Non-trivial Compositions of Differential Operations and Gateaux Directional Derivative

Branko Malešević, Ivana Jovović

Faculty of Electrical Engineering, University of Belgrade,
Bulevar kralja Aleksandra 73, 11000 Belgrade, Serbia

Abstract. This paper is devoted to the enumeration of non-trivial compositions of higher order of differential operations and Gateaux directional derivative in $\mathbb{R}^n$. We present recurrences for counting non-trivial compositions of higher order.

Key words: compositions of differential operations, Gateaux directional derivative, differential forms, exterior derivative, Hodge star operator, enumeration of graphs and maps

1. Non-trivial compositions of differential operations and Gateaux directional derivative of the space $\mathbb{R}^3$

In the three-dimensional Euclidean space $\mathbb{R}^3$ we consider following sets

$$\mathrm{A}_0 = \{ f : \mathbb{R}^3 \longrightarrow \mathbb{R} \,|\, f \in C^{\infty}(\mathbb{R}^3) \} \quad \text{and} \quad \mathrm{A}_1 = \{ \vec{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \,|\, \vec{f} \in \vec{C}^{\infty}(\mathbb{R}^3) \}.$$

Gradient, curl, divergence and Gateaux directional derivative in direction $\vec{e}$, for a unit vector $\vec{e} = (e_1, e_2, e_3) \in \mathbb{R}^3$, are defined in terms of partial derivative operators as follows

E-mails: Branko Malešević <malesevic@etf.rs>, Ivana Jovović <ivana@etf.rs>
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$$\operatorname{grad} f = \nabla_1 f = \frac{\partial f}{\partial x_1}\,\vec{i} + \frac{\partial f}{\partial x_2}\,\vec{j} + \frac{\partial f}{\partial x_3}\,\vec{k}, \quad \nabla_1 : \mathrm{A}_0 \longrightarrow \mathrm{A}_1,$$

$$\operatorname{curl} \vec{f} = \nabla_2 \vec{f} = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)\vec{i} + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right)\vec{j} + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)\vec{k}, \quad \nabla_2 : \mathrm{A}_1 \longrightarrow \mathrm{A}_1,$$

$$\operatorname{div} \vec{f} = \nabla_3 \vec{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}, \quad \nabla_3 : \mathrm{A}_1 \longrightarrow \mathrm{A}_0,$$

$$\operatorname{dir}_{\vec{e}} f = \nabla_0 f = \nabla_1 f \cdot \vec{e} = \frac{\partial f}{\partial x_1}\, e_1 + \frac{\partial f}{\partial x_2}\, e_2 + \frac{\partial f}{\partial x_3}\, e_3, \quad \nabla_0 : \mathrm{A}_0 \longrightarrow \mathrm{A}_0.$$

Let $\mathcal{A}_3 = \{\nabla_1, \nabla_2, \nabla_3\}$ and $\mathcal{B}_3 = \{\nabla_0, \nabla_1, \nabla_2, \nabla_3\}$. The number of compositions of the k^{th} order over the set $\mathcal{A}_3$ is $\mathtt{f}(k) = F_{k+3}$, where F_k is the k^{th} Fibonacci number (see [2] for more details). A composition of differential operations that is not 0 or $\vec{0}$ is called non-trivial. The number of non-trivial compositions of the k^{th} order over the set $\mathcal{A}_3$ is $\mathtt{g}(k) = 3$ (see for instance [1]). In paper [4], it is shown that the number of compositions of the k^{th} order over the set $\mathcal{B}_3$ is $\mathtt{f}^{\mathrm{G}}(k) = 2^{k+1}$. According to the above results, it is natural to try to calculate the number of non-trivial compositions of differential operations from the set $\mathcal{B}_3$. Straightforward verification shows that all compositions of the second order over $\mathcal{B}_3$ are

$$\begin{aligned}
&\operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} f = \nabla_0 \circ \nabla_0 f = \nabla_1\big(\nabla_1 f \cdot \vec{e}\big) \cdot \vec{e},\\
&\operatorname{grad} \operatorname{dir}_{\vec{e}} f = \nabla_1 \circ \nabla_0 f = \nabla_1\big(\nabla_1 f \cdot \vec{e}\big),\\
&\Delta f = \operatorname{div} \operatorname{grad} f = \nabla_3 \circ \nabla_1 f,\\
&\operatorname{curl} \operatorname{curl} \vec{f} = \nabla_2 \circ \nabla_2 \vec{f},\\
&\operatorname{dir}_{\vec{e}} \operatorname{div} \vec{f} = \nabla_0 \circ \nabla_3 \vec{f} = \big(\nabla_1 \circ \nabla_3 \vec{f}\big) \cdot \vec{e},\\
&\operatorname{grad} \operatorname{div} \vec{f} = \nabla_1 \circ \nabla_3 \vec{f},\\
&\operatorname{curl} \operatorname{grad} f = \nabla_2 \circ \nabla_1 f = \vec{0},\\
&\operatorname{div} \operatorname{curl} \vec{f} = \nabla_3 \circ \nabla_2 \vec{f} = 0,
\end{aligned}$$

and that only the last two are trivial. This fact leads us to use the following procedure for determining the number of non-trivial composition over the set $\mathcal{B}_3$. Let us define a binary relation σ on the set $\mathcal{B}_3$ as follows: $\nabla_i \,\sigma\, \nabla_j$ iff the composition $\nabla_j \circ \nabla_i$ is non-trivial.

Relation σ induces Cayley table

σ	∇_0	∇_1	∇_2	∇_3
∇_0	1	1	0	0
∇_1	0	0	0	1
∇_2	0	0	1	0
∇_3	1	1	0	0

For convenience, we extend set $\mathcal{B}_3$ with nowhere-defined function ∇_{-1}, whose domain and range are empty sets, and establish $\nabla_{-1}\,\sigma\,\nabla_i$ for $i = 0, 1, 2, 3$. Thus, graph G of the relation σ is rooted tree with the root ∇_{-1}

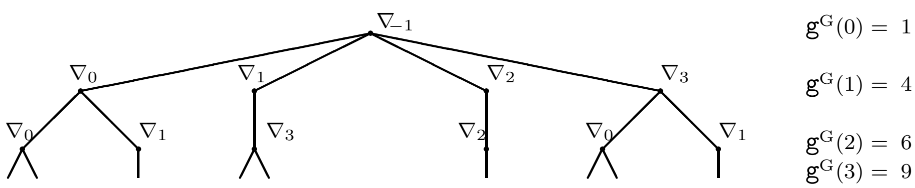


Fig. 1

Here we would like to point out that the child of ∇_i is ∇_j if composition $\nabla_j \circ \nabla_i$ is non-trivial. For any non-trivial composition $\nabla_{i_k} \circ \ldots \circ \nabla_{i_1}$ there is a unique path in the tree (Fig. 1), such that the level of vertex ∇_{i_j} is j, $1 \leq j \leq k$. Let $\mathsf{g}^{\mathrm{G}}(k)$ be the number of non-trivial compositions of the k^{th} order of functions from $\mathcal{B}_3$ and let $\mathsf{g}_i^{\mathrm{G}}(k)$ be the number of non-trivial compositions of the k^{th} order starting with ∇_i. Then we have

$$\mathsf{g}^{\mathrm{G}}(k) = \mathsf{g}_0^{\mathrm{G}}(k) + \mathsf{g}_1^{\mathrm{G}}(k) + \mathsf{g}_2^{\mathrm{G}}(k) + \mathsf{g}_3^{\mathrm{G}}(k).$$

According to the graph G we obtain the equalities

$$\mathsf{g}_0^{\mathrm{G}}(k) = \mathsf{g}_0^{\mathrm{G}}(k-1) + \mathsf{g}_1^{\mathrm{G}}(k-1), \quad \mathsf{g}_1^{\mathrm{G}}(k) = \mathsf{g}_3^{\mathrm{G}}(k-1),$$

$$\mathsf{g}_2^{\mathrm{G}}(k) = \mathsf{g}_2^{\mathrm{G}}(k-1), \quad \mathsf{g}_3^{\mathrm{G}}(k) = \mathsf{g}_0^{\mathrm{G}}(k-1) + \mathsf{g}_1^{\mathrm{G}}(k-1).$$

Since the only child of ∇_2 is ∇_2, we can deduce

$$\mathsf{g}_2^{\mathrm{G}}(k) = \mathsf{g}_2^{\mathrm{G}}(k-1) = \mathsf{g}_2^{\mathrm{G}}(k-2) = \ldots = \mathsf{g}_2^{\mathrm{G}}(1) = 1.$$

Putting things together we obtain the recurrence for $\mathtt{g}^{\mathrm{G}}(k)$:

$$\begin{aligned}\mathtt{g}^{\mathrm{G}}(k) &= \mathtt{g}_0^{\mathrm{G}}(k) + \mathtt{g}_1^{\mathrm{G}}(k) + \mathtt{g}_2^{\mathrm{G}}(k) + \mathtt{g}_3^{\mathrm{G}}(k)\\ &= \big(\mathtt{g}_0^{\mathrm{G}}(k-1)+\mathtt{g}_1^{\mathrm{G}}(k-1)\big)+\mathtt{g}_3^{\mathrm{G}}(k-1)+\mathtt{g}_2^{\mathrm{G}}(k-1)+\big(\mathtt{g}_0^{\mathrm{G}}(k-1)+\mathtt{g}_1^{\mathrm{G}}(k-1)\big)\\ &= \mathtt{g}^{\mathrm{G}}(k-1)+\mathtt{g}_0^{\mathrm{G}}(k-1)+\mathtt{g}_1^{\mathrm{G}}(k-1)\\ &= \mathtt{g}^{\mathrm{G}}(k-1)+\big(\mathtt{g}_0^{\mathrm{G}}(k-2)+\mathtt{g}_1^{\mathrm{G}}(k-2)\big)+\mathtt{g}_3^{\mathrm{G}}(k-2)+\mathtt{g}_2^{\mathrm{G}}(k-2)-\mathtt{g}_2(k-2)\\ &= \mathtt{g}^{\mathrm{G}}(k-1)+\mathtt{g}^{\mathrm{G}}(k-2)-1.\end{aligned}$$

Substituting $\mathtt{t}(k) = \mathtt{g}^{\mathrm{G}}(k) - 1$ into previous formula we obtain recurrence $\mathtt{t}(k) = \mathtt{t}(k-1) + \mathtt{t}(k-2)$. With initial conditions $\mathtt{g}^{\mathrm{G}}(1) = 4, \mathtt{g}^{\mathrm{G}}(2) = 6$, respectively $\mathtt{t}(1) = 3, \mathtt{t}(2) = 5$, we conclude that $\mathtt{g}^{\mathrm{G}}(k) = F_{k+3} + 1$.

2. Non-trivial compositions of differential operations and Gateaux directional derivative of the space $\mathbb{R}^n$

We start this section by recalling some definitions of multivariable calculus.

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and consider set of smooth functions $\mathrm{A}_0 = \{f : \mathbb{R}^n \longrightarrow \mathbb{R} \mid f \in C^\infty(\mathbb{R}^n)\}$. The set of all differential k-forms on $\mathbb{R}^n$ is a free A_0-module of rank $\binom{n}{k}$ with the standard basis $\{dx_I = dx_{i_1} \dots dx_{i_k} \mid 1 \le i_1 < \dots < i_k \le n\}$, denoted $\Omega^k(\mathbb{R}^n)$. Differential k-form ω can be written uniquely as $\omega = \sum_{I \in \mathcal{I}(k,n)} \omega_I dx_I$, where $\omega_I \in \mathrm{A}_0$ and $\mathcal{I}(k,n)$ is the set of multi-indices $I = (i_1, \dots, i_k)$, $1 \le i_1 < \dots < i_k \le n$. The complement of the multi-index I is the multi-index $J = (j_1, \dots, j_{n-k}) \in \mathcal{I}(n-k,n)$, $1 \le j_1 < \dots < j_{n-k} \le n$, where components j_p are elements of the set $\{1, \dots, n\} \backslash \{i_1, \dots, i_k\}$. We have $dx_I dx_J = \sigma(I) dx_1 \dots dx_n$, where $\sigma(I)$ is the signature of the permutation $(i_1, \dots, i_k, j_1, \dots, j_{n-k})$.
Note that $\sigma(J) = (-1)^{k(n-k)}\sigma(I)$. With the notions mentioned above we define $\star_k(dx_I) = \sigma(I) dx_J$. The map $\star_k : \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^{n-k}(\mathbb{R}^n)$ defined by $\star_k(\omega) = \sum_{I \in \mathcal{I}(k,n)} \omega_I \star_k (dx_I)$ is Hodge star operator and it provides natural isomorphism between $\Omega^k(\mathbb{R}^n)$ and $\Omega^{n-k}(\mathbb{R}^n)$. The Hodge star operator twice applied to a differential k-form yields $\star_{n-k}(\star_k \omega) = (-1)^{k(n-k)}\omega$. So for the inverse of the operator $\star_k$ holds ${\star_k}^{-1}(\psi) = (-1)^{k(n-k)} \star_{n-k} (\psi)$, where $\psi \in \Omega^{n-k}(\mathbb{R}^n)$.
A differential 0-form is a function $f(x_1, x_2, \dots, x_n) \in \mathrm{A}_0$. We define df to be the differential 1-form $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$. Given a differential k-form

$\sum_{I\in\mathcal{I}(k,n)} \omega_I dx_I$, the exterior derivative $d_k\omega$ is differential $(k+1)$-form $d_k\omega = \sum_{I\in\mathcal{I}(k,n)} d\omega_I dx_I$. The exterior derivative d_k is a linear map from k-forms to $(k+1)$-forms which obeys Leibnitz rule: If ω is a k-form and ψ is a l-form, then $d_{k+l}(\varphi\psi) = d_k\omega\,\psi + (-1)^k\varphi\, d_l\psi$. The exterior derivative has a property that $d_{k+1}(d_k\omega) = 0$ for any differential k-form ω.
Consider sets of functions

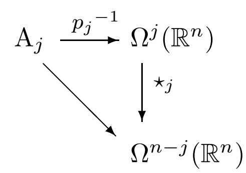
$$\mathrm{A}_k = \{\vec{f}\colon\mathbb{R}^n \longrightarrow \mathbb{R}^{\binom{n}{k}} \mid \vec{f}\in\vec{C}^\infty(\mathbb{R}^n)\},$$

$0 \le k \le m$, $m = [n/2]$. Let $p_k : \Omega^k(\mathbb{R}^n) \to \mathrm{A}_k$ be presentation of differential forms in coordinate notation. Let us define functions φ_i $(0 \le i \le m)$ and φ_{n-j} $(0 \le j < n-m)$ as follows

$$\varphi_i = p_i : \Omega^i(\mathbb{R}^n) \to \mathrm{A}_i$$

and

$$\varphi_{n-j} = p_j \star_j^{-1} : \Omega^{n-j}(\mathbb{R}^n) \to \mathrm{A}_j.$$



Then the combination of the Hodge star operator and the exterior derivative generates differential operations $\nabla_k = \varphi_k\, d_{k-1}\, \varphi_{k-1}^{-1}$, $1 \le k \le n$, in n-dimensional space $\mathbb{R}^n$ (see [3]).

$\mathcal{A}_n$ $(n{=}2m)$:

$$\begin{aligned}
&\nabla_1 = p_1\, d_0\, p_0^{-1} : A_0 \to A_1\\
&\nabla_2 = p_2\, d_1\, p_1^{-1} : A_1 \to A_2\\
&\vdots\\
&\nabla_i = p_i\, d_{i-1}\, p_{i-1}^{-1} : A_{i-1} \to A_i\\
&\vdots\\
&\nabla_m = p_m\, d_{m-1}\, p_{m-1}^{-1} : A_{m-1} \to A_m\\
&\nabla_{m+1} = p_{m-1} \star_{m-1}^{-1} d_m\, p_m^{-1} : A_m \to A_{m-1}\\
&\nabla_{m+2} = p_{m-2} \star_{m-2}^{-1} d_{m+1} \star_{m-1} p_{m-1}^{-1} : A_{m-1} \to A_{m-2}\\
&\vdots\\
&\nabla_{n-j} = p_j \star_j^{-1} d_{n-(j+1)} \star_{j+1} p_{j+1}^{-1} : A_{j+1} \to A_j\\
&\vdots\\
&\nabla_{n-1} = p_1 \star_1^{-1} d_{n-2} \star_2 p_2^{-1} : A_2 \to A_1\\
&\nabla_n = p_0 \star_0^{-1} d_{n-1} \star_1 p_1^{-1} : A_1 \to A_0,
\end{aligned}$$

$\mathcal{A}_n$ $(n{=}2m+1)$:

$$\begin{aligned}
&\nabla_1 = p_1\, d_0\, p_0^{-1} : A_0 \to A_1\\
&\nabla_2 = p_2\, d_1\, p_1^{-1} : A_1 \to A_2\\
&\vdots\\
&\nabla_i = p_i\, d_{i-1}\, p_{i-1}^{-1} : A_{i-1} \to A_i\\
&\vdots\\
&\nabla_m = p_m\, d_{m-1}\, p_{m-1}^{-1} : A_{m-1} \to A_m\\
&\nabla_{m+1} = p_m \star_m^{-1} d_m\, p_m^{-1} : A_m \to A_m\\
&\nabla_{m+2} = p_{m-1} \star_{m-1}^{-1} d_{m+1} \star_m p_m^{-1} : A_m \to A_{m-1}\\
&\nabla_{m+3} = p_{m-2} \star_{m-2}^{-1} d_{m+2} \star_{m-1} p_{m-1}^{-1} : A_{m-1} \to A_{m-2}\\
&\vdots\\
&\nabla_{n-j} = p_j \star_j^{-1} d_{n-(j+1)} \star_{j+1} p_{j+1}^{-1} : A_{j+1} \to A_j\\
&\vdots\\
&\nabla_{n-1} = p_1 \star_1^{-1} d_{n-2} \star_2 p_2^{-1} : A_2 \to A_1\\
&\nabla_n = p_0 \star_0^{-1} d_{n-1} \star_1 p_1^{-1} : A_1 \to A_0.
\end{aligned}$$

List of differential operations in $\mathbb{R}^n$

Formulae for the number of compositions of differential operations from the set $\mathcal{A}_n$ and corresponding recurrences are given by Malešević in [2].
The following theorem provides a natural characterization of the number of non-trivial compositions of differential operations from the set $\mathcal{A}_n$. For the proof we refer reader to [2].

Theorem 2.1. *All non-trivial compositions of differential operations from the set $\mathcal{A}_n$ are given in the following form*

$$(\nabla_i \circ)\nabla_{n+1-i} \circ \nabla_i \circ \cdots \circ \nabla_{n+1-i} \circ \nabla_i$$

where $2i,\ 2(i-1) \neq n,\ 1 \leq i \leq n$*. Term in bracket is included in if the number of differential operations is odd and left out otherwise.*

Theorem 2.2. *Let* $\mathsf{g}(k)$ *be the number of non-trivial compositions of the* k^{th} *order of differential operations from the set* $\mathcal{A}_n$*. Then we have*

$$\mathsf{g}(k) = \begin{cases} n & : \ 2 \nmid n, \\ n & : \ 2 \,|\, n \ , \ k = 1, \\ n-1 & : \ 2 \,|\, n \ , \ k = 2, \\ n-2 & : \ 2 \,|\, n \ , \ k > 2. \end{cases}$$

The Hodge dual to the exterior derivative $d_k : \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^{k+1}(\mathbb{R}^n)$ is codifferential δ_{k-1}, a linear map $\delta_{k-1} : \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^{k-1}(\mathbb{R}^n)$, which is a generalization of the divergence, defined by

$$\delta_{k-1} = (-1)^{n(k-1)+1} \star_{n-(k-1)} d_{n-k} \star_k = (-1)^k \star_{k-1}^{-1} d_{n-k} \star_k .$$

Note that $\nabla_{n-j} = (-1)^{j+1} p_j \, \delta_j \, p_{j+1}^{-1}$, for $0 \leq j < n-m-1$. The codifferential can be coupled with the exterior derivative to construct the Hodge Laplacian, also known as the Laplace-de Rham operator, $\Delta_k : \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^k(\mathbb{R}^n)$, a harmonic generalization of Laplace differential operator, given by $\Delta_0 = \delta_0 d_0$ and $\Delta_k = \delta_k d_k + d_{k-1}\delta_{k-1}$, for $1 \leq k \leq m$. The operator Δ_0 is actually the negative of the Laplace-Beltrami (scalar) operator. A k-form ω is called harmonic if $\Delta_k(\omega) = 0$. We say that $\vec{f} \in A_k$ is a harmonic function if $\omega = {p_k}^{-1}(\vec{f})$ is harmonic k-form. If $k \geq 1$ harmonic function $\vec{f}$ is also called harmonic field. The best general reference here is [5].

Theorem 2.3. *Let* $\vec{f} \in A_k,\ 0 \leq k \leq m$*, be a harmonic function. Then all compositions of order higher than two of differential operations from the set* $\mathcal{A}_n$*,* $n = 2m+1$*, acting on* $\vec{f}$ *are trivial.*

Proof. The proof will be divided into three parts. Let us first examine case $k = 0$. Since $f \in A_0$ is harmonic function we have $\Delta_0 f = \delta_0 d_0 f = 0$, hence $\nabla_n \circ \nabla_1 f = 0$ and finally $(\nabla_1 \circ)\nabla_n \circ \nabla_1 \circ \cdots \circ \nabla_n \circ \nabla_1 f = 0$. So we have proved that all compositions acting on harmonic function f are trivial.
Our next concern will be the behavior of harmonic fields $\vec{f} \in A_k, 1 \leq k < m$. According to Theorem 2.1 we only need to show that compositions of the following form

$$(\nabla_{k+1}\circ)\nabla_{n-k} \circ \nabla_{k+1} \circ \cdots \circ \nabla_{n-k} \circ \nabla_{k+1}\vec{f},$$
$$(\nabla_{n-(k-1)}\circ)\nabla_{k} \circ \nabla_{n-(k-1)} \circ \cdots \circ \nabla_{k} \circ \nabla_{n-(k-1)}\vec{f}$$

are trivial. Since $\vec{f}$ is harmonic field, we have $(\delta_k d_k + d_{k-1}\delta_{k-1})({p_k}^{-1}\vec{f}) = \vec{0}$. From this we see that $\nabla_{n-k} \circ \nabla_{k+1}\vec{f} = \nabla_k \circ \nabla_{n-(k-1)}\vec{f}$, which implies $\nabla_{k+1} \circ (\nabla_{n-k} \circ \nabla_{k+1})\vec{f} = \nabla_{k+1} \circ (\nabla_k \circ \nabla_{n-(k-1)})\vec{f} = (\nabla_{k+1} \circ \nabla_k) \circ \nabla_{n-(k-1)}\vec{f}$. The previous composition is trivial, because $\nabla_{k+1} \circ \nabla_k\, \vec{g} = p_{k+1}\, d_k\, d_{k-1} p_{k-1}^{-1}\, \vec{g} = 0$, for any function $\vec{g} \in A_{k-1}$. In the same manner we can see that composition $\nabla_{n-(k-1)} \circ \nabla_k \circ \nabla_{n-(k-1)}\vec{f}$ is trivial. Therefore all compositions of order higher than two acting on harmonic field $\vec{f}$ are trivial.
It remains to prove the claim for $k = m$. Observe that $\nabla_{m+1} \circ \nabla_{m+1} = p_m \star_m^{-1} d_m \star_m^{-1} d_m\, p_m^{-1} = p_m \star_m^{-1} d_m \star_{m+1} d_m\, p_m^{-1} = (-1)^{m+1}\, p_m\, \delta_m\, d_m\, p_m^{-1}$. The equality $\Delta_m \vec{f} = \delta_m d_m\, \vec{f} + d_{m-1}\delta_{m-1}\, \vec{f} = \vec{0}$ yields $\nabla_{m+1} \circ \nabla_{m+1} = \nabla_m \circ \nabla_{m+2}$. Similarly, we can show that all compositions of order higher than two acting on harmonic field $\vec{f} \in A_m$ are trivial. $\square$

The same conclusion can be drawn for compositions over the set $\mathcal{A}_n$, $n = 2m$, which act on a harmonic function $\vec{f} \in A_k,\ 0 \leq k < m - 1$.

Remark. Some analogous problems can be considered also in Discrete Exterior Calculus [6] (see also [7, 8]) and Combinatorial Hodge Theory [9].

Let $f \in \mathrm{A}_0$ be a scalar function and $\vec{e} = (e_1, \ldots, e_n) \in \mathbb{R}^n$ be a unit vector. The Gateaux directional derivative in direction $\vec{e}$ is defined by

$$\mathrm{dir}_{\vec{e}}\, f = \nabla_0 f = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\, e_k : A_0 \longrightarrow A_0.$$

Let us extend the set of differential operations $\mathcal{A}_n = \{\nabla_1, \ldots, \nabla_n\}$ with Gateaux directional derivative to the set $\mathcal{B}_n = \mathcal{A}_n \cup \{\nabla_0\} = \{\nabla_0, \nabla_1, \ldots, \nabla_n\}$. Recurrences for counting compositions of differential operations from the set $\mathcal{B}_n$ can be found in [4]. For an odd n we can obtain a simpler recurrence $\mathtt{f}^{\mathrm{G}}(k) = 2\mathtt{f}^{\mathrm{G}}(k-1)$, which enable us to find easily explicit formula for the number of compositions of the k^{th} order over the set $\mathcal{B}_n$ $\mathtt{f}^{\mathrm{G}}(k) = 2^{k-1}(n+1)$.

The number of non-trivial compositions of differential operations from the set $\mathcal{B}_n$ is determined by the binary relation ν, defined by:

$\nabla_i \nu \nabla_j$ iff $(i\!=\!0 \wedge j\!=\!0) \vee (i\!=\!0 \wedge j\!=\!1) \vee (i\!=\!n \wedge j\!=\!0) \vee (i\!+\!j\!=\!n\!+\!1 \wedge 2i \neq n)$.

Applying Theorem 2.2 to cases $i = 2, \ldots, n-1$ we conclude that the number of non-trivial compositions of the k^{th} order starting with $\nabla_2, \ldots, \nabla_{n-1}$ can be express by formula

$$\mathtt{j}(k) = \mathtt{g}(k) - 2 = \begin{cases} n-2 & : \ 2 \nmid n, \\ n-2 & : \ 2 \,|\, n \ , \ k = 1, \\ n-3 & : \ 2 \,|\, n \ , \ k = 2, \\ n-4 & : \ 2 \,|\, n \ , \ k > 2. \end{cases}$$

Let $\mathtt{g}^{\mathrm{G}}(k)$ be the number of non-trivial compositions of the k^{th} order of operations from the set $\mathcal{B}_n$. Let $\mathtt{g}_0^{\mathrm{G}}(k)$, $\mathtt{g}_1^{\mathrm{G}}(k)$ and $\mathtt{g}_n^{\mathrm{G}}(k)$ be the numbers of non-trivial the k^{th} order compositions starting with ∇_0, ∇_1 and ∇_n, respectively. Then we have

$$\mathtt{g}^{\mathrm{G}}(k) = \mathtt{g}_0^{\mathrm{G}}(k) + \mathtt{g}_1^{\mathrm{G}}(k) + \mathtt{j}(k) + \mathtt{g}_n^{\mathrm{G}}(k).$$

Denote $\widetilde{\mathtt{g}}^{\mathrm{G}}(k) = \mathtt{g}_0^{\mathrm{G}}(k) + \mathtt{g}_1^{\mathrm{G}}(k) + \mathtt{g}_n^{\mathrm{G}}(k)$. Hence, the following recurrences are true

$$\mathtt{g}_0^{\mathrm{G}}(k) = \mathtt{g}_0^{\mathrm{G}}(k-1) + \mathtt{g}_1^{\mathrm{G}}(k-1), \mathtt{g}_1^{\mathrm{G}}(k) = \mathtt{g}_n^{\mathrm{G}}(k-1), \mathtt{g}_n^{\mathrm{G}}(k) = \mathtt{g}_0^{\mathrm{G}}(k-1) + \mathtt{g}_1^{\mathrm{G}}(k-1).$$

Thus, the recurrence for $\widetilde{\mathtt{g}}^{\mathrm{G}}(k)$ is of the form

$$\begin{aligned} \widetilde{\mathtt{g}}^{\mathrm{G}}(k) &= \mathtt{g}_0^{\mathrm{G}}(k) + \mathtt{g}_1^{\mathrm{G}}(k) + \mathtt{g}_n^{\mathrm{G}}(k) \\ &= \big(\mathtt{g}_0^{\mathrm{G}}(k-1) + \mathtt{g}_1^{\mathrm{G}}(k-1)\big) + \mathtt{g}_n^{\mathrm{G}}(k-1) + \big(\mathtt{g}_0^{\mathrm{G}}(k-1) + \mathtt{g}_1^{\mathrm{G}}(k-1)\big) \\ &= \widetilde{\mathtt{g}}^{\mathrm{G}}(k-1) + \mathtt{g}_0^{\mathrm{G}}(k-1) + \mathtt{g}_1^{\mathrm{G}}(k-1) \\ &= \widetilde{\mathtt{g}}^{\mathrm{G}}(k-1) + \big(\mathtt{g}_0^{\mathrm{G}}(k-2) + \mathtt{g}_1^{\mathrm{G}}(k-2)\big) + \mathtt{g}_n^{\mathrm{G}}(k-2) \\ &= \widetilde{\mathtt{g}}^{\mathrm{G}}(k-1) + \widetilde{\mathtt{g}}^{\mathrm{G}}(k-2). \end{aligned}$$

With initial conditions $\widetilde{\mathtt{g}}^{\mathrm{G}}(1) = 3$, $\widetilde{\mathtt{g}}^{\mathrm{G}}(2) = 5$ we deduce $\widetilde{\mathtt{g}}^{\mathrm{G}}(k) = F_{k+3}$. Therefore, we have proved following theorem.

Theorem 2.4. *The number of non-trivial compositions of the k^{th} order over the set $\mathcal{B}_n$ is*

$$\mathtt{g}^{\mathrm{G}}(k) = F_{k+3} + \mathtt{j}(k) = \begin{cases} F_{k+3} + n - 2 & : \ 2 \nmid n, \\ \quad n+1 & : \ 2 \,|\, n \ , \ k = 1, \\ \quad n+2 & : \ 2 \,|\, n \ , \ k = 2, \\ F_{k+3} + n - 4 & : \ 2 \,|\, n \ , \ k > 2. \end{cases}$$

The values of function $\mathsf{g}^{\mathrm{G}}(k)$ are given in [10] as the following sequences $A001611$ $(n = 3)$, $A000045$ $(n = 4)$, $A157726$ $(n = 5)$, $A157725$ $(n = 6)$, $A157729$ $(n = 7)$, $A157727$ $(n = 8)$.

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