Genericity of Caustics on a corner

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Abstract

We introduce the notions of *the caustic-equivalence* and *the weak caustic-equivalence relations* of reticular Lagrangian maps in order to give a generic classification of caustics on a corner. We give the figures of all generic caustics on a corner in a smooth manifold of dimension 2 and 3.

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1. Introduction

In [1] we investigate the theory of *reticular Lagrangian maps* which can be described stable caustics generated by a hypersurface germ with an r-corner in a smooth manifold. A map germ $\pi \circ i : (\mathbb{L}, 0) \to (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is called *a reticular Lagrangian map* if i is a restriction of a symplectic diffeomorphism germ on $(T^*\mathbb{R}^n, 0)$, where $I_r = \{1, \ldots, r\}$ and $\mathbb{L} = \{(q, p) \in T^*\mathbb{R}^n | q_1 p_1 = \cdots = q_r p_r = q_{r+1} = \cdots = q_n = 0, q_{I_r} \geq 0\}$. For the definitions of caustics and generating families of reticular Lagrangian maps, see [1, p.575-577]. In [2] we investigate the genericity of caustics on an r-corner and give the generic classification for the cases $r = 0$ and 1 by using G.Ishikawa's methods (see [3, Section 5]). We also showed that the method of the paper do not work well for the case $r = 2$, that is the initial hypersurface germ has a corner. In this paper we introduce the two equivalence relations of reticular Lagrangian maps which are weaker than Lagrangian equivalence in order to give a generic classification of caustics on a corner.

2. Caustic-equivalence

We introduce the equivalence relations of reticular Lagrangian maps and their generating families.

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Let $\pi \circ i_j$ be reticular Lagrangian maps for $j = 1, 2$. We say that they are *caustic-equivalent* if there exists a diffeomorphism germ g on $(\mathbb{R}^n, 0)$ such that

$$g(C^1_\sigma) = C^2_\sigma, \;\; g(Q^1_{\sigma,\tau}) = Q^2_{\sigma,\tau} \quad \text{for all } \sigma, \tau \subset I_r \; (\sigma \neq \tau). \tag{1}$$

In order to describe the caustic-equivalence of reticular Lagrangian maps by their generating families, we introduce the following equivalence relation of function germs. We say that $f, g \in \mathcal{E}(r; k)$ are *reticular C-equivalent* if there exist $\phi \in \mathcal{B}(r; k)$ and non-zero number $a \in \mathbb{R}$ such that $g = a \cdot f \circ \phi$. See [1] or [4] for the notations. We construct the theory of unfoldings with respect to the corresponding equivalence relation. Then the relation of unfoldings is given as follows: Two function germs $F(x, y, q), G(x, y, q) \in \mathcal{E}(r; k + n)$ are *reticular $\mathcal{P}$-C-equivalent* if there exist $\Phi \in \mathcal{B}_n(r; k + n)$ and a unit $a \in \mathcal{E}(n)$ and $b \in \mathcal{E}(n)$ and such that $G = a \cdot F \circ \Phi + b$. We define the *stable* reticular ($\mathcal{P}$-)C-equivalence by the ordinary ways (see [1, p.576]). We remark that a reticular $\mathcal{P}$-C-equivalence class includes the reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalence classes.

We review the results of the theory. Let $F(x, y, u) \in \mathfrak{M}(r; k + n)$ be an unfolding of $f(x, y) \in \mathfrak{M}(r; k)$.

We say that F is *reticular $\mathcal{P}$-C-stable* if the following condition holds: For any neighborhood U of 0 in $\mathbb{R}^{r+k+n}$ and any representative $\tilde{F} \in C^\infty(U, \mathbb{R})$ of F, there exists a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ in C^∞-topology such that for any element $\tilde{G} \in N_{\tilde{F}}$ the germ $\tilde{G}|_{\mathbb{H}^r \times \mathbb{R}^{k+n}}$ at $(0, y_0, q_0)$ is reticular $\mathcal{P}$-C-equivalent to F for some $(0, y_0, q_0) \in U$.

We say that F is *reticular $\mathcal{P}$-C-versal* if all unfolding of f is reticular $\mathcal{P}$-C-f-induced from F. That is, for any unfolding $G \in \mathfrak{M}(r; k + n')$ of f, there exist $\Phi \in \mathfrak{M}(r; k + n', r; k + n)$ and a unit $a \in \mathcal{E}(n')$ and $b \in \mathcal{E}(n')$ satisfying the following conditions:
(1) $\Phi(x, y, 0) = (x, y, 0)$ for all $(x, y) \in (\mathbb{H}^r \times \mathbb{R}^k, 0)$ and $a(0) = 1, \; b(0) = 0$,
(2) Φ can be written in the form:

$$\Phi(x, y, q) = (x_1\phi^1_1(x, y, q), \cdots, x_r\phi^r_1(x, y, q), \phi_2(x, y, q), \phi_3(q)),$$

(3) $G(x, y, q) = a(q) \cdot F \circ \Phi(x, y, q) + b(q)$ for all $(x, y, q) \in (\mathbb{H}^r \times \mathbb{R}^{k+n'}, 0)$.

We say that F is *reticular $\mathcal{P}$-C-infinitesimally versal* if

$$\mathcal{E}(r; k) = \langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle_{\mathcal{E}(r;k)} + \langle 1, f, \frac{\partial F}{\partial q}|_{q=0}\rangle_{\mathbb{R}}.$$

We say that F is *reticular $\mathcal{P}$-C-infinitesimally stable* if

$$\mathcal{E}(r; k + n) = \langle x\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\rangle_{\mathcal{E}(r;k+n)} + \langle 1, F, \frac{\partial F}{\partial q}\rangle_{\mathcal{E}(n)}.$$

We say that F is *reticular $\mathcal{P}$-C-homotopically stable* if for any smooth path-germ $(\mathbb{R}, 0) \to \mathcal{E}(r; k + n), t \mapsto F_t$ with $F_0 = F$, there exists a smooth path-germ $(\mathbb{R}, 0) \to \mathcal{B}_n(r; k + n) \times \mathcal{E}(n) \times \mathcal{E}(n), t \mapsto (\Phi_t, a_t, b_t)$ with $(\Phi_0, a_0, b_0) = (id, 1, 0)$ such that each (Φ_t, a_t, b_t) is a reticular $\mathcal{P}$-C-isomorphism from F to F_t, that is $F_t = a_t \cdot F \circ \Phi_t + b_t$ for t around 0.

Theorem 2.1. (cf., [1, Theorem 4.5]) *Let $F \in \mathfrak{M}(r;k+n)$ be an unfolding of $f \in \mathfrak{M}(r;k)$. Then the following are all equivalent.*
(1) *F is reticular $\mathcal{P}$-$\mathcal{C}$-stable.*
(2) *F is reticular $\mathcal{P}$-$\mathcal{C}$-versal.*
(3) *F is reticular $\mathcal{P}$-$\mathcal{C}$-infinitesimally versal.*
(4) *F is reticular $\mathcal{P}$-$\mathcal{C}$-infinitesimally stable.*
(5) *F is reticular $\mathcal{P}$-$\mathcal{C}$-homotopically stable.*

For a non-quasihomogeneous function germ $f(x,y) \in \mathfrak{M}(r;k)$, if $1, f, a_1, \ldots, a_n \in \mathcal{E}(r;k)$ is a representative of a basis of the vector space

$$\mathcal{E}(r;k)/\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle_{\mathcal{E}(r;k)},$$

then the function germ $f + a_1q_1 + \cdots + a_nq_n \in \mathfrak{M}(r;k+n)$ is a reticular $\mathcal{P}$-$\mathcal{C}$-stable unfolding of f. We call n the reticular $\mathcal{C}$-codimension of f. If f is a quasihomogeneous function germ then f is included in $\langle x\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle_{\mathcal{E}(r;k)}$. This means that the reticular $\mathcal{C}$-codimension of a quasihomogeneous function germ is equal to its reticular $\mathcal{R}^+$-codimension.

We define the *simplicity* of function germs under the reticular $\mathcal{C}$-equivalence in the usual way (cf., [2]).

Theorem 2.2. (cf., [2, Theorem 2.1,2.2]) *A reticular $\mathcal{C}$-simple function germ in $\mathfrak{M}(1;k)^2$ is stably reticular $\mathcal{C}$-equivalent to one of the following function germs:*

$$B_l : x^l \ (l \geq 2), \quad C_l^{\varepsilon} : xy + \varepsilon y^l \ (\varepsilon^{l-1} = 1, l \geq 3), \quad F_4 : x^2 + y^3.$$

The relation between reticular Lagrangian maps and their generating families under the caustic-equivalence are given as follows:

Proposition 2.3. *Let $\pi \circ i_j$ be reticular Lagrangian maps with generating families F_j for $j = 1,2$. If F_1 and F_2 are stably reticular $\mathcal{P}$-$\mathcal{C}$-equivalent then $\pi \circ i_1$ and $\pi \circ i_2$ are caustic-equivalent.*

Proof. The function germ F_2 may be written that $F_2(x,y,q) = a(q)F_3(x,y,q)$, where a is a unit and F_1 and F_3 are stably reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent. Then the reticular Lagrangian map $\pi \circ i_3$ given by F_3 and $\pi \circ i_1$ are Lagrangian equivalent and the caustic of $\pi \circ i_2$ and $\pi \circ i_3$ coincide to each other. ■

This proposition shows that it is enough to classify function germs under the stable reticular $\mathcal{P}$-$\mathcal{C}$-equivalence in order to classify reticular Lagrangian maps under the caustic-equivalence. We here give the classification list as the following:

Theorem 2.4. (cf., [1, p.592]) *Let $f \in \mathfrak{M}(2;k)^2$ have the reticular $\mathcal{C}$-codimension≤ 4. Then f is stably reticular $\mathcal{C}$-equivalent to one of the following list.*

k	*Normal form*	*codim*	*Conditions*	*Notation*
0	$x_1^2 \pm x_1x_2 + ax_2^2$	3	$0 < a < \frac{1}{4}$	$B^{\pm,+,1}_{2,2,a}$
	$x_1^2 \pm x_1x_2 + ax_2^2$	3	$a > \frac{1}{4}$	$B^{\pm,+,2}_{2,2,a}$
	$x_1^2 \pm x_1x_2 + ax_2^2$	3	$a < 0$	$B^{\pm,-}_{2,2,a}$
	$x_1^2 \pm x_2^2$	3		$B^{\pm,0}_{2,2}$
	$(x_1 \pm x_2)^2 \pm x_2^3$	3		$B^{\pm,\pm}_{2,2,3}$
	$x_1^2 \pm x_1x_2 \pm x_2^3$	3		$B^{\pm,\pm}_{2,3}$
	$x_1^3 \pm x_1x_2 \pm x_2^2$	3		$B^{\pm,\pm}_{3,2}$
	$x_1^2 \pm x_1x_2^2 \pm x_2^3$	4		$B^{\pm,\pm}_{2,3'}$
	$x_1^3 \pm x_1^2x_2 \pm x_2^2$	4		$B^{\pm,\pm}_{3,2'}$
1	$\pm y_1^3 + x_1y \pm x_2y + x_2^2$	3		$C^{\pm,\pm}_{3,2}$
	$\pm y_1^3 + x_1y \pm x_2y^2 + x_2^2$	4		$C^{\pm,\pm}_{3,2,1}$
	$\pm y_1^3 + x_2y \pm x_1y^2 + x_1^2$	4		$C^{\pm,\pm}_{3,2,2}$

We remark that the stable reticular $\mathcal{C}$-equivalence class $B^{+,+}_{2,3}$ of $x_1^2 + x_1x_2 + x_2^3$ consists of the union of the stable reticular $\mathcal{R}$-equivalence classes of $x_1^2 + x_1x_2 + ax_2^3$ and $-x_1^2 - x_1x_2 - ax_2^3$ for $a > 0$. The same things hold for $B^{\pm,\pm}_{2,2,3}$, $B^{\pm,\pm}_{2,3}$, $B^{\pm,\pm}_{3,2}$, $C^{\pm,\pm}_{3,2}$.

3. Caustic-stability

We define *the caustic-stability* of reticular Lagrangian maps and reduce this to finite dimensional jet spaces of symplectic diffeomorphism germs.

We denote $S(T^*\mathbb{R}^n, 0)$ the set of symplectic diffeomorphism germs on $(T^*\mathbb{R}^n, 0)$ and denote $S(U, T^*\mathbb{R}^n)$ the space of symplectic embeddings from an open set U in $T^*\mathbb{R}^n$ around 0 to $T^*\mathbb{R}^n$ with C^∞-topology.

We say that a reticular Lagrangian map $\pi \circ i$ is *caustic-stable* if the following condition holds: For any extension $S \in S(T^*\mathbb{R}^n, 0)$ of i and any representative $\tilde{S} \in S(U, T^*\mathbb{R}^n)$ of S, there exists a neighborhood $N_{\tilde{S}}$ of $\tilde{S}$ such that for any $\tilde{S}' \in N_{\tilde{S}}$ the reticular Lagrangian map $\pi \circ \tilde{S}'|_{\mathbb{L}}$ at x_0 and $\pi \circ i$ are caustic-equivalent for some $x_0 = (0, \ldots, 0, p^0_{r+1}, \ldots, p^0_n)$.

Definition 3.1. *Let $\pi \circ i$ be a reticular Lagrangian map and l be a non-negative number. We say that $\pi \circ i$ is* caustic l-determined *if the following condition holds: For any extension S of i, the reticular Lagrangian map $\pi \circ S'|_{\mathbb{L}}$ and $\pi \circ i$ are caustic-equivalent for any symplectic diffeomorphism germ S' on $(T^*\mathbb{R}^n, 0)$ satisfying $j^lS(0) = j^lS'(0)$.*

Lemma 3.2. *Let $\pi \circ i : (\mathbb{L}, 0) \to (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a reticular Lagrangian map. If a generating family of $\pi \circ i$ is reticular $\mathcal{P}$-$\mathcal{C}$-stable then $\pi \circ i$ is caustic $(n + 2)$-determined.*

Proof. This is proved by the analogous method of [2, Theorem 5.3]. We give the sketch of proof. Let S be an extension of i. Then we may assume that there exists a function germ $H(Q, p)$ such that the canonical relation P_S has the form:

$$P_S = \{(Q, -\frac{\partial H}{\partial Q}(Q, p), -\frac{\partial H}{\partial p}(Q, p), p) \in (T^*\mathbb{R}^n \times T^*\mathbb{R}^n, (0, 0))\}.$$

Then the function germ $F(x, y, q) = H_0(x, y) + \langle y, q\rangle$ is a reticular $\mathcal{P}$-$\mathcal{C}$-stable generating family of $\pi \circ i$, and H_0 is reticular $\mathcal{R}$-$(n + 3)$-determined, where $H_0(x, y) = H(x, 0, y)$. Let a symplectic diffeomorphism germ S' on $(T^*\mathbb{R}^n, 0)$ satisfying $j^{n+2}S(0) = j^{n+2}S'(0)$ be given. Then there exists a function germ $H'(Q,p)$ such that the canonical relation $P_{S'}$ is given the same form for H' and the function germ $G(x,y,q) = H'_0(x,y) + \langle y, q\rangle$ is a generating family of $\pi \circ S'|_{\mathbb{L}}$. Then it holds that $j^{n+3}H_0(0) = j^{n+3}H'_0(0)$. There exists a function germ G' such that G and G' are reticular $\mathcal{P}$-$\mathcal{R}$-equivalent and F and G' are reticular $\mathcal{P}$-$\mathcal{C}$-infinitesimal versal unfoldings of $H_0(x,y)$. It follows that F and G are reticular $\mathcal{P}$-$\mathcal{C}$-equivalent. Therefore $\pi \circ i$ and $\pi \circ S'|_{\mathbb{L}}$ are caustic-equivalent. ■

For a reticular $\mathcal{P}$-$\mathcal{C}$-stable unfolding $F \in \mathfrak{M}(2;k+n)^2$ with $n \leq 3$, the function germ $f = F|_{q=0}$ has a modality under the reticular $\mathcal{R}$-equivalence (see [1, p.592]). For example, consider the case f is stably reticular $\mathcal{C}$-equivalent to $x_1^2 + x_1x_2 + x_2^3$. Then F is stably reticular $\mathcal{P}$-$\mathcal{C}$-equivalent to $f + q_1x_1 + q_2x_2 + q_3x_2^2$. In this case the function germs $F_a(x,q) = x_1^2 + x_1x_2 + ax_2^3 + q_1x_1 + q_2x_2 + q_3x_2^2 (a > 0)$ are stably reticular $\mathcal{P}$-$\mathcal{C}$-equivalent to F but not stably reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent to each other. Let $S_a^\pm$ be extensions of reticular Lagrangian embeddings defined by F_a and $-F_a$ for $a > 0$ respectively. We define the caustic-equivalence class of S_1 by $[S_1]_c := \cup_{a>0}([S_a^+]_L \cup [S_a^-]_L)$, where $[S_a^\pm]_L$ are the Lagrangian equivalence classes of $S_a^\pm$ respectively. By Proposition 2.3, we have that all reticular Lagrangian maps $\pi \circ S'|_{\mathbb{L}}$ are caustic-equivalent to each other for $S' \in [S_1]_c$. In order to apply the last theorem of this paper, we need to prove that the set consists of the 5-jets of the caustic-equivalence class $[S_1]_c$, we denote this by $[j^5S_1(0)]_c$, is an immersed manifold of $S^5(3)$, where $S^l(n)$ be the set consists of l-jets of elements in $S(T^*\mathbb{R}^n, 0)$. We shall prove that the map germ $(0,\infty) \to S^5(3), a \mapsto j^5S_a(0)$ is not tangent to $[j^5S_a(0)]_L$ for any a, and apply the following lemma:

Lemma 3.3. *Let I be an open interval, N a manifold, and G a Lie group acts on N. Let $x : I \to N$ be a smooth path such that $\frac{dx}{dt}(t)$ is not tangent to $G \cdot x(t)$ for all $t \in I$. Then*

$$\bigcup_{t\in I} G \cdot x(t)$$

is an immersed manifold of N.

We note that we here prove the case $B^{+,+}_{2,3}$. The same method is valid for all $B^{\pm,\pm}_{2,3}, B^{\pm,\pm}_{3,2}$.

We define $G_a \in \mathfrak{M}(6)^2$ by $G_a(Q_1,Q_2,Q_3,q_1,q_2,q_3) = F_a(Q_1,Q_2,q_1,q_2) + Q_3q_3$. Then G_a define the canonical relations P_a and they give symplectic diffeomorphisms S_a of the forms:

$$S_a(Q,P) = (-2Q_1 - Q_2 - P_1, -Q_1 - 3aQ_2^2 - P_2 + 2P_3Q_2, -P_3, Q_1, Q_2, Q_2^2 + Q_3).$$

We have that F_a are generating families of $\pi \circ S_a|_{\mathbb{L}}$. Then $\frac{dS_a}{da} = (0, -3Q_2^2, 0, 0, 0, 0) = X_f \circ S_a$ for $f = -p_2^3$. We suppose that $j^5(\frac{dS_a}{da})(0) \in T_z([z]_L)$ for $z = j^5S_a(0)$. By [2, Lemma 6.2], there exist a fiber preserving function germ $H \in \mathfrak{M}^2_{Q,P}$ and $g \in \langle Q_1P_1, Q_2P_2\rangle_{\mathcal{E}_{Q,P}} + \mathfrak{M}_{Q,P}\langle Q_3\rangle$ such that $j^2(X_f \circ S_a)(0) = j^2(X_H \circ S_a + (S_a)_*X_g)(0)$. This means that $j^3(f \circ S_a)(0) = j^3(H \circ S_a + g)(0)$. It follows that there exist function germs $h_1, h_2, h_3 \in \mathfrak{M}_Q$, $h_0 \in \mathfrak{M}_Q^2$ such that

$$\begin{aligned} f \circ S_a = -Q_2^3 \quad &\equiv \quad h_1(q\circ S_a)Q_1 + h_2(q \circ S_a)Q_2 + h_3(q\circ S_a)(Q_2^2 + Q_3) + h_0(q \circ S_a) \\ &\quad \bmod \langle Q_1P_1, Q_2P_2\rangle_{\mathcal{E}_{Q,P}} + \mathfrak{M}_{Q,P}\langle Q_3\rangle + \mathfrak{M}^4_{Q,P}. \end{aligned}$$

We may reduce this to

$$\begin{aligned} -Q_2^3 \quad \equiv \quad & h_1(-2Q_1 - Q_2, -Q_1 - 3aQ_2^2 - P_2 + 2P_3Q_2, -P_3)Q_1 \\ & + h_2(-2Q_1 - Q_2 - P_1, -Q_1 - 3aQ_2^2 + 2P_3Q_2, -P_3)Q_2 \\ & + h_3(-2Q_1 - Q_2 - P_1, -Q_1, -P_3)Q_2^2 + h_0(-2Q_1 - Q_2 - P_1, -Q_1 - P_2, -P_3) \\ & \bmod \langle Q_1P_1, Q_2P_2\rangle_{\mathcal{E}_{Q,P}} + \mathfrak{M}_{Q,P}\langle Q_3\rangle + \mathfrak{M}^4_{Q,P}. \end{aligned}$$

We show this equation has a contradiction. The coefficients of $P_1^{i_1}P_2^{i_2}P_3^{i_3}$ on the equation depend only on the coefficients of $q_1^{i_1}q_2^{i_2}q_3^{i_3}$ on h_0 respectively. This means that $h_0(q \circ S_a) \equiv 0$. The coefficients of Q_1^2, Q_1P_2, Q_1P_3 on the equation depend only on the coefficients of q_1, q_2, q_3 on h_1 respectively. This means that $j^1(h_1(q \circ S_a)(0) \equiv 0$. The coefficients of Q_2P_1, Q_1Q_2, Q_2P_3 on the equation depend only on the coefficients of q_1, q_1, q_3 on h_2. This means that $j^1(h_2(q \circ S_a))(0) \equiv 0$. So we need only to consider the quadratic part of h_1, h_2 and the linear part of h_3. The coefficients of $Q_2P_1^2, Q_2^2P_1$ on the equation depend only on the coefficient of q_1^2 on h_2 and the coefficient of q_1 on h_3 respectively. This means that their coefficients are all equal to 0. Therefore the coefficient of Q_2^3 on the right hand side of the equation is 0. This contradicts the equation. So we have that $j^5(\frac{dS_a}{da})(0)$ is not included in $T_z([z]_L)$.

We also prove the case $B^{+,+}_{2,2,3}$: We consider the reticular Lagrangian maps $\pi \circ i_a$ with the generating families $F_a(x_1, x_2, q_1, q_2, q_3) = (x_1 + x_2)^2 + ax_2^3 + q_1x_1 + q_2x_2 + q_3x_2^2$. Then the function germs $G_a(Q_1, Q_2, Q_3, q_1, q_2, q_3) = (Q_1 + Q_2)^2 + aQ_2^3 + q_1Q_1 + q_2Q_2 + q_3Q_2^2 + q_3Q_3$ are the generating functions of the canonical relations P_{S_a} and $i_a = S_a|_{\mathbb{L}}$. Then S_a have the forms:

$$S_a(Q, P) = (-(2Q_1 + 2Q_2 + P_1), -(2Q_1 + 2Q_2 + 3aQ_2^2 + P_2 - 2P_3Q_2), -P_3, Q_1, Q_2, Q_2^2 + Q_3).$$

We have that $\frac{dS_a}{da} = (0, -3Q_2^2, 0, 0, 0, 0) = X_f \circ S_a$ for $f = -p_2^3$. Then we consider the following equation:

$$\begin{aligned} f \circ S_a = -Q_2^3 \quad \equiv \quad & h_1(q \circ S_a)Q_1 + h_2(q \circ S_a)Q_2 + h_3(q \circ S_a)(Q_2^2 + Q_3) + h_0(q \circ S_a) \\ & \bmod \langle Q_1P_1, Q_2P_2\rangle_{\mathcal{E}_{Q,P}} + \mathfrak{M}_{Q,P}\langle Q_3\rangle + \mathfrak{M}^4_{Q,P}, \end{aligned}$$

where $h_1, h_2, h_3 \in \mathfrak{M}(Q), h_0 \in \mathfrak{M}^2(Q)$. We may reduce this to

$$\begin{aligned} -Q_2^3 \quad \equiv \quad & h_1(-(2Q_1 + 2Q_2), -(2Q_1 + 2Q_2 + 3Q_2^2 + P_2 - 2Q_2P_3), -P_3)Q_1 \\ & + h_2(-(2Q_1 + 2Q_2 + P_1), -(2Q_1 + 2Q_2 + 3Q_2^2 - 2Q_2P_3), -P_3)Q_2 \\ & + h_3(-(2Q_1 + 2Q_2 + P_1), -(2Q_1 + 2Q_2), -P_3)Q_2^2 \\ & + h_0(-(2Q_1 + 2Q_2 + P_1), -(2Q_1 + 2Q_2 + 3aQ_2^2 + P_2 - 2Q_2P_3), -P_3) \\ & \bmod \langle Q_1P_1, Q_2P_2\rangle_{\mathcal{E}_{Q,P}} + \mathfrak{M}_{Q,P}\langle Q_3\rangle + \mathfrak{M}^4_{Q,P}. \end{aligned}$$

By the same reason in the case $B^{+,+}_{2,3}$, we have that $h_0(q \circ S_a) \equiv 0$. By the consideration of the coefficients of Q_1^2, Q_1P_2, Q_1P_3 and Q_2P_1, Q_2^2, Q_2P_3 on the equation, we have that $j^1(h_1(q \circ S_a)Q_1)(0) \equiv j^1(h_2(q \circ S_a)Q_2)(0) \equiv 0$. The coefficients of $Q_1P_2^2, Q_1P_3^2, Q_1P_2P_3$ on the equation depend only on the coefficients of q_2^2, q_3^2, q_2q_3 on h_1. This means that they are all equal to 0. The coefficients of $Q_1^2P_2, Q_1^2P_3, Q_1^3$ depend only on the coefficients of q_1q_2, q_1q_3, q_1^2 on h_1. This means that they are all equal to 0. We have that $j^2(h_1(q \circ S_a)Q_1)(0) \equiv 0$.

The coefficients of $Q_2P_1^2, Q_2P_3^2, Q_2P_1P_3$ depend only on the coefficients of q_1^2, q_3^2, q_1q_3 on h_2 and they are all equal to 0. We write $h_2 = q_2(bq_1 + cq_2 + dq_3), h_3 = eq_1 + fq_2 + gq_3$. We calculate the coefficients of $Q_1^2Q_2, Q_1Q_2^2, Q_2^2P_1, Q_1Q_2P_3, Q_2^2P_3$, then we have that $-2b - 2c = -8(-2b - 2c) + 2e(-2 - 2f) = 4b - 2e = d = 4d - 2eg = 0$. This is solved that $b = c = d = e = 0$ or $b = \frac{e}{2}, c = -\frac{e}{2}, d = 0, f = -1, g = 0$. This means that the coefficient of Q_2^3 on the right hand side of the equation is $4b + 4c - 2e - 2ef = 0$. This contradicts the equation.

We also prove the case $C^{+,+}_{3,2}$: We consider the reticular Lagrangian maps $\pi \circ i_a$ with the generating families $F_a(y, x_1, x_2, q_1, q_2, q_3) = y^3 + x_1y + x_2y + ax_2^2 + ax_2^3 + q_1y + q_2x_1 + q_3x_2$. Then the function germs $G_a(y, Q_1, Q_2, Q_3, q_1, q_2, q_3) = y^3 + Q_1 y + Q_2 y + aQ_2^2 + q_1 y + q_2 Q_1 + q_3 Q_2 + yQ_3$ are the generating families of the canonical relations P_{S_a} and $i_a = S_a|_{\mathbb{L}}$. Then S_a have the forms:

$$S_a(Q, P) = (-(3P_3^2 + Q_1 + Q_2 + Q_3), P_3 - P_1, P_3 - 2aQ_2 - P_2, -P_3, Q_1, Q_2).$$

We have that $\frac{dS_a}{da} = (0, 0, -2Q_2, 0, 0, 0) = X_f \circ S_a$ for $f = -p_3^2$. Then we consider the following equation:

$$\begin{aligned} f \circ S_a = -Q_2^2 \quad \equiv \quad & h_1(q \circ S_a)(-P_3) + h_2(q \circ S_a)Q_1 + h_3(q \circ S_a)Q_2 + h_0(q \circ S_a) \\ & \bmod \langle Q_1P_1, Q_2P_2\rangle_{\mathcal{E}_{Q,P}} + \mathfrak{M}_{Q,P}\langle Q_3\rangle + \mathfrak{M}_{Q,P}^3. \end{aligned}$$

We may reduce this to

$$\begin{aligned} -Q_2^2 \quad \equiv \quad & h_1(-(Q_1 + Q_2), P_3 - P_1, P_3 - 2aQ_2 - P_2)(-P_3) \\ & +h_2(-(Q_1 + Q_2), P_3, P_3 - 2aQ_2 - P_2)Q_1 \\ & +h_3(-(Q_1 + Q_2), P_3 - P_1, P_3 - 2aQ_2)Q_2 \\ & +h_0(-(Q_1 + Q_2), P_3 - P_1, P_3 - 2aQ_2 - P_2) \\ & \bmod \langle Q_1P_1, Q_2P_2\rangle_{\mathcal{E}_{Q,P}} + \mathfrak{M}_{Q,P}\langle Q_3\rangle + \mathfrak{M}_{Q,P}^3. \end{aligned}$$

Since the coefficients of $P_1^{i_2}P_2^{i_3}$ on the equation depend only on the coefficients of $q_2^{i_2}q_3^{i_3}$ on h_0, it follows that they are all equal to 0. Since the coefficients of P_1P_3, P_2P_3 depend only on the coefficients of q_2, q_3 on h_1, it follows that they are all equal to 0.

Therefore we may set $h_1 = bq_1$, $h_2 = cq_1 + dq_2 + eq_3$, $h_3 = fq_1 + gq_2 + hq_3$, $h_0 = q_1(iq_1 + jq_2 + hq_3)$. By the calculation of the equation, we have that the coefficient of Q_2^2 on the right hand side of the equation is 0. This contradicts the equation.

Lemma 3.4. *Let $\pi \circ i : (\mathbb{L}, 0) \to (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a reticular Lagrangian map, S be an extension of i. Suppose that the caustic-equivalence class $[j_0^{n+2}S(0)]_c$ be an immersed manifold of $S^{n+2}(n)$. If a generating family of $\pi \circ i$ is reticular $\mathcal{P}$-C-stable and $j_0^{n+2}S$ is transversal to $[j_0^{n+2}S(0)]_c$ at 0, then $\pi \circ i$ is caustic stable.*

This is proved by the analogous method of [2, Theorem 6.6 (t)&(is)⇒(s)]. By this lemma, we have that the caustic-stability of reticular Lagrangian maps is reduced to the transversality of finite dimensional jets of extensions of their reticular Lagrangian embeddings.

4. Weak Caustic-equivalence

There exist modalities in the classification list of Section 2. This means that the caustic-equivalence is still too strong for a generic classification of caustics on a corner. In order to obtain the generic classification, we need to admit the following equivalence relations:

We say that reticular Lagrangian maps $\pi \circ i_1$ and $\pi \circ i_2$ are *weakly caustic-equivalent* if there exists a homeomorphism germ g on $(\mathbb{R}^n, 0)$ such that g is smooth on all C_σ^1, $Q_{\sigma,\tau}^1$, and satisfies (1).

We say that two function germs in $\mathfrak{M}(r; k + n)^2$ are *weakly reticular $\mathcal{P}$-C-equivalent* if they are generating families of weakly caustic-equivalent reticular Lagrangian maps. We define the *stable* weakly reticular $\mathcal{P}$-C-equivalence by the ordinary way.

We here investigate the reticular $\mathcal{C}$-equivalence classes $B^{+,+,2}_{2,2,a}$ of function germs. The same methods are valid for the classes $B^{\pm,+,1}_{2,2,a}$, $B^{\pm,+,2}_{2,2,a}$, $B^{\pm,-}_{2,2,a}$. So we prove only to the classes $B^{+,+,2}_{2,2,a}$.

We consider the reticular Lagrangian maps $\pi \circ i_a : (\mathbb{L}, 0) \to (T^*\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ with the generating families $F_a(x_1, x_2, q_1, q_2) = x_1^2 + x_1x_2 + ax_2^2 + q_1x_1 + q_2x_2$ $(a > \frac{1}{4})$. We give the caustic of $\pi \circ i_a$ and $\pi \circ i_b$ for $\frac{1}{4} < a < b$. In these figures $Q_{1,I_2}, Q_{2,I_2}, Q_{\emptyset,2}$ are in the same positions. Suppose that there exists a diffeomorphism germ g on $(\mathbb{R}^2, 0)$ such that $Q_{1,I_2}, Q_{2,I_2}, Q_{\emptyset,2}$ are invariant under g. Then g can not map $Q_{\emptyset,1}$ from one to the other. This implies that caustic-equivalence is too strong for generic classifications. But these caustic are equivalent under the weak caustic-equivalence. This implies that the reticular Lagrangian map $\pi \circ i_a$ is weakly caustic equivalent to $\pi \circ i_1$ for any $a > \frac{1}{4}$ and hence F_a is weakly reticular $\mathcal{P}$-$\mathcal{C}$-equivalent to F_1. We remark that a homeomorphism germ g_a, which gives the weak caustic-equivalence of $\pi \circ i_1$ and $\pi \circ i_a$, may be chosen to be smooth outside 0 and depends smoothly on a. This means that the weak caustic-equivalence relation is naturally extended for the (caustic) stable reticular Lagrangian maps with the generating families $F'_a(x_1, x_2, q_1, q_2, q_3) = x_1^2 + x_1x_2 + ax_2^2 + q_1x_1 + q_2x_2 + q_3x_2^2$ and F'_a is weakly reticular $\mathcal{P}$-$\mathcal{C}$-equivalent to $F'(x_1, x_2, q_1, q_2, q_3) = x_1^2 + x_1x_2 + x_2^2 + q_1x_1 + q_2x_2$. The figure of the corresponding caustic is given in [1, p.602 $B^{+,+,\overset{<}{a}}_{2,2}$]. We also remark that the functions $x_1^2 + x_1x_2 + x_2^2 + q_1x_1 + q_2x_2$ and $x_1^2 + x_1x_2 + \frac{1}{5}x_2^2 + q_1x_1 + q_2x_2$ are not weakly reticular $\mathcal{P}$-$\mathcal{C}$-equivalent because $Q_{\emptyset,1}$ and $Q_{\emptyset,1}$ of their caustics are in the opposite positions to each other.

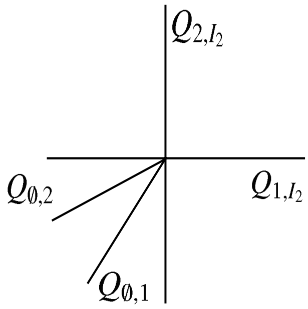

Figure 1: the caustics of $\pi \circ i_a$

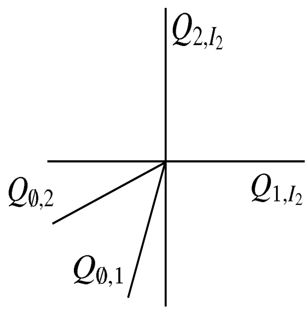

Figure 2: the caustics of $\pi \circ i_b$

By the above consideration, we regard the function germ $f_a(x) = x_1^2 + x_1x_2 + ax_2^2 (a > \frac{1}{4})$ are all equivalent. We say this equivalence relation *the weak reticular $\mathcal{C}$-equivalence*. Since $\frac{df_a}{da} = x^2$ is not included in $\langle x\frac{\partial f_a}{\partial x}\rangle_{\mathcal{E}(x)}$, it follows that the l-jets of the weak reticular $\mathcal{C}$-equivalence class of f_a consists an immersed manifold of $J^l(2, 1)$ for $l \geq 2$.

We classify function germs in $\mathfrak{M}(2; k)^2$ with respect to the weak reticular $\mathcal{C}$-equivalence with the codimension≤ 3. Then we have the following list:

k	Normal form	codim	Notation
0	$x_1^2 \pm x_1x_2 + \frac{1}{5}x_2^2$	2	$B_{2,2}^{\pm,+,1}$
	$x_1^2 \pm x_1x_2 + x_2^2$	2	$B_{2,2}^{\pm,+,2}$
	$x_1^2 \pm x_1x_2 - x_2^2$	2	$B_{2,2}^{\pm,-}$
	$x_1^2 \pm x_2^2$	3	$B_{2,2}^{\pm,0}$
	$(x_1 \pm x_2)^2 \pm x_2^3$	3	$B_{2,2,3}^{\pm,\pm}$
	$x_1^2 \pm x_1x_2 \pm x_2^3$	3	$B_{2,3}^{\pm,\pm}$
	$x_1^3 \pm x_1x_2 \pm x_2^2$	3	$B_{3,2}^{\pm,\pm}$
1	$\pm y_1^3 + x_1y \pm x_2y + x_2^2$	3	$C_{3,2}^{\pm,\pm}$

Proposition 4.1. *Let $\pi \circ i_a : (\mathbb{L}, 0) \to (T^*\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be the reticular Lagrangian map with the generating family $x_1^2 + x_1x_2 + ax_2^2 + q_1x_1 + q_2x_2$. Let $S_a \in S(T^*\mathbb{R}^2, 0)$ be extensions of i_a. Then the weak caustic-equivalence class*

$$[j^l S_1(0)]_w := \bigcup_{a>\frac{1}{4}} [j^l S_a(0)]_c$$

is an immersed manifold in $S^l(2)$ for $l \geq 1$.

Proof. The function germ $G_a(Q_1, Q_2, q_1, q_2) = Q_1^2 + Q_1Q_2 + aQ_2^2 + q_1Q_1 + q_2Q_2$ is a generating function of the canonical relation P_{S_a} and we have that

$$S_a(Q, P) = (-(2Q_1 + Q_2 + P_1), -(Q_1 + 2aQ_2 + P_2), Q_1, Q_2).$$

This means that $\frac{dS_a}{da} = (0, -2Q_2, 0, 0) = X_f \circ S_a$ for $f = -p_2^2$. Suppose that $j^1(\frac{dS_a}{da})(0)$ is included in $T_z(rLa^1(2) \cdot z)$. Then there exist $h_1, h_2 \in \mathfrak{M}_{Q,P}$ and $h_0 \in \mathfrak{M}_{Q,P}^2$ such that

$$-Q_2^2 \equiv h_1(q \circ S_a)Q_1 + h_2(q \circ S_a)Q_2 + h_0(q \circ S_a) \bmod \langle Q_1P_1, Q_2P_2 \rangle_{\mathcal{E}_{Q,P}} + \mathfrak{M}_{Q,P}^3.$$

We need only to consider the linear parts of h_1, h_2 and the quadratic part of h_0. The coefficients of P_1^2, P_2^2, P_1P_2 depend only on the coefficients of Q_1^2, Q_2^2, Q_1Q_2 on h_0 respectively. This means that $h_0 \equiv 0$. We set $h_1 = bq_1 + cq_2, h_2 = dq_1 + eq_2$ and calculate the coefficients of $Q_1^2, Q_1Q_2, Q_1P_2, Q_2P_1$ in the equation. Then we have that $-2b - c = 0, -b - 2d - e - 2ca = 0, c = 0, d = 0$. This means that $e = 0$. Then we have that the coefficient of Q_2^2 of the right hand side of the equation is equivalent to $-d - ae = 0$. This contradicts the equation. ∎

If we consider the (caustic) stable reticular Lagrangian map $\pi \circ i_a : (\mathbb{L}, 0) \to (T^*\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ with the generating family $x_1^2 + x_1x_2 + ax_2^2 + q_1x_1 + q_2x_2 + q_3x_2^2$ and take an extension $S'_a \in S(T^*\mathbb{R}^2, 0)$ of i_a, then we have by the analogous method that:

Corollary 4.2. *Let S'_a be as above. Then*

$$[j^l S'_1(0)]_w := \bigcup_{a>\frac{1}{4}} [j^l S'_a(0)]_c$$

is an immersed manifold in $S^l(3)$ for $l \geq 1$.

Since the caustic of $\pi \circ i_a$ is given by the restrictions of $\pi \circ i_a$ to $L^0_\sigma \cap L^0_\tau$ for $\sigma \neq \tau$ in this case, it follows that the caustic is determined by the linear part of i_a. This means that $\pi \circ i_a$ is 1-determined with respect to the weak caustic-equivalence (cf., Definition 3.1).

Theorem 4.3. *The function germ $F(x_1, x_2, q_1, q_2) = x_1^2 + x_1x_2 + x_2^2 + q_1x_1 + q_2x_2$ is a weakly reticular $\mathcal{P}$-$\mathcal{C}$-stable unfolding of $f(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2$*

Proof. We define $F' \in \mathfrak{M}(2;3)^2$ by $F'(x_1, x_2, q_1, q_2, q_3) = F(x_1, x_2, q_1, q_2) + q_3x_2^2$ Then F' is a reticular $\mathcal{P}$-$\mathcal{R}^+$-stable unfolding of f. It follows that for any neighborhood U' of 0 in $(\mathbb{R}^5, 0)$ and any representative $\tilde{F'} \in C^\infty(U, \mathbb{R})$, there exists a neighborhood $N_{\tilde{F}'}$ such that for any $\tilde{G'} \in N_{\tilde{F}'}$ the function germ $\tilde{G'}|_{\mathbb{H}^2\times\mathbb{R}^3}$ at p'_0 is reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent to F' for some $p'_0 = (0, 0, q^0_1, q^0_2, q^0_3) \in U'$.

Let a neighborhood U of 0 in $(\mathbb{R}^4, 0)$ and a representative $\tilde{F} \in C^\infty(U, \mathbb{R})$ be given. We set the open interval $I = (-0.5, 0.5)$ and set $U' = U \times I$. Then there exists $N_{\tilde{F}'}$ for which the above condition holds. We can choose a neighborhood $N_{\tilde{F}}$ of $\tilde{F}$ such that for any $\tilde{G} \in N_{\tilde{F}}$ the function $\tilde{G} + q_3x_2^2 \in N_{\tilde{F}'}$. Let a function $\tilde{G} \in N_{\tilde{F}}$ be given. Then the function germ $G' = (\tilde{G} + q_3x_2^2)|_{\mathbb{H}^2\times\mathbb{R}^3}$ at p'_0 is reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent to F' for some $p'_0 = (0, 0, q^0_1, q^0_2, q^0_3) \in U'$. We define $G \in \mathfrak{M}(2;2)^2$ by $\tilde{G}|_{\mathbb{H}^2\times\mathbb{R}^2}$ at $p_0 = (0, 0, q^0_1, q^0_2) \in U$. Then it holds that $G'(x, q) = G(x, q_1, q_2) + (q_3 + q^0_3)x_2^2$, and $G'|_{q=0} = G(x, 0) + q^0_3x_2^2$ is reticular $\mathcal{R}$-equivalent to f. Let (Φ, a) be the reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalence from G' to F'. We write $\Phi(x, q) = (x\phi_1(x, q), \phi^2_1(q), \phi^2_2(q), \phi^2_3(q))$. By shrinking U if necessary, we may assume that the map germ

$$(q_1, q_2) \mapsto (\phi^2_1(q_1, q_2, 0), \phi^2_2(q_1, q_2, 0)) \text{ on } (\mathbb{R}^2, 0)$$

is a diffeomorphism germ. Then F is reticular $\mathcal{P}$-$\mathcal{R}^+$-equivalent to $G_1 \in \mathfrak{M}(2;2)^2$ given by $G_1(x, q) = G(x_1, x_2, q_1, q_2) + (\phi^2_3(q_1, q_2, 0) + q^0_3)x_2^2$. It follows that the reticular Lagrangian maps defined by F and G_1 are Lagrangian equivalent. We have that

$$j^2(G + q^0_3x_2^2)(0) = j^2G_1(0), \quad q^0_3 > -0.5.$$

This means that the caustic of G_1 is weakly caustic-equivalent to the caustic of G because the reticular Lagrangian maps of G_1 and F are the same weak caustic-equivalence class that is 1-determined under the weak caustic-equivalence. This means that F and G are weakly reticular $\mathcal{P}$-$\mathcal{C}$-equivalent. Therefore F is weakly reticular $\mathcal{P}$-$\mathcal{C}$-stable. ■

By the above consideration, we have that: For each singularity $B^{\pm,+,1}_{2,2}, B^{\pm,+,2}_{2,2}, B^{\pm,-}_{2,2}$, if we take the symplectic diffeomorphism germ $S_a(S'_a)$ as the above method, then the weak caustic-equivalence class $[j^lS_a(0)]_w([j^lS'_a(0)]_w)$ is one class and immersed manifold in $S^l(2)(S^l(3))$ for $l \geq 1$ respectively.

Theorem 4.4. *Let $n = 2$ or 3, and U a neighborhood of 0 in $T^*\mathbb{R}^n$. Then there exists a residual set $O \subset S(U, T^*\mathbb{R}^n)$ such that for any $\tilde{S} \in O$ and $x \in U$, the reticular Lagrangian map $\pi \circ \tilde{S}_x|_{\mathbb{L}}$ is weakly caustic-stable or caustic-stable, where $\tilde{S}_x \in S(T^*\mathbb{R}^n, 0)$ be defined by the map $x_0 \mapsto \tilde{S}(x_0 + x) - \tilde{S}(x)$.*

A reticular Lagrangian map $\pi \circ \tilde{S}_x|_{\mathbb{L}}$ for any $\tilde{S} \in O$ and $x \in U$ has a generating family F which is a weakly reticular $\mathcal{P}$-$\mathcal{C}$-stable unfolding of $B^{\pm,+,1}_{2,2}, B^{\pm,+,2}_{2,2}, B^{\pm,-}_{2,2}$, or a reticular $\mathcal{P}$-$\mathcal{C}$-stable unfolding of $B^{\pm,0}_{2,2}, B^{\pm,\pm}_{2,2,3}, B^{\pm,\pm}_{2,3}, B^{\pm,\pm}_{3,2}, C^{\pm,\pm}_{2,3}$, that is F is weakly reticular $\mathcal{P}$-$\mathcal{C}$-equivalent to one of

$B^{\pm,+,1}_{2,2}$: $F(x_1, x_2, q_1, q_2) = x_1^2 \pm x_1x_2 + \frac{1}{5}x_2^2 + q_1x_1 + q_2x_2$,
$B^{\pm,+,2}_{2,2}$: $F(x_1, x_2, q_1, q_2) = x_1^2 \pm x_1x_2 + x_2^2 + q_1x_1 + q_2x_2$,
$B^{\pm,-}_{2,2}$: $F(x_1, x_2, q_1, q_2) = x_1^2 \pm x_1x_2 - x_2^2 + q_1x_1 + q_2x_2$,
or F is reticular $\mathcal{P}$-$\mathcal{C}$-equivalent to one of
$B^{\pm,0}_{2,2}$: $F(x_1, x_2, q_1, q_2, q_3) = x_1^2 \pm x_2^2 + q_1x_1 + q_2x_2 + q_3x_1x_2$,
$B^{\pm,\pm}_{2,2,3}$: $F(x_1, x_2, q_1, q_2, q_3) = (x_1 \pm x_2)^2 \pm x_2^3 + q_1x_1 + q_2x_2 + q_3x_2^2$,
$B^{\pm,\pm}_{2,3}$: $F(x_1, x_2, q_1, q_2, q_3) = x_1^2 \pm x_1x_2 \pm x_2^3 + q_1x_1 + q_2x_2 + q_3x_2^2$,
$B^{\pm,\pm}_{3,2}$: $F(x_1, x_2, q_1, q_2, q_3) = x_1^3 \pm x_1x_2 \pm x_2^2 + q_1x_1 + q_2x_2 + q_3x_1^2$,
$C^{\pm,\pm}_{3,2}$: $F(y, x_1, x_2, q_1, q_2, q_3) = \pm y_1^3 + x_1y \pm x_2y + x_2^2 + q_1y + q_2x_1 + q_3x_2$.

Proof. We choose the weakly caustic-stable reticular Lagrangian maps $\pi \circ i_X : (\mathbb{L}, 0) \to (T^*\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ for

$$X = B^{\pm,+,1}_{2,2}, B^{\pm,+,2}_{2,2}, B^{\pm,-}_{2,2}. \tag{2}$$

We also choose the caustic-stable reticular Lagrangian maps $\pi \circ i_X : (\mathbb{L}, 0) \to (T^*\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ for

$$X = B^{\pm,0}_{2,2}, B^{\pm.\pm}_{2,2,3}, B^{\pm,\pm}_{2,3}, B^{\pm,\pm}_{3,2}, C^{\pm,\pm}_{2,3}. \tag{3}$$

Then other reticular Lagrangian maps are not caustic-stable since other singularities have reticular $\mathcal{C}$-codimension > 3. We choose extensions $S_X \in S(T^*\mathbb{R}^n, 0)$ of i_X for all X. We define that

$$O'_1 = \{\tilde{S} \in S(U, T^*\mathbb{R}^n) \mid j_0^{n+2}\tilde{S} \text{ is transversal to } [j^{n+2}S_X(0)]_w \text{ for all } X \text{ in (2)}\},$$
$$O'_2 = \{\tilde{S} \in S(U, T^*\mathbb{R}^n) \mid j_0^{n+2}\tilde{S} \text{ is transversal to } [j^{n+2}S_X(0)]_c \text{ for all } X \text{ in (3)}\},$$

where $j_0^l\tilde{S}(x) = j^l\tilde{S}_x(0)$. Then O'_1 and O'_2 are residual sets. We set

$$Y = \{j^{n+2}S(0) \in S^{n+2}(n) \mid \text{the codimension of } [j^{n+2}S(0)]_L > 8\}.$$

Then Y is an algebraic set in $S^{n+2}(n)$ by [2, Theorem 6.6 (a')]. Therefore we can define that

$$O'' = \{\tilde{S} \in S(U, T^*\mathbb{R}^n) \mid j_0^{n+2}\tilde{S} \text{ is transversal to } Y\}.$$

For any $S \in S(T^*\mathbb{R}^n, 0)$ with $j^{n+2}S(0)$ and any generating family F of $\pi \circ S|_{\mathbb{L}}$, the function germ $F|_{q=0}$ has the reticular $\mathcal{R}^+$-codimension > 4. This means that $F|_{q=0}$ has the reticular C-codimension > 3. It follows that $j^{n+2}S(0)$ does not belong to the above equivalence classes. Then Y has codimension > 6. Then we have that

$$O'' = \{\tilde{S} \in S(U, T^*\mathbb{R}^n) \mid j_0^{n+2}\tilde{S}(U) \cap Y = \emptyset\}.$$

We define $O = O'_1 \cap O'_2 \cap O''$. Since all $\pi \circ i_X$ for X in (2) are weak caustic 1-determined, and all $\pi \circ i_X$ in (3) are caustic 5-determined by Lemma 3.2. Then O has the required condition. ■

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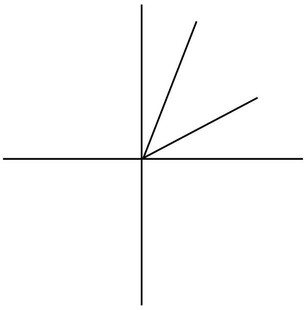

Figure 3: $B_{2,2}^{+,+,1}$, $B_{2,2}^{+,+,2}$

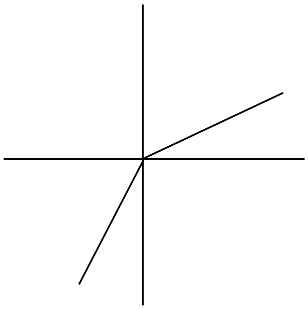

Figure 4: $B_{2,2}^{-,+,1}$, $B_{2,2}^{-,+,2}$

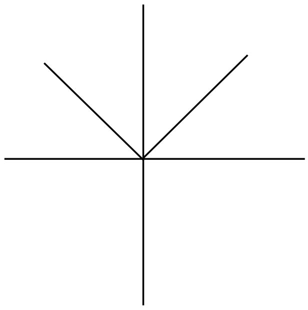

Figure 5: $B_{2,2}^{+,-}$, $B_{2,2}^{-,-}$

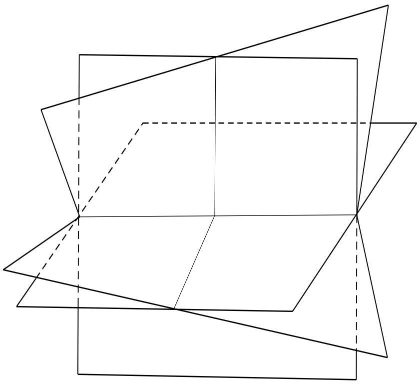

Figure 6: $B_{2,2}^{+,0}$

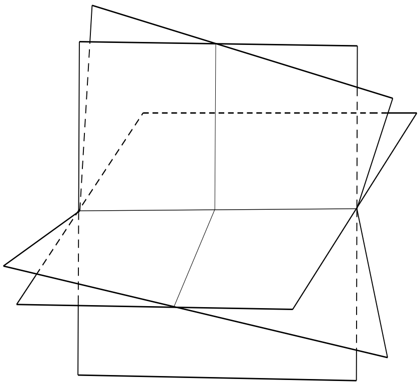

Figure 7: $B_{2,2}^{-,0}$

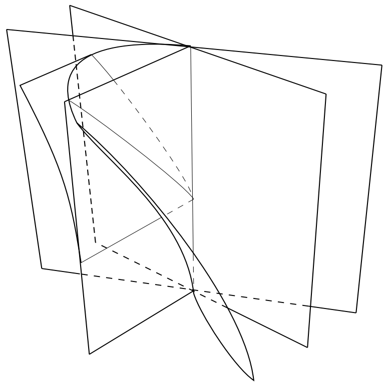

Figure 8: $B^{+,+}_{2,2,3}$

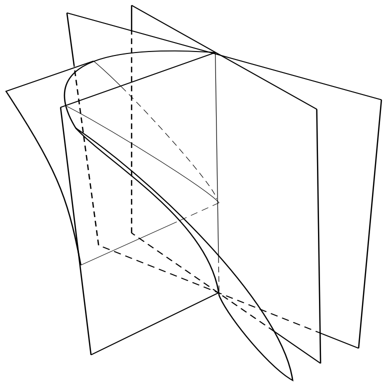

Figure 9: $B^{+,-}_{2,2,3}$

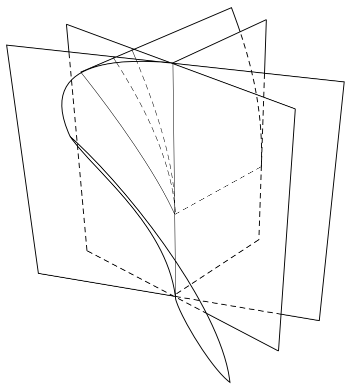

Figure 10: $B^{-,+}_{2,2,3}$

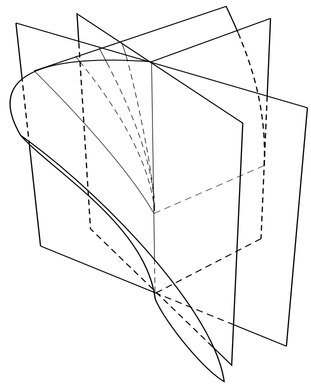

Figure 11: $B^{-,-}_{2,2,3}$

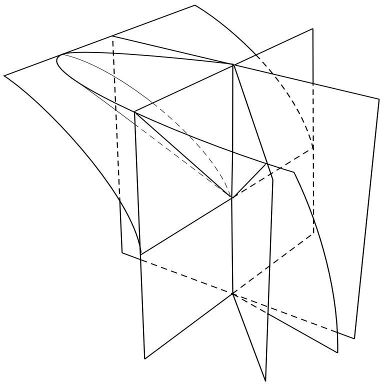

Figure 12: $B^{+,+}_{2,3}$

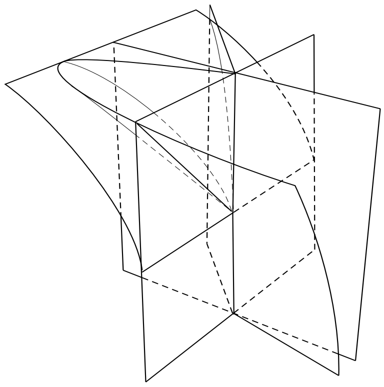

Figure 13: $B^{+,-}_{2,3}$

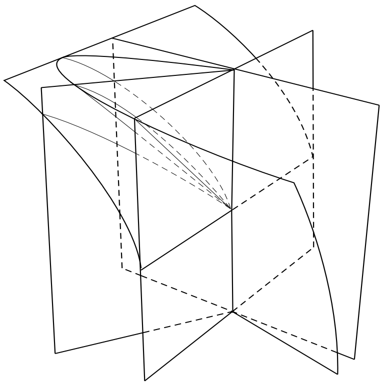

Figure 14: $B^{-,+}_{2,3}$

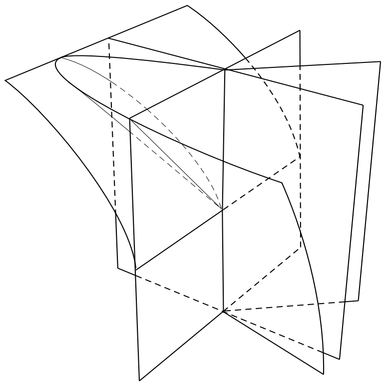

Figure 15: $B^{-,-}_{2,3}$

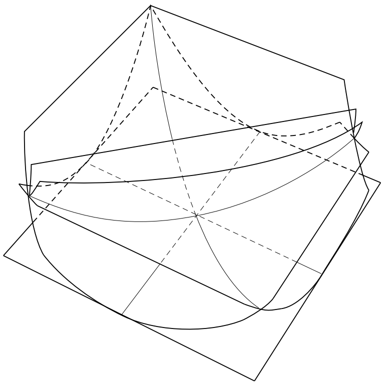

Figure 16: $C_{3,2}^{+,+}$

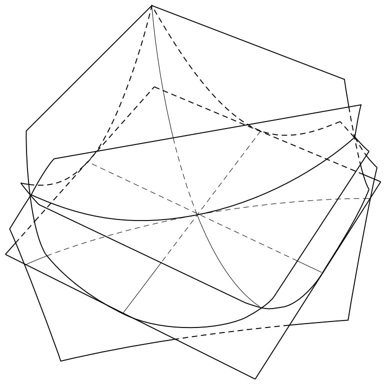

Figure 17: $C_{3,2}^{+,-}$

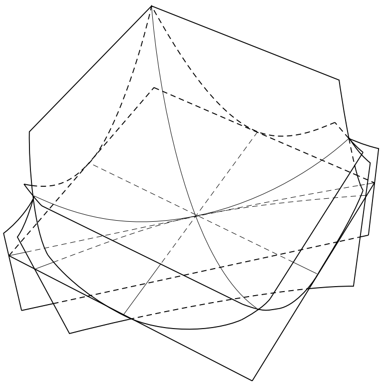

Figure 18: $C_{3,2}^{-,+}$

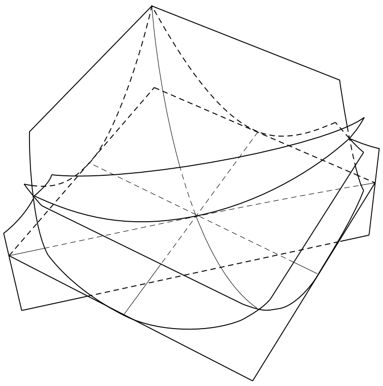

Figure 19: $C_{3,2}^{-,-}$