CONCERNING THE L^4 NORMS OF TYPICAL EIGENFUNCTIONS ON COMPACT SURFACES

CHRISTOPHER D. SOGGE AND STEVE ZELDITCH

ABSTRACT. Let (M,g) be a two-dimensional compact boundaryless Riemannian manifold with Laplacian, Δ_g. If e_λ are the associated eigenfunctions of $\sqrt{-\Delta_g}$ so that $-\Delta_g e_\lambda = \lambda^2 e_\lambda$, then it has been known for some time [18] that $\|e_\lambda\|_{L^4(M)} \lesssim \lambda^{1/8}$, assuming that e_λ is normalized to have L^2-norm one. This result is sharp in the sense that it cannot be improved on the standard sphere because of highest weight spherical harmonics of degree k. On the other hand, we shall show that the average L^4 norm of the standard basis for the space $\mathcal{H}_k$ of spherical harmonics of degree k on S^2 merely grows like $(\log k)^{1/4}$. We also sketch a proof that the average of $\sum_{j=1}^{2k+1} \|e_\lambda\|_{L^4}^4$ for a random orthonormal basis of $\mathcal{H}_k$ is $O(1)$. We are not able to determine the maximum of this quantity over all orthonormal bases of $\mathcal{H}_k$ or for orthonormal bases of eigenfunctions on other Riemannian manifolds. However, under the assumption that the periodic geodesics in (M,g) are of measure zero, we are able to show that for *any* orthonormal basis of eigenfunctions we have that $\|e_{\lambda_{j_k}}\|_{L^4(M)} = o(\lambda_{j_k}^{1/8})$ for a density one subsequence of eigenvalues λ_{j_k}. This assumption is generic and it is the one in the Duistermaat-Gullemin theorem [6] which gave related improvements for the error term in the sharp Weyl theorem. The proof of our result uses a recent estimate of the first author [20] that gives a necessary and sufficient condition that $\|e_\lambda\|_{L^4(M)} = o(\lambda^{1/8})$.

1. **Introduction.**

The purpose of this note is to introduce a new problem on L^p norms of eigenfunctions on compact Riemannian manifolds (M,g). We prove some initial results on the problem, and also include some conjectures and heuristic remarks.

The problem, roughly speaking, is to determine the asymptotic average of the L^4 norms $||e_\lambda||_4$ of the elements of an orthonormal basis of eigenfunctions

$$-\Delta_g e_\lambda = \lambda^2 e_\lambda$$

of the associated Laplace-Beltrami operator. In practice it is simpler to consider the fourth power Weyl sums,

$$\frac{1}{N(\lambda)} \sum_{j:\lambda_j \le \lambda} ||e_{\lambda_j}||_4^4 \tag{1.1}$$

where

$$N(\lambda) = \#\{\lambda_j \le \lambda\},$$

2000 *Mathematics Subject Classification.* Primary, 35F99; Secondary 35L20, 42C99.
Key words and phrases. Eigenfunction estimates, spherical harmonics.
The authors were supported in part by the NSF.

is the Weyl counting function. The asymptotics of (1.1) depend on the entire orthonormal basis and, as will be seen below, can behave quite differently from the behavior of individual eigenfunctions in the basis.

Before stating our results, let us recall the results on L^p norms of individual eigenfunctions. In 1988, one of us showed in [18] that for $2 < q \le \infty$ and

$$\sigma(q) = \max\big(2(1/2-1/q)-1/2, \tfrac{1}{2}(1/2-1/q)\big) = \begin{cases} 2(1/2-1/q)-1/2, & q \ge 6 \\ \tfrac{1}{2}(1/2-1/q), & 2 < q \le 6, \end{cases} \tag{1.2}$$

we have

$$\|e_\lambda\|_{L^q(M)} \lesssim \lambda^{\sigma(q)}, \tag{1.3}$$

assuming as we shall do throughout that the eigenfunctions are L^2-normalized so that

$$\|e_\lambda\|_{L^2(M)} = 1,$$

where the norms are taken with respect to the volume element, dV. This result is sharp since certain spherical harmonics on the sphere, S^2, with the round metric saturate the estimate (1.3). Specifically, when $q \ge 6$, L^2-normalized zonal functions, Z_k satisfy

$$\|Z_k\|_{L^q(S^2)} \approx k^{2(1/2-1/q)-1/2}, \quad q \ge 6,$$

while the L^2-normalized highest weight spherical harmonics, $Q_k = c_k(x_1+ix_2)^k$ satisfy

$$\|Q_k\|_{L^q(S^2)} \approx k^{\frac{1}{2}(1/2-1/q)}, \quad q \ge 2.$$

Both are eigenfunctions of the standard Laplacian on S^2 with eigenvalue $\lambda^2 = k(k+1)$ in the above notation. Also, we are taking S^2 to be $\{(x_1,x_2,x_3) : x_1^2+x_2^2+x_3^2 = 1\}$, so that, as $k \to \infty$, the Q_k become highly concentrated on the equator where $x_3 = 0$. The orthonormal basis of joint eigenfunctions of Δ_g and of x_3-axis rotations are generally denoted by $Y_m^k, m = -k, \ldots, k$; in particular, $Z_k = Y_0^k$ and $Q_k = Y_k^k$.

Even though (1.2) cannot be improved on the sphere, it is thought that for generic manifolds one has at least

$$\|e_\lambda\|_{L^q(M)} = o(\lambda^{\sigma(q)}), \quad \text{as} \;\; \lambda \to \infty, \tag{1.4}$$

for a given $q > 2$. In [22] we showed that for generic (M,g) this is true for $q > 6$ (and also corresponding results for higher dimensions). This just followed from showing that under a certain generic condition on (M,g) one can improve the L^∞ estimate in (1.3) to be $\|e_\lambda\|_\infty = o(\lambda^{1/2})$, which implies (1.4) for all $q > 6$ by interpolating with (1.3) for $q = 6$. The results in [22] were recently improved in [21]. The key point was to show that the bound $\|e_\lambda\|_\infty = O(\lambda^{1/2})$ can only be obtained on (M,g) possessing a "peak point" or "pole" z_0 with the property that a positive measure of directions in $S^*_{z_0}M$ exponentiate to geodesic loops which return to z_0 at some time. This behavior occurs at poles of a surface of revolution, since all meridians are closed geodesics through the pole, and in particular explains why the sup norm bounds are attained by zonal functionsZ_k on the round sphere (see §3 below).

Even though there are satisfactory results concerning (1.4) for relatively large exponents $q > 6$, much less is known for relatively small exponents $2 < q < 6$. In this case, it is thought that the enemy for (1.4) is maximal concentration along periodic geodesics, as occurs for the highest weight spherical harmonics. Using the formula for the Q_k one checks that they have L^2-mass bounded below on shrinking $k^{-1/2}$ neighborhoods of the equator $\gamma = \{(x_1, x_2, x_3) \in S^2 : x_3 = 0\}$. In [20], (following an earlier result in [3]), the first author proved that for $2 < q < 6$, (1.4) is valid if and only if this type of concentration does not occur. Specifically, a necessary and sufficient condition for (1.4) for this range of exponents is that

$$\sup_{\gamma \in \Pi} \int_{\mathrm{dist}_g(\gamma, y) \le \lambda^{-1/2}} |e_\lambda(y)|^2 \, dV = o(1), \tag{1.5}$$

where Π is the space of all unit-length geodesics in M, and $\mathrm{dist}_g(\,\cdot\,,\,\cdot\,)$ is the geodesic distance associated to the metric g.

The goal of this paper is to show that even though on some manifolds there are eigenfunctions e_λ having L^4-norms of maximal size $\approx \lambda^{1/8}$ as $\lambda \to \infty$, they are very sparse. Our first result of this type says that given any orthonormal basis $\{e_{\lambda_j}\}$ of eigenfunctions with eigenvalues $\lambda_1 \le \lambda_2 \le \dots$ on a two-dimensional compact Riemannian manifold (M, g) with a zero measure of periodic geodesics, one can find a density one subsequence of eigenvalues, $\{\lambda_{j_k}\}$, for which

$$\|e_{\lambda_{j_k}}\|_{L^4(M)} = o(\lambda_{j_k}^{1/8}). \tag{1.6}$$

By interpolation with the L^6-estimate in (1.3) and the trivial L^2-estimate, this implies that we also have $\|e_{\lambda_{j_k}}\|_{L^q} = o(\lambda_{j_k}^{\sigma(q)})$ for every $2 < q < 6$. Presently, we do not how to prove the corresponding results for $q \ge 6$, or how to obtain any results like this for higher dimensions $n \ge 3$. The condition that (M, g) have a zero set of periodic geodesics is generic and it is the assumption in the Duistermatt-Guillemin theorem [6], which involved a similar o-improvement of the error term in the Weyl formula.

The assumptions that the periodic orbits are of measure zero of course is not valid for the sphere. Nonetheless, we can prove a much stronger result for the standard basis $\{Y_m^k\}$ on S^2, even though, as we pointed out before, this eigen-basis has functions saturating (1.3) for each $2 < q \le \infty$.

To be more specific, we recall that the Laplace-Beltrami operator on $S^2 \subset \mathbb{R}^3$ with the standard round metric has eigenvalues $\lambda^2 = k(k+1)$ repeating with multiplicity $2k+1$, meaning that the corresponding eigenspace $\mathcal{H}_k$ of spherical harmonics of degree k has this dimension. If we use longitudinal coordinates $\phi \in [0, \pi]$ and latitudinal ones $\theta \in [0, 2\pi]$ so that $S^2 \ni x = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, then in these coordinates the standard basis for $\mathcal{H}_k$ has elements

$$Y_m^k(\phi, \theta) = c_{m,k} P_k^m(\cos\phi) e^{im\theta}, \quad -k \le m \le k, \tag{1.7}$$

where P_k^m are Legendre functions and $c_{m,k}$ are L^2-normalizing constants. When $m = 0$, Y_0^k is the zonal function Z_k, and when $m = \pm k$ it is a highest weight spherical harmonic of degree k. For this basis, we shall show that the average L^4-norm is of size $\approx (\log k)^{1/4}$, as $k \to \infty$, i.e.,

$$\frac{1}{2k+1} \sum_{m=-k}^{k} \int_{S^2} |Y_m^k|^4 \, dV \approx \log k, \quad k \ge 2, \tag{1.8}$$

which of course is much stronger than (1.6) since it shows that there must be a density one sequence of eigenfunctions among this basis with L^4-norms growing logarithmically with respect to the eigenvalues. It seems somewhat paradoxical at first that (1.8) is valid for the standard basis on the sphere, while the same basis is the worst case for (1.3). But this holds because the left side of (1.8) is a functional of an orthonormal basis rather than of individual eigenfunctions, and most elements Y^k_m have relatively small L^4 norms. It is doubtful that the $\{Y^k_m\}$ maximize this functional among orthonormal bases of spherical harmonics. In Section 4 we explain this further.

These observations raise the following

Problem Let $\dim M = 2$. For which (M, g) (if any) does there exist an orthonormal basis of eigenfunctions for which there exists a positive density subsequence $e_{\lambda_{j_k}}$ so that $||e_{\lambda_{j_k}}||_{L^4} = \Omega(\lambda_{j_k}^{1/8})$.? Or is a result like (1.6) is valid on any compact surface?

We prove (1.8) by obtaining pointwise bounds for the $\ell^4(m)$ norms of the basis elements of $\mathcal{H}_k$. Specifically, we shall prove sharp estimates for

$$\|Y^k_m(x)\|_{\ell^4(m)} = \left(\sum_{m=-k}^{k} |Y^k_m(x)|^4 \right)^{1/4} .$$

By the inclusion $\ell^4 \subset \ell^2$, this quantity is bounded by the corresponding $\ell^2(m)$-norm. On the sphere, the $\ell^2(m)$ norm is independent of x, and, in fact,

$$\|Y^k_m(x)\|_{\ell^2(m)} = \sqrt{(2k+1)/4\pi}. \tag{1.9}$$

The $\ell^4(m)$ norm is of this order of of magnitude for points x of distance $O(1/k)$ from the poles where $\phi = 0$ or π, but in order to obtain (1.8) much better estimates are needed. We shall obtain such an improvement, which turns out to be sharp, by using (1.7) and well known asymptotics for the kernel of the projection onto the spherical harmonics of degree k, $\mathcal{H}_k$. Thus, we are very much using here special properties of S^2. Our results can be thought of as a natural analog for S^2 of Zygmund's [27] theorem for the two-torus $\mathbb{T}^2$, which says that the eigenfunctions of its Laplace-Beltrami operator have uniformly bounded L^4-norms. As we pointed out before, this is far from true on S^2, but in an averaged sense it is almost true since the average L^4-norms just grow like powers of logs of the eigenvalues.

For general Riemannian manifolds of dimension n, the local Weyl formula says that if N is large enough and fixed then

$$\left(\sum_{|\lambda_j - \lambda| \le N} |e_{\lambda_j}(x)|^2 \right)^{1/2} \approx \lambda^{(n-1)/2}. \tag{1.10}$$

It would be interesting to see to what extent there is an improvement in the general case when one replaces this ℓ^2-norm by ℓ^q norms with $q > 2$ and to what extent results of this type perhaps depend on properties of the geodesic flow starting at x. In a future work, we intend to carry out the analysis for round spheres of dimension $n \ge 3$ and certain surfaces of revolution. Understanding the case of general manifolds and to what extent these results might depend on x seems difficult at present. On the other hand, by using estimates like (1.3), one can see that for most points $x \in M$, once can improve on the trivial consequence of (1.10) that

$$\Bigl(\sum_{|\lambda_j-\lambda|\le N} |e_{\lambda_j}(x)|^q \Bigr)^{1/q} \lesssim \lambda^{(n-1)/2}.$$

For instance, if $q=4$ and $n=2$, then using (1.3) and Tchebyschev's inequality one sees that if $C<\infty$ is fixed then

$$\Bigl| \Bigl\{ x\in M :\, \bigl(\sum_{|\lambda_j-\lambda|\le N} |e_{\lambda_j}(x)|^4 \bigr)^{1/4} \ge C\lambda^{1/2} \Bigr\} \Bigr| = O(\lambda^{-1/2}),$$

which, not surprisingly, is exactly the size of the sets on which the highest weight spherical harmonics are concentrated.

This paper is organized as follows. In the next section we shall present the proof of (1.6). Then we shall turn our attention to the sphere S^2 and prove the much stronger bounds (1.8) for S^2.

2. L^4 norms of generic eigenfunctions.

In this section we shall establish (1.6). Specifically, we shall prove the following

Theorem 2.1. *Let (M,g) be a two-dimensional compact Riemannian manifold. If $\Phi_t : S^*M \to S^*M$ is geodesic flow on the cosphere bundle, assume that the set*

$$\mathcal{P} = \{(x,\xi)\in S^*M : \Phi_t(x,\xi)=(x,\xi),\ \textit{some } t>0\} \tag{2.1}$$

has measure zero in S^*M with respect to the volume element. Then if e_{λ_j} is an orthonormal basis of eigenfunctions, $-\Delta e_{\lambda_j} = \lambda_j^2$, with $\lambda_1 \le \lambda_2 \le \dots$ there is a subsequence of eigenvalues λ_{j_k} satisfying

$$\lim_{\lambda\to\infty} \frac{\#\{\lambda_{j_k}\le\lambda\}}{N(\lambda)} = 1, \tag{2.2}$$

so that

$$\|e_{\lambda_{j_k}}\|_{L^4(M)} = o(\lambda_{j_k}^{1/8}). \tag{2.3}$$

To prove this we shall use an estimate from [20] and arguments from [5] and [26]. The estimate from [20] says given (M,g) as above there is a uniform constant C so that if $-\Delta e_\lambda = \lambda^2 e_\lambda$ and $N=1,2,3,\dots$ then

$$\begin{aligned}\int_M |e_\lambda(x)|^4\, dV \le CN^{-1/2}\lambda^{1/2}\|e_\lambda\|^4_{L^2(M)} \\ + CN\lambda^{1/2}\|e_\lambda\|_{L^2(M)} \Bigl[\sup_{\gamma\in\Pi} \int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |e_\lambda(x)|^2\, dV \Bigr] + C\|e_\lambda\|^4_{L^2(M)}.\end{aligned}$$

Here $dV = dV_g$ is the volume element, Π is the space of all unit-length geodesics, and

$$\mathcal{T}_\varepsilon(\gamma) = \{y\in M :\, \mathrm{dist}_g(y,\gamma)\le\varepsilon\},$$

denotes an ε-tube about γ. By optimizing the choice of N, we see that the preceding inequality implies that

$$\|e_\lambda\|_{L^4(M)} \le C\lambda^{1/8}\|e_\lambda\|_{L^2(M)}^{5/6} \sup_{\gamma\in\Pi} \|e_\lambda\|_{L^2(\mathcal{T}_{\lambda^{-1/2}})}^{1/6} + C\|e_\lambda\|_{L^2(M)}. \tag{2.4}$$

In addition to this we require the following result which is a simple consequence of the local Weyl law (see [11]).

Lemma 2.2. *Let (M,g) be a compact Riemannian manifold and let $A \in \Psi^0_{cl}(M)$ be a classical pseudo-differential operator on M of order zero. Then if A_0 is the principal symbol of A,*

$$\sum_{\lambda_j\le\lambda}\int_M |Ae_{\lambda_j}(x)|^2 dV = (2\pi)^{-n}\lambda^n\int_{T^*B}|A_0(x,\xi)|dV d\xi + O(\lambda^{n-1}). \tag{2.5}$$

Here, $T^*B\subset T^*M$ is the ball bundle, $\{(x,\xi) : \sum_{jk} g^{jk}(x)\xi_j\xi_k \le 1\}$, where g^{jk} is the cometric, i.e., $(g^{jk}(x))^{-1} = (g_{jk}(x))$. Note that (2.5) with A being the identity operator is the sharp Weyl formula ([1], [12], [9]), and the proof of the more general case just follows from a straightforward modifications of that of this special case.

As a first step in the proof of Theorem 2.1, let us use some ideas from the proof of the Duistermaat-Guillemin theorem [6] (see also [8]). Given $(x,\xi)\in S^*M$ we define $L(x,\xi)$ for $(x,\xi)\in\mathcal{P}$ to be the minimal $t>0$ so that $\Phi_t(x,\xi)=(x,\xi)$ and we define $L(x,\xi)$ to be $+\infty$ if $(x,\xi)\notin\mathcal{P}$, where $\mathcal{P}$ is as in (2.1). Then $L(x,\xi)$ is clearly a lower semicontinuous function on S^*M. As a result, since we are assuming that $\mathcal{P}$ has measure zero, it follows that for a given $T>0$

$$\mathcal{P}_T = \{(x,\xi)\in S^*M : L(x,\xi)\le T\}$$

is a closed subset of S^*M which is of measure zero since $\mathcal{P}_T\subset\mathcal{P}$. Therefore, given $\varepsilon>0$, we can find a pseudodifferential operator $b\in\Psi^0_{cl}(M)$ whose principal symbol satisfies $0\le b_0(x,\xi)\le 1$, $b_0(x,\xi)=1$ for $(x,\xi)\in\mathcal{N}(\mathcal{P}_T)$, where $\mathcal{N}(\mathcal{P}_T)$ is a neighborhood of Π in S^*M and

$$\int_{B^*M}|b_0(x,\xi)|^2 dV d\xi < \varepsilon/3.$$

By Lemma 2.2, we conclude from this that

$$\frac{1}{N(\lambda)}\sum_{\lambda_j\le\lambda}\int_M |be_{\lambda_j}|^2\,dV < \varepsilon/3 + O_b(\lambda^{-1}), \tag{2.6}$$

since we are assuming that $|T^*B|=1$.

If we let $B = Id - b \in \Psi^0_{cl}(M)$ then we claim that there is a uniform constant C, which is independent of ε and T above so that

$$\sup_{\gamma\in\Pi}\int_{\mathcal{T}_{\lambda_j^{-1/2}}(\gamma)} |Be_{\lambda_j}|^2\,dV \le C/T + C'_{B,T}\lambda^{-1/2}. \tag{2.7}$$

If T is chosen large enough so that $C/T<\varepsilon/3$, the preceding inequalities imply that there is an $\lambda_0=\lambda_0(\varepsilon)$ so that

$$\frac{1}{N(\lambda)}\sum_{\lambda_j\le\lambda}\sup_{\gamma\in\Pi}\int_{\mathcal{T}_{\lambda_j^{-1/2}}(\gamma)} |e_{\lambda_j}(x)|^2\,dV < \varepsilon, \quad \text{if } \lambda>\lambda_0. \tag{2.8}$$

As we shall see, this and (2.4) immediately yield Theorem 2.1.

Our main estimate (2.7) would follow from showing that there is a constant C as above so that

$$\sup_{\gamma\in\Pi}\int_\gamma |Be_\lambda|^2\,ds \le CT^{-1}\lambda^{1/2}+C'_{T,B}. \tag{2.9}$$

Here ds is the geodesic arclength measure on γ. Estimate (2.9) yields (2.10) due to the simple fact that if $f\ge 0$ then for any $\gamma_0\in\Pi$

$$\int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma_0)} f\,dV \le C\lambda^{-1/2}\sup_{\gamma\in\Pi}\int_\gamma f\,ds$$

for a uniform constant C since $\mathcal{T}_{\lambda^{-1/2}}(\gamma_0)$ is a tube of width $\lambda^{-1/2}$ about γ_0. [1] Let Π_{cl} be the set of unit geodesics that are part of a periodic geodesic. In [20] one of us showed that

$$\int_\gamma |e_\lambda|^2\,ds = o(\lambda^{1/2}),\quad \text{if } \gamma\in\Pi\backslash\Pi_{cl},$$

which was an o-improvement of the restriction bounds in [4]. The proof of (2.9) is an adaptation of the one used to establish this result.

To prove (2.9), let us fix a real-valued even function $\chi\in\mathcal{S}(\mathbb{R})$ with $\chi(0)=1$ and $\hat\chi(t)=0$, $|t|>1/4$, where $\hat\chi$ denotes the Fourier transform of χ. We then have that

$$\chi(T(P-\lambda))e_\lambda = e_\lambda$$

if $P=\sqrt{-\Delta_g}$. Therefore, in order to prove (2.9), it suffices to show that

$$\int_\gamma |B\chi(T(P-\lambda))f|^2\,ds \le CT^{-1}\lambda^{1/2}\|f\|^2_{L^2(M)}+C'_{T,B}\|f\|^2_{L^2(M)},\quad \gamma\in\Pi, \tag{2.10}$$

where C (but not $C'_{T,B}$) is a uniform constant independent of T and B. We shall assume in what follows that T is fixed but large, in particular $T>10$.

Note that Π is compact. Therefore, in order to prove (2.10), it suffices to show that given $\gamma_0\in\Pi$ there is a neighborhood $\mathcal{N}(\gamma_0)$ of γ_0 in Π on which the analog of (2.10) holds with constants independent of $\gamma\in\mathcal{N}(\gamma_0)$. Different arguments are needed for the cases where γ_0 is or is not part of a periodic geodesic of period $\le T$, where T is as above.

Given $\gamma\in\Pi$ we let $T^*\gamma\subset T^*M$ and $S^*\gamma\subset S^*M$ be the cotangent and unit cotangent bundles over γ, respectively. Thus, if $(x,\xi)\in T^*\gamma$ then $\xi_\sharp$ is a tangent vector to γ at x if $T^*M\ni\xi\to\xi_\sharp\in TM$ is the standard musical isomorphism, which, in local coordinates, sends $\xi=(\xi_1,\xi_2)\in T^*_xM$ to $\xi_\sharp=(\xi^1_\sharp,\xi^2_\sharp)$ with $\xi^j_\sharp=\sum_k g^{jk}(x)\xi_k$. Note that if $\gamma\in\Pi_{cl}$ then $L(x,\xi)\equiv t(\gamma)<\infty$ for $(x,\xi)\in S^*\gamma$. With this in mind, we shall let $\Pi_{cl}(T)$ denote those $\gamma\in\Pi_{cl}$ for which $L(x,\xi)\le T$ if $(x,\xi)\in S^*\gamma$.

[1] Note that, in $\mathbb{R}^2$, the integral of $f\ge 0$ over an $1\times\lambda^{-1/2}$ rectangle is dominated by $\lambda^{-1/2}$ times the supremum of integrals over the line segments in the rectangle that are parallel to the center segment, and a similar argument works for the above tubes if one uses Fermi normal coordinates about a geodesic which intersects γ_0 orthogonally.

Let us first see that a stronger version of (2.9) must be valid whenever $\gamma \in \Pi_{cl}(T)$. We first note that if $A \in \Psi^0_{cl}(M)$ then

$$A\chi(T(P-\lambda))f(x) = T^{-1}\int \hat{\chi}(t/T)\, e^{-i\lambda t}\Big(Ae^{itP}f\Big)(x)\, dt, \tag{2.11}$$

and recall that because of the support properties of $\hat{\chi}(t)$, the integral vanishes when $|t| \geq T/2$. The operator

$$f \to \Big(Ae^{itP}f\Big)(x)$$

is a Fourier integral operator with wave front set

$$\big\{ (x,t,\xi,\tau,y,-\eta) :\ \Phi_t(x,\xi) = (y,\eta),\ \pm\tau = p(x,\xi),\ (x,\xi) \in \text{supp } A(x,\xi) \big\}, \tag{2.12}$$

where $p(x,\xi)$ is the principal symbol of $P = \sqrt{-\Delta_g}$ and $A(x,\xi)$ is the symbol of A. If R_γ denotes the restriction to γ then we are really concerned with the operator

$$f \to R_\gamma A e^{itP} f. \tag{2.13}$$

Regarded as an operator from $C^\infty(M) \to C^\infty(\gamma_0 \times [-T/2, T/2])$, if supp $A(x,\xi) \cap S^*\gamma = \emptyset$, this is a Fourier integral operator of order zero which is locally a canonical graph [2]. If $\gamma = \gamma_0 \in \Pi_{cl}(T)$ and we take $A = B$, where B is as above, then this is automatically the case since $B = Id - b$ and b has a symbol which equals one in a neighborhood of $S^*\gamma_0$ if $\gamma_0 \in \Pi_{cl}(T)$. Therefore, by Hörmander's [10] L^2-estimates for nondegenerate Fourier integral operators we have

$$\int_{-T/4}^{T/4}\int_{\gamma_0} |Be^{itP}f|^2\, ds dt \leq C\|f\|^2_{L^2(M)}.$$

The constant C here of course depends on T and γ_0 (with its main dependence being on dist$(S^*\gamma_0$, supp $B(x,\xi))$. Since the Fourier integral (2.13) with $A = B$ will also be nondegenerate if γ is close to γ_0, we conclude that whenever $\gamma_0 \in \Pi_{cl}(T)$, there must be a neighborhood $\mathcal{N}(\gamma_0)$ in Π and a constant $C_{\gamma_0,B,T}$ so that

$$\int_{-T/4}^{T/4}\int_{\gamma} |Be^{itP}f|^2\, ds dt \leq C_{\gamma_0,B,T}\|f\|^2_{L^2(M)}, \quad \gamma \in \mathcal{N}(\gamma_0).$$

If we use the Schwarz inequality and (2.11) we conclude from this that

$$\int_\gamma |B\chi(T(P-\lambda))f|^2\, ds \leq TC_{\gamma_0,B,T}\|f\|^2_{L^2(M)}, \quad \gamma \in \mathcal{N}(\gamma_0), \tag{2.14}$$

which is stronger than (2.10) for these γ.

Let us now see that we also have favorable bounds on $\Pi\backslash\Pi_{cl}(T)$. If we fix a γ_0 in this set and choose a $C \in \Psi^0_{cl}(M)$ whose symbol vanishes on a conic neighborhood of $T^*\gamma_0$ then by the above arguments there must be a conic neighborhood of γ_0 on which we have the analog of (2.14) when B is replaced by $C \circ B$. This fact is independent of whether or not γ_0 is periodic. It is just our earlier observation that (2.13) is a nondegenerate Fourier integral operator when the symbol of A vanishes in a conic neighborhood of $T^*\gamma_0$.

[2]Since, for fixed t, $e^{itP} : C^\infty(M) \to C^\infty(M)$ is a nondegenerate Fourier integral operator, one needs only to check this assertion for $t = 0$, in which case it is an easy calculation using any parametrix for the half-wave operator.

Thus, in order to show that we have uniform bounds as in (2.10) on a neighborhood of such a $\gamma_0 \in \Pi\backslash\Pi_{cl}(T)$, it is enough to show that if $A \in \Psi^0_{cl}(M)$ has a symbol supported in a small neighborhood of $T^*\gamma_0$ then we have

$$\int_\gamma |A\chi(T(P-\lambda))f|^2\, ds \le CT^{-1}\lambda^{1/2}\|f\|^2_{L^2(M)} + C'_{T,A,\gamma_0}\|f\|^2_{L^2(M)} \tag{2.15}$$

for every $\gamma \in \Pi$.

Note that for every $x \in \gamma_0$, $T_x\gamma_0$ is one-dimensional and if $\xi \in T^*_x\gamma$ then $-\xi \in T^*_x\gamma_0$, since $\pm\xi_\sharp \in T_x\gamma$ are the corresponding tangent vectors to γ_0 at x pointing in opposite directions. Thus, $T^*\gamma_0$ naturally splits into two components, which we shall denote by $T^*_\pm\gamma_0$, and in order to prove (2.15), it suffices to show that the estimate holds if the symbol of A is supported in a small neighborhood of one of them, say, $T^*_+\gamma_0$, since the same argument will apply to $T^*_-\gamma_0$.

We shall assume in what follows that the injectivity radius of (M,g) is 10 or more. If not than we can subdivide γ into a finite number of segments of length smaller than one tenth of the injectivity radius and use the argument that follows to prove the analog of (2.15) for each of these, which in turn yields (2.15) for all of γ.

Let $Sf = A\chi(T(P-\lambda))f|_\gamma$ then we wish to show that

$$\left(\|S\|_{L^2(M)\to L^2(\gamma)}\right)^2 \le CT^{-1}\lambda^{1/2} + C_{T,A,\gamma_0}.$$

This is equivalent to saying that the dual operator $S^* : L^2(\gamma) \to L^2(M)$ with the same norm, and since

$$\|S^*g\|^2_{L^2(M)} = \int_\gamma SS^*g\,\overline{g}\,ds \le \|SS^*g\|_{L^2(\gamma)}\|g\|_{L^2(\gamma)},$$

we would be done if we could show that

$$\|SS^*g\|_{L^2(\gamma)} \le \Big(CT^{-1}\lambda^{1/2} + C_{T,A,\gamma_0}\Big)\|g\|_{L^2(\gamma)}. \tag{2.16}$$

But the kernel of SS^* is $K(\gamma(s),\gamma(s'))$ where $\gamma(s)$ parameterizes γ by arclength and $K(x,y)$, $x,y \in M$ is the kernel of the operator $A \circ \rho(T(P-\lambda)) \circ A^*$ with $\rho = (\chi(\tau))^2$ being the square of χ. Its Fourier transform $\hat\rho$ is the convolution of $\hat\chi$ with itself and thus $\hat\rho(t) = 0$, $|t| \ge 1/2$. Consequently, we can write

$$A \circ \rho(T(P-\lambda)) \circ A^* = T^{-1}\int_{-T/2}^{T/2} \hat\rho(t/T)\, e^{-it\lambda}\Big(A \circ e^{itP} \circ A^*\Big)\,dt. \tag{2.17}$$

The wave front set of the kernel of

$$A \circ e^{itP} \circ A^*$$

regarded as an operator from $C^\infty(M)$ to $C^\infty(M\times\mathbb{R})$ is contained in

$$\{(x,t,\xi,\tau;y,-\eta) : \Phi_t(x,\xi) = (y,\eta), \tau = \pm p(y,\eta),\ (x,\xi),(y,\eta) \in \operatorname{supp} A\}. \tag{2.18}$$

Our assumption that $\gamma_0 \notin \Pi_{cl}(T)$ implies that if $(x,\xi) \in S^*\gamma_0$ then $\{\Phi_t(x,\xi) : 1 \le |t| \le T\}$ must be a closed subset of S^*M which is disjoint from $\{(x,\xi)\}$. If we assume also that (x,ξ) and (y,η) belong to the same component $S^*_+\gamma_0$ of $S^*\gamma_0$ then we have that

$\Phi_{t_0}(x,\xi)=(y,\eta)$ for some $|t_0|\le 1$, and therefore $\Phi_t(x,\xi)\neq(y,\eta)$, for $|t|\in[2,T-1]$ since if $\Phi_t(x,\xi)=(y,\eta)$ then we must also have $\Phi_{t-t_0}(x,\xi)=(x,\xi)$ for this t_0. Consequently,

$$\{(x,\xi,\Phi_t(x,\xi)):\,(x,\xi)\in S^*_+\gamma_0,\,2\le|t|\le T-1\}\cap S^*_+\gamma_0\times S^*_+\gamma_0=\emptyset,$$

and since both are compact subsets of $S^*M\times S^*M$, we deduce from (2.18) that if the symbol of A is supported in a small conic neighborhood of $S^*_+\gamma_0$, then $K(t,x,y)$ will be C^∞ when $|t|\in[2,T-1]$.

Therefore, for such A, if if $\alpha\in C_0^\infty(\mathbb{R})$ equals one if $|t|\le 3$ and zero for $|t|\ge 4$, the difference between the kernel $K(x,y)$ in (2.17) and

$$K_0(x,y)=T^{-1}\int\alpha(t)\hat\rho(t/T)e^{-it\lambda}\Bigl(A\circ e^{itP}\circ A^*\Bigr)(x,y)\,dt,$$

must be bounded, by a constant which is independent of x and y (but depends on T, γ_0 and A). Since we are assuming that the injectivity radius of (M,g) is 10 or more one can use the Hadamard parametrix construction for the wave equation and standard stationary phase arguments (cf. Chapter 5 in [19] or the proof of Lemma 4.1 in [4]) to see that for $x,y\in M$ we have

$$|K_0(x,y)|\le CT^{-1}\lambda^{1/2}\,(\,\mathrm{dist}_g(x,y)\,)^{-1/2}+C_A.$$

Since this kernel restricted to $\gamma\times\gamma$ gives rise to an integral operator satisfying the estimates in (2.16), we conclude that we also have uniform bounds of the form (2.10), when A is as above.

This completes the proof that the analog of (2.10) holds for all γ in some neighborhood of γ_0 when $\gamma_0\in\mathit{\Pi}\backslash\mathit{\Pi}_{cl}(T)$.

Combining what we have done for $\mathit{\Pi}_{cl}(T)$ and $\mathit{\Pi}\backslash\mathit{\Pi}_{cl}(T)$, since $\mathit{\Pi}$ is compact, we conclude that (2.10) must be valid with uniform constants for every $\gamma\in\mathit{\Pi}$. This completes the proof of (2.9) and hence (2.8). Since the latter holds for all $\varepsilon>0$, we conclude from (2.4) that

$$\limsup_{\lambda\to\infty}\frac{1}{N(\lambda)}\sum_{\lambda_j\le\lambda}\bigl(\,\lambda_j^{-1/8}\|e_{\lambda_j}\|_{L^4(M)}\,\bigr)^{12}=0. \tag{2.19}$$

We can now finish the proof of Theorem 2.1 using a counting argument from [5] and [26]. If $S\subset\{\lambda_j\}$, we define its density to be

$$D(S)=\liminf_{\lambda\to\infty}\frac{\#\{\lambda_j\in S:\,\lambda_j\le\lambda\}}{N(\lambda)}.$$

Then if we use (2.19) we conclude that for every $n=2,3,\dots$ we can find a subset S_n of the eigenvalues $\{\lambda_j\}$ so that

$$D(S_n)\ge 1-\frac1n\;\text{ and }\;\lambda_j^{-1/8}\|e_{\lambda_j}\|_{L^4(M)}\le\frac1n,\;\lambda_j\in S_n.$$

Using this we conclude that there must be a set $S_\infty=\{\lambda_{j_k}\}\subset\{\lambda_j\}$ of density 1 so that

$$\limsup_{k\to\infty}\lambda_{j_k}^{-1/8}\|e_{\lambda_{j_k}}\|_{L^4(M)}=0.$$

Indeed, by the above, we can choose increasing $N_\nu\in\mathbb{N}$, $\nu=2,3,\dots$ so that

$$\#\{\lambda_j\in S_\nu:\,\lambda_j\le\lambda\}/N(\lambda)\ge 1-2/\nu,\quad\forall\,\lambda\ge N_{n-1}.$$

Consequently,

$$S_\infty = \bigcup_{\nu=2}^{\infty} S_\nu \cap \{\lambda_j \le N_\nu\}$$

will have the desired properties. □

3. **Average L^4 norms of spherical harmonics.**

In this section we shall prove (1.8):

Theorem 3.1. *Let $\{Y_m^k(x)\}_{m=-k}^k$, $x = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$, be the orthonormal basis of spherical harmonics of degree k defined in* (1.7). *Then there is a uniform constant C so that*

$$\frac{1}{2k+1}\sum_{m=-k}^{k}\int_{S^2}|Y_m^k|^4 dV \le C\log k, \quad k\ge 2. \tag{3.1}$$

Moreover, if $\mathbb{1} = (0,0,1)$ and if $r = \min_{\pm} dist(x, \pm\mathbb{1})$

$$\Big(\sum_{m=-k}^{k}|Y_m^k(x)|^4\Big)^{1/4} \le \begin{cases} Ck^{1/4}r^{-1/4}\big(\log(kr)\big)^{1/4}, & r\ge 2/k \\ \\ Ck^{1/2}, & r\le 2/k. \end{cases} \tag{3.2}$$

Clearly (3.2) implies (3.1), and so we just need to prove the second inequality in the theorem. To prove this, we first realize that by Parseval's theorem we have

$$2\pi\sum_{m=-k}^{k}|Y_m^k(x)|^4 = \int_0^{2\pi}\Big|\sum_{m=-k}^{k}|Y_m^k(x)|^2e^{im\theta}\Big|^2\,d\theta.$$

The kernel $\Pi_k(x,y)$ for projection on to spherical harmonics of degree k is given my the formula

$$\Pi_k(x,y) = \sum_{m=-k}^{k} Y_m^k(x)\overline{Y_m^k(y)},$$

which means that

$$2\pi\sum_{m=-k}^{k}|Y_m^k(x)|^4 = \int_0^{2\pi}|\Pi_k(x, e^{i\theta}x)|^2\,d\theta, \tag{3.3}$$

if we abuse notation a bit and let $e^{i\theta}x$ denote rotation of our vector $x = (x_1, x_2, x_3)$ by angle θ about the x_3-axis, i.e., $e^{i\theta}x = (\cos\theta x_1, \sin\theta x_2, x_3)$. Using the well known bounds (see [23], [17]) for Π_k,

$$|\Pi_k(x,y)| \le Ck^{1/2}(k^{-1} + \mathrm{dist}(x,y))^{-1/2}, \quad x,y\in S^2$$

we conclude that

$$\sum_{m=-k}^{k}|Y_m^k(x)|^4 \le Ck\int_0^{2\pi}\big(k^{-1} + \mathrm{dist}(x, e^{i\theta}x)\big)^{-1}\,d\theta. \tag{3.4}$$

Since $\mathrm{dist}(x, e^{i\theta}) \le Cr|\sin\theta|$, we conclude that the right side of (3.4) is $\le Ckr^{-1}\log(kr)$ if $r\ge 2/k$, and $\le Ck$ if $r\le 2/k$, for some uniform constant C when $k\ge 2$, which is just (3.2). □

We believe this estimate to be sharp, but defer the analysis to the future.

4. L^4 **norms of orthonormal bases of spherical harmonics.** The space of $\mathcal{O}NB_k$ of Hermitian orthonormal bases of $\mathcal{H}_k$ may be identified with the unitary group $U(2k+1)$. Any orthonormal basis $\Phi_k = \{\phi_1^k, \dots, \phi_{2k+1}^k\}$ can be obtained by applying an element $U \in U(2k+1)$ to the standard orthonormal basis $\{Y_m^k\}$. We then consider the functional on $\mathcal{O}NB_k$ defined by,

$$\Lambda_k^4(\Phi_k) = \sum_{m=1}^{2k+1} ||\phi_m^k||_{L^4}^4.$$

We have just proved that Λ_k^4 of the standard orthonormal basis is bounded by $k \log k$.

One may consider similar functionals on orthonormal bases of all eigenspaces with $\lambda_k \leq \lambda$, i.e. the direct sum $\bigoplus_{k \leq \lambda} \mathcal{H}_k$. The functional then has a natural generalization to any (M, g) and is essentially the one studied in previous sections.

5. **Random orthonormal bases of spherical harmonics.** We now consider the Λ_k^4 functional on a random basis of spherical harmonics. The question we pose is, what is the average value of the functional on random orthonormal bases? In [SZ] we considered problems of this kind for L^p norms of individual eigenfunctions, but there is a new dimension to the problem for random orthonormal bases. For background on random orthonormal bases we refer to [SZ].

We introduce the probability space $(\mathcal{O}NB, d\nu)$, where $\mathcal{O}NB$ is the infinite product of the sets, and $\nu = \prod_{k=1}^\infty \nu_k$, where ν_k is Haar probability measure on $\mathcal{O}NB_k$. A point of $\mathcal{O}NB$ is thus a sequence $\mathbf{\Phi} = \{(\phi_1^k, \dots, \phi_{2k+1}^k)\}_{k \geq 1}$ of orthonormal bases of $\mathcal{H}_k$.

The functionals we are interested in are

$$\Lambda_k^4(\mathbf{\Phi}) = \sum_{j=1}^{2k+1} \int_{S^2} \|\phi_j^k(x))\|^4 dV \ . \tag{5.5}$$

If we fix the standard ONB $Y_k = \{Y_m^k\}$ and express every other as $U_k Y^k$, then our functional is

$$\Lambda_k^4(U) = \sum_{j=1}^{2k+1} \int_{S^2} \|(UY^k)_j(z)\|^4 dV. \tag{5.6}$$

Let $d\mu_k$ be normalized Haar measure on $U(k)$ and let $\mathbf{E}_k$ denote expectation with respect to this measure. We conjecture that

$$\mathbf{E}_k \Lambda_k^4 = 2(2k+1), \tag{5.7}$$

i.e. the elements on average have L^4 norm equal to 2.

We briefly sketch the proof. We start from the fact that

$$\begin{aligned} \mathbf{E}\Lambda_k^4(U) \quad = \quad & \textstyle\sum_{j=1}^{2k+1} \sum_{m_1, m_2, m_3, m_4 = -k}^k \left(\int_{S^2} Y_{m_1}^k(x) \overline{Y}_{m_2}^k(x) Y_{m_3}^k(x) \overline{Y}_{m_4}^k(x) dV \right) \\ & \left(\int_{U(2k+1)} U_{m_1}^j \overline{U}_{m_2}^j U_{m_3}^j \overline{U}_{m_4}^j d\mu_k(U) \right). \end{aligned} \tag{5.8}$$

In fact, the sum over j is constant, so the right side equals $(2k+1)$ times

$$\sum_{m_1,m_2,m_3,m_4=-k}^{k} \left(\int_{S^2} Y^k_{m_1}(x)\overline{Y}^k_{m_2}(x) Y^k_{m_3}(x)\overline{Y}^k_{m_4}(x) dV \right) \times \left(\int_{U(2k+1)} U^1_{m_1}\overline{U}^1_{m_2} U^1_{m_3}\overline{U}^1_{m_4} d\mu_k(U) \right).$$

The integrals $\left(\int_{U(2k+1)} U^j_{m_1}\overline{U}^j_{m_2} U^j_{m_3}\overline{U}^j_{m_4} d\mu_k(U) \right)$ were first studied by Weingarten [W]. The main result is that the random variables $\{\sqrt{2k+1}U_{ij}\}$ behave asymptotically like independent complex Gaussian random variables of mean zero and variance one. Exact formulae are given in [CS], and the latter can be used to determine the asymptotics of our sums over $2k+1$ indices with different coefficients as $k \to \infty$. The dominant terms come from the cases where all m_j are equal (then one has the fourth moment of the Gaussian) or when the indices m_j are paired into couples (one barred and one unbarred). Then we have,

(5.9)

$$\begin{aligned} &(2k+1)^2 \mathbf{E}\Lambda^4_{Nk}(U) \\ &\sim (2k+1)\left(2\sum_{m=-k}^{k} \int_{S^2} |Y^k_m(x)|^4 dV + 2\sum_{m_1\neq m_2=-k}^{k} \left(\int_{S^2} |Y^k_{m_1}(x)|^2 |Y^k_{m_2}|^2 dV \right) \right) \\ &= 2(2k+1)\int_{S^2} |\Pi_k(x,x)|^2 dV = 2(2k+1)^3. \end{aligned}$$

Here, we use that the 4th moment of the complex normal Gaussian equals 2 and that there are two ways to pair the indices in the off diagonal terms. Dividing by $(2k+1)^2$ then implies the result.

6. **Other orthonormal bases.** Theorem 3.1 shows that the Λ^4_k functional on the standard basis $\{Y^k_m\}$ is only $\log k$ higher than for a random orthonormal basis. Hence it is doubtful that it does not maximize Λ^4_k. We do not know which ONB maximizes the functional, but in this section we suggest a possible construction of one which has a higher Λ^4_k value than the standard basis.

As mentioned above, the highest wight spherical harmonic Y^k_k has L^4_4 equal to $k^{1/2}$ on S^2, thus maximizing the norm functional. This suggests constructing orthonormal bases $\phi^k_{\gamma_j}$ consisting in part of highest weight spherical harmonics with respect to a well-separated set of closed geodesics γ_j. That is, for each closed geodesic γ, one introduces the subgroup G_γ of rotations fixing γ (as a set) and then constructs Y^k_m's with respect to this circle action.

Of course, the $\phi^k_{\gamma_j}$ are not orthogonal, and their inner products $\langle \phi^k_{\gamma_j}, \phi^k_{\gamma_i} \rangle$ depend on the angle $\vartheta_{j,k}$ between the geodesics γ_j, γ_i. To construct an orthonormal basis it would be necessary to apply Gram-Schmidt to such $\phi^k_{\gamma_j}$, and in the process one may destroy the high L^4 norms of the resulting eigenfunctions. The question is, how many $\phi^k_{\gamma_j}$ can be used in such a construction while preserving the high L^4 norms of these Gaussian beams?

The geodesics γ_j are points in the space $G(S^2, g_0)$ of geodesics of S^2. A well-separated set of $2k+1$ geodesics (i.e. a basis) would only have separation of order $k^{-\frac{1}{2}}$. To beat the bound $k \log k$ for the standard basis one it would suffice to construct a partial orthonormal basis containing $k^{1-\delta}$ roughly Gaussian beams with roughly $||\phi^k_{\gamma_j}||_4^4 \simeq \sqrt{k}$ and with $\delta < \frac{1}{2}$. One would then complete it with an arbitrary orthonormal basis of the ortho-complement of the span. It would be interesting to see how far separated the γ_j would need to be so that Gram-Schmidt would not destroy the bounds $||\phi^k_{\gamma_j}||_4^4 \simeq \sqrt{k}$ too much.

References

[1] V. I. Avakumovič, Über die Eigenfunktionen auf geshchlossenen Riemannschen Mannigfaltigkeiten. Math. Z. **65** (1956), 327–344.

[2] V. M. Babič and V. F. Lazutkin, *The eigenfunctions which are concentrated near a closed geodesic*, (Russian) Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov, (LOMI) **9** 1967 15–25.

[3] J. Bourgain, *Geodesic restrictions and L^p-estimates for eigenfunctions of Riemannian surfaces*, Linear and complex analysis, 27–35, Amer. Math. Soc. Tranl. Ser. 2, 226, Amer. Math. Soc., Providence, RI, 2009.

[4] N. Burq, P. Gérard and N. Tzvetkov: *Restriction of the Laplace-Beltrami eigenfunctions to submanifolds*, Duke Math. J. **138** (2007), 445–486.

[5] Y. Colin de Verdière, *Ergodicité et fonctions propres du laplacien*, Comm. Math. Phys. 102 (1985), 497–502.

[CS] B. Collins and P. Sniady, Integration with respect to the Haar measure on unitary, orthogonal and symplectic group, Comm. Math. Phys. 264 (2006), no. 3, 773795.

[6] J. J. Duistermaat and V. W. Guillemin, *The spectrum of positive elliptic operators and periodic bicharacteristics*, Invent. Math. **29** (1975), 39–79.

[7] A. Greenleaf and A. Seeger, *Fourier integrals with fold singularities*, J. Reine Angew. Math. **455** (1994), 35–56.

[8] V. Ivrii, *The second term of the spectral asymptotics for a Laplace-Beltrami operator on manifolds with boundary* (Russian) Funktsional. Anal. i Prilozhen. **14** (1980), 25–34.

[9] L. Hörmander, The spectral function of an elliptic operator. Acta Math. **121** (1968), 193–218.

[10] L. Hörmander: *Fourier integral operators. I*, Acta Math. **127** (1971), 79–183.

[11] L. Hörmander, *The Analysis of Linear Partial Differential Operators, Volume IV*, Springer-Verlag Berlin Heidelberg, 1983.

[12] B. M. Levitan, On the asymptoptic behavior of the spectral function of a self-adjoint differential equaiton of second order. Isv. Akad. Nauk SSSR Ser. Mat. **16** (1952), 325–352.

[13] J. V. Ralston, *On the construction of quasimodes associated with stable periodic orbits*, Comm. Math. Phys. **51** (1976), 219–242.

[14] A. Reznikov, *Norms of geodesic restrictions for eigenfunctions on hyperbolic surfaces and representation theory*, arXiv:math.AP/0403437.

[15] A. Schnirelman, *Ergodic properties of eigenfunctions*, Usp. Math. Nauk. **29** (1974), 181–182.

[SZ] B. Shiffman and S. Zelditch, Random polynomials of high degree and Levy concentration of measure. Volume in honor of Y. T. Siu, Asian J. Math. 7 (2003), no. 4, 627–646.

[16] H. Smith and C. D. Sogge, *On the L^p norm of spectral clusters for compact manifolds with boundary*, Acta Math. **198** (2007), 107–153.

[17] C. D. Sogge, *Oscillatory integrals and spherical harmonics*, Duke Math. J. **53** (1986), 43–65.

[18] C. D. Sogge, *Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds*, J. Funct. Anal. **77** (1988), 123–138.

[19] C. D. Sogge, *Fourier integrals in classical analysis*, Cambridge Tracts in Math., Cambridge Univ. Press, Cambridge, 1993.

[20] C. D. Sogge, *Kakeya-Nikodym averages and L^p-norms of eigenfunctions*, to appear Tohoku Math. J.

[21] C. D. Sogge, J. Toth and S. Zelditch, *About the blowup of quasimodes on Riemannian manifolds*, to appear, J. Geom. Anal.

[22] C. D. Sogge and S. Zelditch, *Riemannian manifolds with maximal eigenfunction growth*, Duke Math. J. **114** (2002), 387–437.

[23] G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI, 4th ed., 1974.
[24] J. VanderKam, *L^∞ norms and quantum ergodicity on the sphere*, Internat. Math. Res. Notices 1997, 329–347.
[25] D. Tataru, *On the regularity of boundary traces for the wave equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **26** (1998), 185–206.
[W] D. Weingarten, Asymptotic behavior of group integrals in the limit of infinite rank. J. Math. Phys. 19(5), 999–1001 (1978).
[26] S. Zelditch, *Uniform distribution of eigenfunctions on compact hyperbolic surfaces*, Duke Math. J. **55** (1987), 919–941.
[27] A. Zygmund, *On Fourier coefficients and transforms of two variables*, Studia Math. **50** (1974), 189–201.

Johns Hopkins University, Baltimore, MD

Northwestern University, Evanston, IL