ON THE SUM OF A PRIME AND A FIBONACCI NUMBER

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Abstract. We show that the set of the numbers that are the sum of a prime and a Fibonacci number has positive lower asymptotic density.

1. Introduction

Suppose $\mathcal{S}$ is a set of positive integers. We denote the number of positive integers in S not exceeding N by $\mathcal{S}(N)$. This function is called the *counting function* of the set $\mathcal{S}$. The sumset, $\mathcal{S}+\mathcal{T}$, is the collection of the numbers of the form $s+t$ where $s \in \mathcal{S}$ and $t \in \mathcal{T}$.

Suppose $\mathcal{A} = \{p + 2^i : p \text{ a prime}, i \geq 1\}$. In 1934, Romanoff [7] published the following interesting result. For N sufficiently large, we have $\mathcal{A}(N) \geq cN$ for some $c > 0$. In other words, the set $\mathcal{A}$ has a positive lower asymptotic density. Romanoff showed that a positive proportion of positive integers can be decomposed into the form $p+2^i$.

Let $u_1 = 1, u_2 = 1, u_{i+2} = u_{i+1} + u_i$ where i is a positive integer. Denote by $\mathcal{U}$ the collection of Fibonacci numbers, namely $\mathcal{U} = \{u_i\}_{i\geq 2}$. Furthermore, let $\mathcal{P}$ denote the set of primes. For convenience, we stipulate that p and p' (with or without subscripts) are primes, and u and u' (with or without subscripts) are Fibonacci numbers. Throughout this paper, we use the Vinogradov symbol $\ll$ and the Landau symbol O with their usual meanings.

In this manuscript, we study the set of integers that are the sum of a prime and a bounded number of Fibonacci numbers. In view of Romanoff's theorem, a key element in the proof is

$$\sum_{\substack{d=1 \\ (2,d)=1}}^{\infty} \frac{\mu^2(d)}{de(d)} \ll 1$$

where $e(d)$ is the exponent of 2 modulo d.

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By using an estimate ([8], [9]) of the number of times the residue t appeared in a full period of $u_i \pmod p$, we are able to substitute the period $k(d)$ for $e(d)$ and prove that $\mathcal{P} + \mathcal{U}$ has a positive lower asymptotic density.

Theorem 1. *Suppose*

$$\mathcal{F} = \mathcal{P} + \mathcal{U} = \{p + u : p \in \mathcal{P}, u \in \mathcal{U}\}.$$

Then there is a positive constant c such that

$$\mathcal{F}(N) \geq cN$$

for all sufficiently large N.

As a consequence, the set $\mathcal{P} + k\mathcal{U}$ has a positive lower asymptotic density for each $k \geq 1$, since $1 \in \mathcal{U}$.

2. Proof of the Theorem

For our convenience, we let $L = [\log_\tau N]$ for a given N and use this throughout this paper. Let $\tau = (1 + \sqrt{5})/2$. It is well-known that

$$\left| u_i - \frac{\tau^i}{\sqrt{5}} \right| < \frac{1}{2}$$

for all $i \geq 1$. Thus $u_i = \tau^i/\sqrt{5} + O(1)$. A routine computation yields that

$$\mathcal{U}(N) = L + O(1).$$

Denote by $r'(N) = \sum_{p+u=N} 1$ the number of solutions of the equation $N = p + u$ for $N \geq 1$. We begin with the following lemma.

Lemma 1. *For N a large number, we have*

$$\sum_{n \leq N} r'(n) \sim \frac{NL}{\log N}.$$

Proof. Note that

$$\pi(N - \frac{N}{L})\mathcal{U}(\frac{N}{L}) \leq \sum_{n \leq N} r'(n) \leq \pi(N)\mathcal{U}(N).$$

The lemma then follows from the prime number theorem. □

Properties of Fibonacci numbers can be found in standard texts such as [4], [11], and [12]. For our discussions, we recall some properties of Fibonacci numbers without providing proofs. Given a positive integer n, there is a unique decomposition of n into the sum of non-consecutive Fibonacci numbers, namely,

$$n = u_{i_1} + u_{i_2} + \cdots + u_{i_r}$$

where $2 \le i_r$ and $2 \le i_j - i_{j+1}$. This is called Zeckendorf representation [1] (or canonical representation). In other words, if $2 \le i_r$ and $2 \le i_j - i_{j+1}$, the set of integers $(i_1, i_2, \dots, i_r)$ is uniquely determined by n and conversely. It is well-known that $u_i \pmod d$ forms a purely periodic series [13]. Let $k(d)$ denote the period of Fibonacci numbers modulo d. That is to say $k(d)$ is the smallest positive integers m such that $u_{i+m} \equiv u_i \pmod d$ for all i. In particular, $d | u_{k(d)}$. Furthermore, the period $k(d)$ is equal to the least common multiple of $\{k(p_1^{r_1}), k(p_2^{r_2}), \dots, k(p_t^{r_t})\}$ where $d = p_1^{r_1} \cdots p_t^{r_t}$. We also have that $k(d) | k(m)$ if $d \,| m$.

Let us investigate the following example. The table below presents one period of the residues for $u_i \pmod 6$, $u_i \pmod 2$, and $u_i \pmod 3$, respectively, where $i \ge 2$.

i	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$u_i \pmod 6$	1	2	3	5	2	1	3	4	1	5	0	5	5	4	3	1	4	5	3	2	5	1	0	1
$u_i \pmod 2$	1	0	1																					
$u_i \pmod 3$	1	2	0	2	2	1	0	1																

From the table, we see that $k(6) = 24, k(2) = 3$, and $k(3) = 8$. We note that $k(6) = LCM[k(2), k(3)]$. For any modulus $d \ge 2$, and residue $y \pmod d$, denote by $\nu(d, y)$ the number of occurrences of y as a residue in one full period of $u_i \pmod d$. Let us explore the case $y \equiv 5 \pmod d$. From the table, we have $\nu(6, 5) = 6$, $\nu(2, 5) = \nu(2, 1) = 2$, and $\nu(3, 5) = \nu(3, 2) = 3$. It is clear that $\nu(d, y) \le \dfrac{k(d)}{k(p)} \nu(p, y)$ where $p|d$. We now return to our proof.

Lemma 2. *Let $-N \le h \le N$ and $f(h)$ be the number of solutions of the equation:*

$$u - u' = h$$

where $u, u' \le N$. Then

(1) *$f(0) \sim L$ and $f(h) \le 2$ if $h \ne 0$;*
(2) *Suppose $d > 1$ is an integer and $p|d$. Then*

$$\sum_{d|h} f(h) \le 4L\left(1 + \frac{L}{k(p)}\right).$$

Proof. Without loss of generality, we can assume $h \ge 0$. (1) Clearly we have $f(0) = \mathcal{U}(N) \sim L$. Next we claim that $f(h) \le 2$ when $h > 0$. Assume that $h = u_j - u_i = u_t - u_s$ where $j > i$ and $t > s$. If $j - i = 1$, then $h = u_j - u_{j-1} = u_{j-2} = u_{j-1} - u_{j-3}$. Suppose $t > j$. If $t - s = 1$, we have $u_{t-2} = u_{j-2}$, a contradiction. If $t - s > 1$, we have $u_t - u_s > u_t - u_{t-1} = u_{t-2}$, a contradiction again! Suppose now $t < j - 1$. This forces $u_t - u_s < u_{j-2} = h$. Therefore, there are only two decompositions of h into the difference of two Fibonacci numbers, namely $u_j - u_{j-1}$ and $u_{j-1} - u_{j-3}$.

Now suppose $j - i \geq 2$. From the definition of Fibonacci numbers, we derive that

$$u_j - u_i = \begin{cases} u_{j-1} + u_{j-3} + \cdots + u_{j-(2v-1)}, & \text{if } i = j - 2v; \\ u_{j-1} + u_{j-3} + \cdots + u_{j-(2v-1)} + u_{j-(2v+2)}, & \text{if } i = j - (2v+1); \end{cases}$$

where $v \geq 1$. Clearly these are Zeckendorf representations. By the same token, $u_t - u_s$ has similar decompositions. The uniqueness of Zeckendorf representation implies that $t = j$ and thus $s = i$. As a consequence, $f(h) \leq 2$.

(2) The sum $\sum_{d|h} f(h)$ is the number of solutions of the congruence $u \equiv u' \pmod{d}$. Note that $u \equiv u' \pmod{p}$ if $p|d$. However, Schinzel [8] and Somer [9] showed that $\nu(p, y) \leq 4$, namely, there are at most 4 choices for u in any interval of length $k(p)$ such that $u \equiv y \pmod{p}$. This implies within an interval of length L there are at most $4(1 + \frac{L}{k(p)})$ solutions to $u \equiv y \pmod{d}$. Thus $p|d$ implies

$$\sum_{d|h} f(h) \leq 4L\left(1 + \frac{L}{k(p)}\right).$$

□

Lemma 3. *For $k \geq 1$ and N sufficiently large, we have*

$$\sum_{n \leq N} (r'(n))^2 \leq c\frac{NL^2}{(\log N)^2}$$

where $c > 0$.

Proof. In the following, we assume that $p, p', u, u' \leq N$. We first break the sum into three parts.

$$\begin{aligned} \sum_{n \leq N} (r'(n))^2 &= \sum_{n \leq N} \Bigg(\sum_{p+u=n} 1 \Bigg)^2 \\ &= \sum_{n \leq N} \sum_{\substack{p+u=n \\ p'+u'=n}} 1 \\ &= \sum_{-N \leq h \leq N} \Bigg(\sum_{\substack{p-p'=h \\ p,p' \leq N}} 1 \Bigg) f(h). \end{aligned}$$

Let

$$\sum(h) = \Big(\sum_{\substack{p-p'=h\\ p,p'\le N}} 1\Big) f(h),$$

where $h = 0$, $h > 0$, $h < 0$. We investigate these three cases respectively. First, suppose $h = 0$. From Lemma 2, we have

$$\sum(0) = \Big(\sum_{p\le N} 1\Big) f(0) \sim \frac{NL}{\log N}.$$

Next, we suppose $h > 0$ and is odd. This implies $p' = 2$, since $p-p' = h$. Thus

$$\sum_{\substack{0<h\le N\\ 2\nmid h}} \sum(h) = \sum_{\substack{0<h\le N\\ 2\nmid h}} \Big(\sum_{\substack{p=h+2\\ p\le N}} 1\Big) f(h).$$

Therefore, we have

$$\sum_{\substack{0<h\le N\\ 2\nmid h}} \sum(h) \ll \sum_{\substack{0<h\le N\\ 2\nmid h}} \Big(\sum_{\substack{p=h+2\\ p\le N}} 1\Big) \ll \frac{N}{\log N}.$$

We now assume $h > 0$ is even. Recall that the number of primes $p \le N$ such that $p+h$ is also a prime is given by (cf. [3, p.102], [5, p.97] , and [6, p.190])

$$O\Big(\frac{N}{(\log N)^2} \prod_{p\mid h} \Big(1+\frac{1}{p}\Big)\Big).$$

By using Lemma 2, we obtain that

$$\begin{aligned}
\sum_{\substack{0<h\le N\\ 2\mid h}} \sum(h) &\ll \frac{N}{(\log N)^2} \sum_{0<h\le N} f(h) \prod_{p\mid h} \Big(1+\frac{1}{p}\Big)\\
&\ll \frac{N}{(\log N)^2} \sum_{d\le N} \frac{\mu^2(d)}{d} \sum_{\substack{0<h\le N\\ d\mid h}} f(h)\\
&\ll \frac{NL^2}{(\log N)^2} + \frac{NL}{(\log N)^2} \sum_{1<d\le N} \frac{\mu^2(d)}{d} \Big(1+\frac{L}{k(p)}\Big),
\end{aligned}$$

where p is a prime factor of d. For our investigation, we let the function $LP(d) = \max\{p|d : k(p) \ge k(p') \text{ for } p'|d\}$. We are to show that

$$\sum_{\substack{d\le N\\ p=LP(d)}} \frac{\mu^2(d)}{dk(p)} \ll 1.$$

We define

$$E(x)=\sum_{g\le x}\sum_{\substack{p=LP(d)\\k(p)=g}}\frac{\mu^2(d)}{d}. \qquad (*)$$

In 1974, Catlin [2] showed that if $k(m)<2t$ then $m<L_t$ where L_t is the t-th Lucas number. Therefore, for a fixed number g, there are only finitely many solutions p to the equation $k(p)=g$. Furthermore, there can only be a finite number of primes having period less than or equal to $k(p)$, and thus there are only finitely many squarefree d having $p=LP(d)$. This means $E(x)$ is well-defined. Let

$$D(x)=\prod_{i\le x}u_i.$$

Without loss of generality, we assume that d, appearing in the sum (*), is squarefree. Note that $p'|d$ implies $p'|u_{k(p')}|D(x)$. We then have $d|D(x)$ since $k(p)\le x$ and $p=LP(d)$. It is also clear that the number d appears in (*) once. Let $n=\omega(D(x))$ be the number of distinct prime factors of $D(x)$. Then

$$2^n\le D(x)\ll\prod_{i\le x}\tau^i\ll\tau^{x^2}.$$

In other words, we have $n\ll x^2$, and thus $\log p_n\ll\log n\ll x$ (where p_i is the i-th prime). Immediately, we have

$$E(x)\ll\sum_{d|D(x)}\frac{\mu^2(d)}{d}=\prod_{p|D(x)}\left(1+\frac{1}{p}\right)\ll\prod_{i=1}^{n}\left(1+\frac{1}{p_i}\right).$$

Apply Merten's formula to the last term to obtain

$$E(x)\ll\log p_n\ll\log x.$$

By partial summation, we have

$$\sum_{g\le x}\frac{1}{g}\sum_{\substack{p=LP(d)\\k(p)=g}}\frac{\mu^2(d)}{d}=\frac{E(x)}{x}+\int_1^x\frac{E(x)}{t^2}dt\ll 1.$$

This implies

$$\lim_{x\to\infty}\sum_{\substack{d\le x\\p=LP(d)}}\frac{\mu^2(d)}{dk(p)}=\lim_{x\to\infty}\sum_{g\le x}\frac{1}{g}\sum_{\substack{p=LP(d)\\k(p)=g}}\frac{\mu^2(d)}{d}\ll 1.$$

As a consequence, we have

$$\sum_{0<h\le N}\sum(h) \ll \frac{NL^2}{(\log N)^2}.$$

By symmetry,

$$\sum_{-N\le h<0}\sum(h) \ll \frac{NL^2}{(\log N)^2}.$$

Combining the above estimations, we obtain

$$\sum_{n\le N}(r'(n)) \ll \frac{NL^2}{(\log N)^2}.$$

□

Invoking the Cauchy-Schwarz inequality, we have

$$\left(\sum_{n\le N} r'(n)\right)^2 \le \mathcal{F}(N)\sum_{n\le N}(r'(n))^2.$$

However, Lemma 1 and Lemma 3 imply

$$\mathcal{F}(N) \ge \frac{\left(\sum_{n\le N} r'(n)\right)^2}{\sum_{n\le N}(r'(n))^2} \ge \frac{1}{c}N.$$

This proves the theorem.

3. Remarks

To conclude our paper, we post the following questions related to our quest.

(1) Is $r'(n) \ll 1$? The referee notices that for any fixed $k \ge 2$, we can choose distinct Fibonacci numbers $u_{m_1}, u_{m_2}, \cdots, u_{m_k}$ such that for any prime p there exists $1 \le d_p \le p$ satisfying $u_{m_i} \not\equiv d_p \pmod p$ for each $1 \le i \le k$ (see Schinzel [8, Corollary 1]). Then by the widely believed prime k-tuple conjecture (see [5]), there exist infinitely many n such that $n - u_{m_1}, n - u_{m_2}, \cdots, n - u_{m_k}$ are all primes. That is, $r'(n) \ge k$. Thus the referee suggests that $\limsup_{n\to\infty} r'(n) = +\infty$ instead.

(2) Find an infinite sequence (or an arithmetic progression) of positive integers that each of the terms cannot be of the form $p+u$. Note Wu and Sun [14] constructed a class that does not contain integers representable as the sum of a prime and half of a Fibonacci number.

(3) Is there a positive integer k such that n can be decomposed into a sum of a prime and k Fibonacci numbers for n sufficiently large? Note that Sun [10] has recently conjectured that every integer (> 4) can be written as the sum of an odd prime and two positive Fibonacci numbers.

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