mu_{12}(1 - g_{12}^2), x_{12}\right]_1 &\in I_{\prec x_{12}^3} \\ [0, x_{12}]_{q_{1,12}q_{12,12}} - [x_1, 4\mu_1 q_{21} x_2^2 + \mu_{12}(1 - g_{12}^2)]_{q_{1,12}^2} &\in I_{\prec x_1 x_{12}^2}, \\ [x_{12}, 0]_{q_{12,12}q_{12,2}} - [4\mu_1 q_{21} x_2^2 + \mu_{12}(1 - g_{12}^2), x_2]_{q_{12,2}^2} &\in I_{\prec x_{12}^2 x_2}. \end{aligned}$$

Again, if $\mu_1 \neq 0$, then $q_{12}^2 = q_{21}^2 = 1$, hence $q_{1,12}^2 = 1$ and $q_{2,12}^2 = 1$. If $\mu_{12} \neq 0$, then $\chi_{12}^2 = \varepsilon$ and $q_{1,12}^2 = 1$; in this case also $q_{12}^2 = q_{21}^2 = 1$. Thus modulo $I_{\prec x_{12}^3}$ we have

$$\begin{aligned}\big[4\mu_1 q_{21} x_2^2 + \mu_{12}(1 - g_{12}^2), x_{12}\big]_1 &= 4\mu_1 q_{21}\big[x_2^2, x_{12}\big]_1 - \mu_{12}(q_{12,12}^2 - 1)x_{12}g_{12}^2 \\ &= 4\mu_1\mu_2 q_{21}\big[1 - g_2^2, x_{12}\big]_1 = -4\mu_1\mu_2 q_{21}(q_{2,12}^2 - 1)x_{12}g_2^2 = 0.\end{aligned}$$

Further modulo $I_{\prec x_1 x_{12}^2}$ we get

$$\begin{aligned}[x_1, 4\mu_1 q_{21} x_2^2 + \mu_{12}(1 - g_{12}^2)]_1 &= 4\mu_1 q_{21}[x_1, x_2^2]_1 + \mu_{12}[x_1, 1 - g_{12}^2]_1 \\ &= 4\mu_1 q_{21} c_{122}^{\rho} - \mu_{12}(1 - q_{12,1}^2)x_1 g_{12}^2 = 0,\end{aligned}$$

which means that the second condition is fulfilled. The third one of Eq. (9.8) works analogously.

The last condition is Eq. (9.4), or equivalently condition Eq. (9.9) since $q_{11} = q_{22}$:

$$\big[0, x_2\big]_q - \big[x_1, 0\big]_q = 0 \in I_{\prec x_1 x_{12} x_2}.$$

(1b) Either $\lambda_{112} = \lambda_{122} = 0$, or $\chi_{112} = \varepsilon$ and/or $\chi_{122} = \varepsilon$, from where we conclude $q := q_{11} = q_{12} = q_{21} = q_{22}$. We start with Eq. (9.4): Since $q^3 = 1$ we have $\big[\lambda_{112}(1 - g_{112}), x_2\big]_1 - \big[x_1, \lambda_{122}(1 - g_{122})\big]_1 = 0$. We continue with Eq. (9.5): Either $\mu_1 = 0$ or $\chi_1^3 = \varepsilon$, hence $q_{21}^3 = 1$ and then also $q_{12}^3 = (q_{12}q_{21})^3 = q_{11}^{-3} = 1$. Then $\big[x_1, \lambda_{112}(1 - g_{112})\big]_1 - [\mu_1(1 - g_1^3), x_2]_1 = 0$. Next, Eq. (9.6): In the same way, $\mu_2 \neq 0$ or $q_{21}^3 = q_{12}^3 = 1$. Then $\big[\lambda_{122}(1 - g_{122}), x_2\big]_1 - [x_1, \mu_2(1 - g_2^3)]_1 = 0$. For Eq. (9.8) we have $q_{1,12}^3 = 1$ if $\mu_{12} \neq 0$. Thus $q_{12}^3 = 1$, moreover $q_{21}^3 = (q_{12}q_{21})^3 = q_{11}^{-3} = 1$. Hence modulo $I_{\prec x_1 x_{12}^3}$ we have

$$\begin{aligned}&\big[[\lambda_{112}(1 - g_{112}), x_{12}]_{q_{1,12}q_{12,12}}, x_{12}\big]_{q_{1,12}q_{12,12}^2} \\ &\qquad - \big[x_1, -(1 - q_{11})q_{11}\lambda_{112}\lambda_{122}(1 - g_{122}) + \mu_1(1 - q_{11})^3 x_2^3 + \mu_{12}(1 - g_{12}^3)\big]_{q_{1,12}^3} = 0,\end{aligned}$$

since each summand is zero. Further a straightforward calculation shows

$$\begin{aligned}&\big[x_{12}, [x_{12}, \lambda_{122}(1 - g_{122})]_{q_{12,12}q_{12,2}}\big]_{q_{12,12}^2 q_{12,2}} \\ &\qquad - \big[-(1 - q_{11})q_{11}\lambda_{112}\lambda_{122}(1 - g_{122}) + \mu_1(1 - q_{11})^3 x_2^3 + \mu_{12}(1 - g_{12}^3), x_2\big]_{q_{12,2}^2} = 0.\end{aligned}$$

Finally, an easy calculation shows that

$$\big[-(1 - q_{11})q_{11}\lambda_{112}\lambda_{122}(1 - g_{122}) + \mu_1(1 - q_{11})^3 x_2^3 + \mu_{12}(1 - g_{12}^3), x_{12}\big]_1 = 0$$

modulo $I_{\prec x_{12}^4}$, again by definition of the coefficients.

(1c) is a generalization of (1a) (and (1b) if $\lambda_{112} = \lambda_{122} = 0$) and works completely in the same way (only the Serre-relations $[x_1x_1x_2] = [x_1x_2x_2] = 0$ are not redundant, as they are (1a)). We leave this to the reader.

(2a) We leave this to the reader and prove the little more complicated (2b): Since we have $N_2=2$, as in (1a) we deduce from Eq. (9.7), that $c_{122}=[x_1,\mu_2(1-g_2^2)]_{q_{12}^2}=\mu_2(q_{21}^2-1)x_1g_2^2$ and the condition Eq. (9.6) is fulfilled.

If $\lambda_{112}\neq 0$ then $q_{11}=q_{21}$ of order 4, $q_{12}=q_{22}=-1$; if $\mu_1\neq 0$ then $q_{12}^4=1$. Then Eq. (9.5) is fulfilled: $\big[x_1,[x_1,\lambda_{112}(1-g_{112})]_1\big]_{q_{11}}-[\mu_1(1-g_1^4),x_2]_1=0$, since both summands are zero.

It is $q_{11}\neq q_{22}$, $\mathrm{ord} q_{12,12}=2$ and c_{112}^ρ resp. c_{122}^ρ are linear combinations of monomials of length 0 resp. 1. By the discussion before Eq. (9.10), we see that $[x_1x_2]^2-d_{12}$ is redundant and for

$$\begin{aligned} d_{12}^\rho &:= -(q_{1,12}-q_{12,2})^{-1}\Big(\big[\lambda_{112}(1-g_{112}),x_2\big]_{-1}-\big[x_1,\mu_2(q_{21}^2-1)x_1g_2^2\big]_{q_{11}}\Big)\\ &= -q_{12}^{-1}(q_{11}+1)^{-1}\big(\lambda_{112}2x_2-\mu_2\underbrace{(q_{21}^2-1)(1-q_{11}q_{21}^2)}_{=:q}x_1^2g_2^2\big)\end{aligned}$$

the condition Eq. (9.4) is fulfilled. We are left to show the conditions Eq. (9.8) $\big[d_{12}^\rho,x_{12}\big]_1\in I_{\prec x_{12}^3}$,

$$\big[c_{112}^\rho,x_{12}\big]_{q_{112,12}}-\big[x_1,d_{12}^\rho\big]_{q_{1,12}^2}\in I_{\prec x_1x_{12}^2}\quad\text{and}\quad \big[x_{12},c_{122}^\rho\big]_{q_{12,122}}-\big[d_{12}^\rho,x_2\big]_{q_{12,2}^2}\in I_{\prec x_{12}^2x_2}.$$

We calculate the first one: Modulo $I_{\prec x_{12}^3}$ we get

$$\big[d_{12}^\rho,x_{12}\big]_1=-q_{12}^{-1}(q_{11}+1)^{-1}\big(-\lambda_{112}2\underbrace{\big[x_{12},x_2\big]_1}_{=c_{122}^\rho}-\mu_2q\underbrace{\big[x_1^2g_2^2,x_{12}\big]_1}_{=q_{21}^2[x_1^2,x_{12}]_{q_{1,12}^2}g_2^2}\big).$$

Now by the q-derivation property $[x_1^2,x_{12}]_{q_{1,12}^2}=x_1c_{112}^\rho+q_{1,12}c_{112}^\rho x_1=\lambda_{112}(1-q_{11})x_1$. Because of the coefficient λ_{112} the two summands in the parentheses have the coefficient $\pm 4\lambda_{112}\mu_2$, hence cancel. (3) works exactly as (2).

(4a) Since we have $N_1=N_2=2$, as in (1a) we deduce from Eq. (9.7), that

$$c_{112}=[\mu_1(1-g_1^2),x_2]_{q_{12}^2}=\mu_1(1-q_{12}^2)x_2\quad\text{and}\quad c_{122}=[x_1,0]_{q_{12}^2}=0$$

and the conditions Eqs. (9.5) and (9.6) are fulfilled.

For the second condition of Eq. (9.8) one can easily show by induction

$$\begin{aligned}&\big[\dots[c_{112}^\rho,\underbrace{x_{12}]_{q_{1,12}q_{12,12}}\dots,x_{12}}_{N-1}\big]_{q_{1,12}q_{12,12}^{N-1}}\\ &=\mu_1(1-q_{12}^2)\big[\dots[x_2,\underbrace{x_{12}]_{q_{11}q_{12}^2=q_{21}}\dots,x_{12}}_{N-1}\big]_{q_{11}q_{12}^Nq_{21}^{N-1}}=\mu_1\prod_{i=0}^{N-1}(1-q_{12}^{i+2}q_{21}^i)x_2x_{12}^{N-1}=0.\end{aligned}$$

The last equation holds since for $i=N-2$ we have $1-q_{12}^Nq_{21}^{N-2}=0$: if $\mu_1\neq 0$ then $q_{21}^2=1$ and $(q_{12}q_{21})^N=q_{12,12}^N=1$. Further also $[x_1,d_{12}^\rho]_{q_{1,12}^N}=[x_1,\mu_{12}(1-g_{12}^N)]_1=$

$-\mu_{12}(1-q_{12,1}^N)x_1g_{12}^N = 0$, since either $\mu_{12}=0$ or $q_{12}^N = q_{21}^N = (-1)^N$ such that $q_{12,1}^N = (-1)^N(-1)^N = 1$. This proves the second condition of Eq. (9.8). The third of Eq. (9.8) is easy since $c_{122}=0$, and the first of Eq. (9.8) is a direct computation.

Finally, Eq. (9.4) is Eq. (9.9), since $q_{11}=q_{22}$: $\big[\mu_1(1-q_{12}^2)x_2, x_2\big]_{q_{112,2}} - \big[x_1, 0\big]_{q_{1,122}} = 0$ because of the relation $x_2^2=0$.

(4b) works analogously to (4a). Note that here $c_{112}=0$ and $c_{122} = [x_1, \mu_2(1-g_2^2)]_1 = \mu_2(q_{21}^2-1)x_1g_2^2$. □

9.3 PBW basis for $L = \{x_1 < x_1x_1x_2 < x_1x_2 < x_2\}$

This PBW basis $[L]$ occurs in the Nichols algebras with Cartan matrix B_2 and their liftings [9, Prop. 5.11,Thm. 5.13]. More generally, we list the conditions when $[L]$ is a PBW Basis of $(T(V)\#\Bbbk[\Gamma])/I$ where I is generated by

$$\begin{aligned} &[x_1x_1x_1x_2] - c_{1112}, && x_1^{N_1} - d_1,\\ &[x_1x_1x_2x_1x_2] - c_{11212}, && [x_1x_1x_2]^{N_{112}} - d_{112},\\ &[x_1x_2x_2] - c_{122}, && [x_1x_2]^{N_{12}} - d_{12},\\ & && x_2^{N_2} - d_2. \end{aligned}$$

In $\Bbbk\langle x_1, x_{112}, x_{12}, x_2\rangle\#\Bbbk[\Gamma]$ we have the following $c^\rho_{(u|v)}$ ordered by $\ell(uv)$, $u,v\in L$: If $\mathrm{Sh}(uv) = (u|v)$ then

$$\begin{aligned} &c^\rho_{(1|2)} = x_{12}, && c^\rho_{(12|2)} = c^\rho_{122}, && c^\rho_{(112|12)} = c^\rho_{11212},\\ &c^\rho_{(1|12)} = x_{112}, && c^\rho_{(1|112)} = c^\rho_{1112}, \end{aligned}$$

and for $\mathrm{Sh}(1122) \neq (112|2)$ by Eq. (4.1)

$$\begin{aligned} c^\rho_{(112|2)} &= \partial_1^\rho(c^\rho_{(12|2)}) + q_{12,2}c^\rho_{(1|2)}x_{12} - q_{1,12}x_{12}c^\rho_{(1|2)},\\ &= \partial_1^\rho(c^\rho_{122}) + (q_{12,2}-q_{1,12})x_{12}^2. \end{aligned}$$

We have for $1<112<2$, $1<112<12$ and $112<12<2$ the following q-Jacobi conditions (note that we can leave out $1<12<2$):

$$\begin{aligned} &\big[c^\rho_{1112}, x_2\big]_{q_{1112,2}} - \big[x_1, c^\rho_{(112|2)}\big]_{q_{1,1122}}\\ &\qquad + q_{1,112}x_{112}[x_1,x_2] - q_{112,2}[x_1,x_2]x_{112} \in I_{\prec x_1x_{112}x_2}\\ \Leftrightarrow &\big[c^\rho_{1112}, x_2\big]_{q_{1112,2}} - \big[x_1, \partial_1^\rho(c^\rho_{122})\big]_{q_{1,1122}}\\ &\quad - (q_{12,2}-q_{1,12})c^\rho_{11212} - (q_{12,2}-q_{1,12})q_{1,12}(q_{12,12}+1)x_{12}x_{112}\\ &\qquad + q_{1,112}c^\rho_{11212} + q_{112,2}(q_{1,112}q_{112,1}-1)x_{12}x_{112} \in I_{\prec x_1x_{112}x_2}\\ \Leftrightarrow &\big[c^\rho_{1112}, x_2\big]_{q_{1112,2}} - \big[x_1, \partial_1^\rho(c^\rho_{122})\big]_{q_{1,1122}} + \underbrace{q_{12}(q_{11}^2 - q_{22} + q_{11})}_{=:q} c^\rho_{11212}\\ &\quad + \underbrace{q_{12}^2(q_{22}(q_{11}^4q_{12}q_{21}-1) - q_{11}(q_{22}-q_{11})(q_{12,12}+1))}_{=:q'} x_{12}x_{112} \in I_{\prec x_1x_{112}x_2} \end{aligned} \tag{9.11}$$

If $q \neq 0$, we see that $[x_1x_1x_2x_1x_2] - c_{11212} \in \big([x_1x_1x_1x_2] - c_{1112}, [x_1x_2x_2] - c_{122}\big)$ is redundant with

$$c_{11212} = -q^{-1}\big(\big[c_{1112}, x_2\big]_{q_{1112,2}} - \big[x_1, \partial_1(c_{122})\big]_{q_{1,1122}} + q'[x_1x_2][x_1x_1x_2]\big)$$

by Corollary 5.6 and the q-Jacobi identity of Proposition 1.2. We have $\deg_{\widehat{\Gamma}}(c_{11212}) = \chi_{11212}$; suppose that $c_{11212} \prec_L [x_1x_1x_2x_1x_2]$ (e.g. c_{1112} resp. c_{122} are linear combinations of monomials of length < 4 resp. < 3) then condition Eq. (9.11) is fulfilled for

$$c^\rho_{11212} := -q^{-1}\big(\big[c^\rho_{1112}, x_2\big]_{q_{1112,2}} - \big[x_1, \partial^\rho_1(c^\rho_{122})\big]_{q_{1,1122}} + q' x_{12}x_{112}\big).$$

There are three cases, where the coefficients q, q' are of a better form for our setting: Since

$$q = q_{12}\big((3)_{q_{11}} - (2)_{q_{22}}\big), \quad q' = q_{12}\big(q(1 + q^2_{11}q_{12}q_{21}q_{22}) - q_{11}q_{12}(2)_{q_{22}}\big),$$

we have

$$\begin{aligned}
&q = q_{12}q_{11} \neq 0, && q' = -q_{12}q^2_{11}q(1 - q^2_{11}q_{12}q_{21}), && \text{if } q^2_{11} = q_{22},\\
&q = q_{12}(3)_{q_{11}}, && q' = q_{12}q(1 - q^2_{11}q_{12}q_{21}), && \text{if } q_{22} = -1,\\
&q = -q_{12}(2)_{q_{22}}, && q' = -q_{12}q(1 + q_{11} + q^2_{11}q_{12}q_{21}q_{22}), && \text{if } \mathrm{ord} q_{11} = 3.
\end{aligned}$$

The second q-Jacobi condition for $1 < 112 < 12$ reads

$$\begin{aligned}
&\big[c^\rho_{1112}, x_{12}\big]_{q_{1112,12}} - \big[x_1, c^\rho_{11212}\big]_{q_{1,11212}}\\
&\qquad + q_{1,112}x_{112}[x_1, x_{12}] - q_{112,12}[x_1, x_{12}]x_{112} \in I_{\prec x_1x_{112}x_{12}}\\
\Leftrightarrow &\big[c^\rho_{1112}, x_{12}\big]_{q_{1112,12}} - \big[x_1, c^\rho_{11212}\big]_{q_{1,11212}} + \underbrace{q^2_{11}q_{12}(1 - q_{12}q_{21}q_{22})}_{=:q''} x^2_{112} \in I_{\prec x_1x_{112}x_{12}}
\end{aligned} \tag{9.12}$$

If $q'' \neq 0$ then we see that $[x_1x_1x_2]^2 - d_{112} \in \big([x_1x_1x_1x_2] - c_{11212}, [x_1x_1x_2x_1x_2] - c_{11212}\big)$ is redundant with $d_{112} = -q''^{-1}\big(\big[c_{1112}, [x_1x_2]\big]_{q_{1112,12}} - \big[x_1, c_{11212}\big]_{q_{1,11212}}\big)$ by Corollary 5.6 and the q-Jacobi identity of Proposition 1.2. It is $\deg_{\widehat{\Gamma}}(d_{112}) = \chi^2_{112}$; suppose that $d_{112} \prec_L [x_1x_1x_2]^2$ then condition Eq. (9.13) is fulfilled for

$$d^\rho_{112} := -q''^{-1}\big(\big[c^\rho_{1112}, x_{12}\big]_{q_{1112,12}} - \big[x_1, c^\rho_{11212}\big]_{q_{1,11212}}\big)$$

If further $\mathrm{ord} q_{112,112} = 2$ then we have to consider the restricted q-Leibniz conditions for d^ρ_{112} (see below).

The last q-Jacobi condition for $112 < 12 < 2$ is

$$\begin{aligned}
&\big[c^\rho_{11212}, x_2\big]_{q_{11212,2}} - \big[x_{112}, c^\rho_{122}\big]_{q_{112,122}}\\
&\qquad + q_{112,12}x_{12}[x_{112}, x_2] - q_{12,2}[x_{112}, x_2]x_{12} \in I_{\prec x_{112}x_{12}x_2}\\
\Leftrightarrow &\big[c^\rho_{11212}, x_2\big]_{q_{11212,2}} - \big[x_{112}, c^\rho_{122}\big]_{q_{112,122}}\\
&\quad + q_{112,12}x_{12}\partial^\rho_1(c^\rho_{122}) - q_{12,2}\partial^\rho_1(c^\rho_{122})x_{12}\\
&\qquad + \underbrace{q^2_{12}q_{22}(q_{22} - q_{11})(q^2_{11}q_{12}q_{21} - 1)}_{=:q'''} x^3_{12} \in I_{\prec x_{112}x_{12}x_2}
\end{aligned} \tag{9.13}$$

If $q''' \neq 0$ then we see that $[x_1x_2]^3 - d_{12} \in \big([x_1x_1x_2x_1x_2] - c_{11212}, [x_1x_2x_2] - c_{122}\big)$ is redundant with $d_{12} := -q'''^{-1}\big(\big[c_{11212}, x_2\big]_{q_{11212,2}} - \big[[x_1x_1x_2], c_{122}\big]_{q_{112,122}} + q_{112,12}[x_1x_2]\partial_1(c_{122}) - q_{12,2}\partial_1(c_{122})[x_1x_2]\big)$ by Corollary 5.6 and the q-Jacobi identity of Proposition 1.2. It is $\deg_{\widehat{\Gamma}}(d_{12}) = \chi_{12}^3$; suppose that $d_{12} \prec_L [x_1x_1]^3$ (e.g., c_{11212} resp. c_{122} are linear combinations of monomials of length < 5 resp. < 3) then condition Eq. (9.13) is fulfilled for

$$\begin{aligned} d_{12}^\rho := -q''^{-1}\big(\big[c_{11212}^\rho, x_2\big]_{q_{11212,2}} - \big[x_{112}, c_{122}^\rho\big]_{q_{112,122}} \\ + q_{112,12}x_{12}\partial_1^\rho(c_{122}^\rho) - q_{12,2}\partial_1^\rho(c_{122}^\rho)x_{12}\big) \end{aligned}$$

If further $\operatorname{ord} q_{12,12} = 3$ then we have to consider the q-Leibniz conditions for d_{12}^ρ (see below).

There are the following restricted q-Leibniz conditions: If $N_1 < \infty$, then $\big[d_1^\rho, x_1\big]_1 \in I_{\prec x_1^{N_1+1}}$ and for $1 < 2$ (we can omit $1 < 12, 1 < 112$)

$$\big[\underbrace{x_1, \dots [x_1}_{N_1-3}, c_{1112}^\rho]_{q_{11}^3 q_{12}} \dots\big]_{q_{11}^{N_1-1} q_{12}} - [d_1^\rho, x_2]_{q_{12}^{N_1}} \in I_{\prec x_1^{N_1} x_2}. \tag{9.14}$$

If $N_2 < \infty$, then $\big[d_2^\rho, x_2\big]_1 \in I_{\prec x_2^{N_2+1}}$ and for $1 < 2$ (we can omit $12 < 2, 112 < 2$)

$$\big[\dots [c_{122}^\rho \underbrace{x_2]_{q_{12}q_{22}^2} \dots, x_2}_{N_2-2}\big]_{q_{12}q_{22}^{N_2-1}} - [x_1, d_2^\rho]_{q_{12}^{N_2}} \in I_{\prec x_1 x_2^{N_2}}. \tag{9.15}$$

If $N_{12} < \infty$, then $\big[d_{12}^\rho, x_{12}\big]_1 \in I_{\prec x_{12}^{N_{12}+1}}$ and for $1 < 12$, $12 < 2$ (we can omit $112 < 12$)

$$\begin{aligned} &\big[\dots [c_{112}^\rho, \underbrace{x_{12}]_{q_{1,12}q_{12,12}} \dots, x_{12}}_{N_{12}-1}\big]_{q_{1,12}q_{12,12}^{N_{12}-1}} - [x_1, d_{12}^\rho]_{q_{1,12}^{N_{12}}} \in I_{\prec x_1 x_{12}^{N_{12}}}, \\ &\big[\underbrace{x_{12}, \dots [x_{12}}_{N_{12}-1}, c_{122}^\rho]_{q_{12,12}q_{12,2}} \dots\big]_{q_{12,12}^{N_{12}-1} q_{12,2}} - [d_{12}^\rho, x_2]_{q_{12,2}^{N_{12}}} \in I_{\prec x_{12}^{N_{12}} x_2}. \end{aligned} \tag{9.16}$$

If $N_{112} < \infty$, then $\big[d_{112}^\rho, x_{112}\big]_1 \in I_{\prec x_{112}^{N_{112}+1}}$ and for $1 < 112$, $112 < 12$, $112 < 2$

$$\begin{aligned} &\big[\dots [c_{1112}^\rho, \underbrace{x_{112}]_{q_{1,112}q_{112,112}} \dots, x_{112}}_{N_{112}-1}\big]_{q_{1,112}q_{112,112}^{N_{112}-1}} - [x_1, d_{112}^\rho]_{q_{1,112}^{N_{112}}} \in I_{\prec x_1 x_{112}^{N_{112}}} \\ &\big[\underbrace{x_{112}, \dots [x_{112}}_{N_{112}-1}, c_{11212}^\rho]_{q_{112,112}q_{112,12}} \dots\big]_{q_{112,112}^{N_{112}-1} q_{112,12}} - [d_{112}^\rho, x_{12}]_{q_{112,12}^{N_{112}}} \in I_{\prec x_{112}^{N_{112}} x_{12}} \\ &\big[\underbrace{x_{112}, \dots [x_{112}}_{N_{112}-1}, c_{(112|2)}^\rho]_{q_{112,112}q_{112,2}} \dots\big]_{q_{112,112}^{N_{112}-1} q_{112,2}} - [d_{112}^\rho, x_2]_{q_{112,2}^{N_{112}}} \in I_{\prec x_{112}^{N_{112}} x_2} \end{aligned} \tag{9.17}$$

The proof that the liftings of [9, Thm. 5.13] have the PBW basis $[L]$ consists in replacing the c_{uv}^ρ and d_u^ρ in the conditions above, like it was done before in Proposition 9.3. We leave this to the reader.

9.4 PBW basis for $L = \{x_1 < x_1x_1x_2 < x_1x_2 < x_1x_2x_2 < x_2\}$

This PBW basis $[L]$ appears in the Nichols algebras of non-standard type and their liftings of [9, Thm. 5.17 (1)]. Generally, we ask for the conditions when $[L]$ is a PBW Basis of $(T(V)\#\Bbbk[\Gamma])/I$ where I is generated by

$$\begin{aligned}
&[x_1x_1x_1x_2] - c_{1112}, && x_1^{N_1} - d_1,\\
&[x_1x_1x_2x_2] - c_{1122}, && [x_1x_1x_2]^{N_{112}} - d_{112},\\
&[x_1x_1x_2x_1x_2] - c_{11212}, && [x_1x_2]^{N_{12}} - d_{12},\\
&[x_1x_2x_1x_2x_2] - c_{12122}, && [x_1x_2x_2]^{N_{122}} - d_{122},\\
&[x_1x_2x_2x_2] - c_{1222}, && x_2^{N_2} - d_2.
\end{aligned}$$

In $\Bbbk\langle x_1, x_{112}, x_{12}, x_{122}, x_2\rangle\#\Bbbk[\Gamma]$ we have the following $c^{\rho}_{(u|v)}$ ordered by $\ell(uv)$, $u, v \in L$: If $\mathrm{Sh}(uv) = (u|v)$ then

$$\begin{aligned}
c^{\rho}_{(1|2)} &= x_{12}, & c^{\rho}_{(1|112)} &= c^{\rho}_{1112}, & c^{\rho}_{(112|12)} &= c^{\rho}_{11212},\\
c^{\rho}_{(1|12)} &= x_{112}, & c^{\rho}_{(1|122)} &= c^{\rho}_{1122}, & c^{\rho}_{(12|122)} &= c^{\rho}_{12122},\\
c^{\rho}_{(12|2)} &= x_{122}, & c^{\rho}_{(122|2)} &= c^{\rho}_{1222},
\end{aligned}$$

and for $\mathrm{Sh}(1122) \neq (112|2)$ and $\mathrm{Sh}(112122) \neq (112|122)$ by Eq. (4.1)

$$\begin{aligned}
c^{\rho}_{(112|2)} &= \partial^{\rho}_1(c^{\rho}_{(12|2)}) + q_{12,2}c^{\rho}_{(1|2)}x_{12} - q_{1,12}x_{12}c^{\rho}_{(1|2)}\\
&= c^{\rho}_{1122} + (q_{12,2} - q_{1,12})x_{12}^2,\\
c^{\rho}_{(112|122)} &= \partial^{\rho}_1(c^{\rho}_{12122}) + q_{12,122}c^{\rho}_{1122}x_{12} - q_{1,12}x_{12}c^{\rho}_{1122}.
\end{aligned}$$

We have to check the q-Jacobi conditions for $1 < 112 < 2$ (like Eq. (9.11)), $1 < 112 < 12$ (like Eq. (9.12)), $1 < 112 < 122$, $1 < 122 < 2$, $112 < 12 < 2$ (like Eq. (9.13)), $112 < 12 < 122$, $112 < 122 < 2$, $12 < 122 < 2$ (note that we can omit $1 < 12 < 2$, $1 < 12 < 122$). The restricted q-Leibniz conditions are treated like before (note that we can leave out those for $1 < 112$, $1 < 12$, $1 < 122$ if $N_1 < \infty$, $112 < 12$, $12 < 122$ if $N_{12} < \infty$, $112 < 2$, $12 < 2$, $122 < 2$ if $N_2 < \infty$).

Both types of conditions detect many redundant relations like before. The proof that the given ideals of the Nichols algebras and their liftings of [9, Thm. 5.17 (1)] admit the PBW basis $\{x_1, [x_1x_1x_2], [x_1x_2], [x_1x_2x_2], x_2\}$ is again a straightforward but rather expansive calculation.

9.5 PBW basis for $L = \{x_1 < x_1x_1x_2 < x_1x_1x_2x_1x_2 < x_1x_2 < x_2\}$

This PBW basis $[L]$ shows up in the Nichols algebras of non-standard type and their liftings of [9, Thm. 5.17 (2),(4)]. More generally, we examine when $[L]$ is a PBW Basis of

$(T(V)\#\Bbbk[\Gamma])/I$ where I is generated by

$$\begin{aligned} &[x_1x_1x_1x_2] - c_{1112}, && x_1^{N_1} - d_1,\\ &[x_1x_1x_1x_2x_1x_2] - c_{111212}, && [x_1x_1x_2]^{N_{112}} - d_{112},\\ &[x_1x_1x_2x_1x_1x_2x_1x_2] - c_{11211212}, && [x_1x_1x_2x_1x_2]^{N_{11212}} - d_{11212},\\ &[x_1x_1x_2x_1x_2x_1x_2] - c_{1121212}, && [x_1x_2]^{N_{12}} - d_{12},\\ &[x_1x_2x_2] - c_{122}, && x_2^{N_2} - d_2. \end{aligned}$$

In $\Bbbk\langle x_1, x_{112}, x_{11212}, x_{12}, x_2\rangle\#\Bbbk[\Gamma]$ we have the following $c^\rho_{(u|v)}$ ordered by $\ell(uv)$, $u, v \in L$: If $\mathrm{Sh}(uv) = (u|v)$ then

$$\begin{aligned} c^\rho_{(1|2)} &= x_{12}, & c^\rho_{(1|112)} &= c^\rho_{1112}, & c^\rho_{(11212|12)} &= c^\rho_{1121212},\\ c^\rho_{(1|12)} &= x_{112}, & c^\rho_{(112|12)} &= x_{11212}, & c^\rho_{(112|11212)} &= c^\rho_{11211212},\\ c^\rho_{(12|2)} &= c^\rho_{122}, & c^\rho_{(1|11212)} &= c^\rho_{111212}, \end{aligned}$$

and for $\mathrm{Sh}(1122) \neq (112|2)$ and $\mathrm{Sh}(112122) \neq (11212|2)$ by Eq. (4.1)

$$\begin{aligned} c^\rho_{(112|2)} &= \partial^\rho_1(c^\rho_{(12|2)}) + q_{12,2}c^\rho_{(1|2)}x_{12} - q_{1,12}x_{12}c^\rho_{(1|2)}\\ &= c^\rho_{1122} + (q_{12,2} - q_{1,12})x_{12}^2,\\ c^\rho_{(11212|2)} &= \partial^\rho_{112}(c^\rho_{122}) + q_{12,2}c^\rho_{(112|2)}x_{12} - q_{112,12}x_{12}c^\rho_{(112|2)}\\ &= \partial^\rho_{112}(c^\rho_{122}) + q_{12,2}c^\rho_{1122}x_{12} - q_{112,12}x_{12}c^\rho_{1122}\\ &\quad + (q_{12,2} - q_{112,12})(q_{12,2} - q_{1,12})x_{12}^3. \end{aligned}$$

Again we have to consider all q-Jacobi conditions and restricted q-Leibniz conditions, from where we detect again many redundant relations. Like before, we leave the concrete calculations for the cases of [9, Thm. 5.17 (2),(4)] to the reader.

9.6 PBW basis for $L = \{x_1 < x_1x_1x_1x_2 < x_1x_1x_2 < x_1x_2 < x_2\}$

The Nichols algebras of non-standard type and their liftings in [9, Thm. 5.17 (3),(5)] have this PBW basis $[L]$. We study the situation, when $[L]$ is a PBW Basis of $(T(V)\#\Bbbk[\Gamma])/I$ where I is generated by

$$\begin{aligned} &[x_1x_1x_1x_1x_2] - c_{11112}, && x_1^{N_1} - d_1,\\ &[x_1x_1x_1x_2x_1x_1x_2] - c_{1112112}, && [x_1x_1x_1x_2]^{N_{1112}} - d_{1112},\\ &[x_1x_1x_2x_1x_2] - c_{11212}, && [x_1x_1x_2]^{N_{112}} - d_{112},\\ &[x_1x_2x_2] - c_{122}, && [x_1x_2]^{N_{12}} - d_{12},\\ & && x_2^{N_2} - d_2. \end{aligned}$$

In $\Bbbk\langle x_1, x_{112}, x_{11212}, x_{12}, x_2\rangle \# \Bbbk[\Gamma]$ we have the following $c^\rho_{(u|v)}$ ordered by $\ell(uv)$, $u, v \in L$: If $\mathrm{Sh}(uv) = (u|v)$ then

$$
\begin{aligned}
c^\rho_{(1|2)} &= x_{12}, & c^\rho_{(1|112)} &= x_{1112}, & c^\rho_{(1112|112)} &= c^\rho_{1121212},\\
c^\rho_{(1|12)} &= x_{112}, & c^\rho_{(112|12)} &= c^\rho_{11212}, & &\\
c^\rho_{(12|2)} &= c^\rho_{122}, & c^\rho_{(1|1112)} &= c^\rho_{11112}, & &
\end{aligned}
$$

and for $\mathrm{Sh}(1122) \neq (112|2)$, $\mathrm{Sh}(11122) \neq (1112|2)$ and $\mathrm{Sh}(111212) \neq (1112|12)$ by Eq. (4.1)

$$
\begin{aligned}
c^\rho_{(112|2)} &= \partial^\rho_1(c^\rho_{(12|2)}) + q_{12,2}c^\rho_{(1|2)}x_{12} - q_{1,12}x_{12}c^\rho_{(1|2)}\\
&= \partial^\rho_1(c^\rho_{122}) + (q_{12,2} - q_{1,12})x_{12}^2,\\
c^\rho_{(1112|2)} &= \partial^\rho_1(c^\rho_{(112|2)}) + q_{112,2}c^\rho_{(1|2)}x_{112} - q_{1,112}x_{112}c^\rho_{(1|2)},\\
&= \partial^\rho_1(\partial^\rho_1(c^\rho_{122})) + (q_{12,2} - q_{1,12})(x_{112}x_{12} + q_{1,12}x_{12}[x_1, x_{12}])\\
&\quad + q_{112,2}x_{12}x_{112} - q_{1,112}x_{112}x_{12},\\
&= \partial^\rho_1(\partial^\rho_1(c^\rho_{122})) + q_{12}(q_{22} - q_{11} - q_{11}^2)x_{112}x_{12}\\
&\quad + q_{12}^2(q_{11}(q_{22} - q_{11}) + q_{22})x_{12}x_{112},\\
c^\rho_{(1112|12)} &= \partial^\rho_1(c^\rho_{11212}) + (q_{112,2} - q_{1,112})x_{112}^2.
\end{aligned}
$$

Note that for the fifth equation we used the relation $[x_1, x_{12}] - x_{112}$. The assertion concerning the PBW basis and the redundant relations of [9, Thm. 5.17 (3),(5)] are again straightforward to verify.

References

[1] N. Andruskiewitsch and H.-J. Schneider. Lifting of quantum linear spaces and pointed Hopf algebras of order p^3. *J. Algebra*, 209:658–691, 1998.

[2] N. Andruskiewitsch and H.-J. Schneider. Lifting of Nichols algebras of type A_2 and pointed Hopf algebras of order p^4. In S. Caenepeel and F. van Oystaeyen, editors, *Hopf algebras and quantum groups: Proceedings of the Brussels Conference*, volume 209 of *Lecture Notes in Pure and Appl. Math.*, pages 1–14. Marcel Dekker, 2000.

[3] N. Andruskiewitsch and H.-J. Schneider. On the classification of finite-dimensional pointed Hopf algebras. 2007. to appear in *Ann. Math.*, `arXiv math.QA/0502157`.

[4] R. Berger. The quantum Poincaré-Birkhoff-Witt theorem. In *Communications in Mathematical Physics*, volume 143, pages 215–234. Springer, 1992.

[5] G.M. Bergman. The diamond lemma for ring theory. *Advances in Mathematics*, 29:178–218, 1978.

[6] L.A. Bokut and G.P. Kukin. *Algorithmic and Combinatorial Algebra*, volume 255 of *Mathematics and Its Applications*. Kluwer Academic Publishers, Dordrecht-Boston-London, 1994.

[7] L.A. Bokut. Unsolvability of the word problem and subalgebras of finitely presented Lie algebras. *Izv. Akad. Nauk. Ser. Mat.*, 36 N6:1173–1219, 1972.

[8] M. Helbig. *Lifting of Nichols algebras*. Südwestdeutscher Verlag für Hochschulschriften, 2009. available at `http://edoc.ub.uni-muenchen.de/10378/`.

[9] M. Helbig. On the lifting of Nichols algebras. preprint, available at `arxiv.org: 1003.5882`, 2010.

[10] M. Helbig. On the presentation of pointed Hopf algebras. preprint, available at `arxiv.org: 1003.5879`, 2010.

[11] V. Kharchenko. A quantum analog of the Poincaré-Birkhoff-Witt theorem. *Algebra and Logic*, 38:259–276, 1999.

[12] M. Lothaire. *Combinatorics on Words*, volume 17 of *Encyclopedia of Mathematics*. Addison-Wesley, 1983.

[13] G. Lusztig. Canonical bases arising from quantized enveloping algebras. *J. of Amer. Math. Soc.*, 3:447–498, 1990.

[14] G. Lusztig. *Introduction to quantum groups*, volume 110 of *Progress in Mathematics*. Birkhäuser, 1993.

[15] D. Radford. Finite-dimensional simple-pointed Hopf algebras. *J. Algebra*, 211:686–710, 1999.

[16] C. Reutenauer. *Free Lie Algebras*, volume 7 of *London Mathematical Society Monographs, New Series*. Clarendon Press, London, 1993.

[17] C.M. Ringel. PBW-bases of quantum groups. *J. Reine Angew. Math.*, 470:51–88, 1996.

[18] M. Rosso. An anlogue of the Poincaré-Birkhoff-Witt theorem and the universal R-matrix of $U_q(sl(N+1))$. *Comm. Math. Phys.*, 124:307–318, 1989.

[19] H. Yamane. A Poincaré-Birkhoff-Witt theorem for quantized universal enveloping algebas of type A_N. *J. Reine Angew. Math.*, 470:51–88, 1996.