

Thermodynamics of quantum measurements

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Quantum measurement of a system can change its mean energy, as well as entropy. A selective measurement (classical or quantum) can be used as a “Maxwell’s demon” to power a single-temperature heat engine, by decreasing the entropy. Quantum mechanically, so can a *non-selective* measurement, despite *increasing* the entropy of a thermal state. The maximal amount of work extractable following the measurement is given by the change in free energy: $W_{max}^{(non-)\text{sel.}} = \Delta E_{meas} - T_{Bath} \Delta S_{meas}^{(non-)\text{sel.}}$. This follows from the “generalized 2nd law for nonequilibrium initial state” [Hasegawa et. al, *PLA* (2010)], of which an elementary reduction to the standard law is given here. It is shown that $W_{max}^{\text{sel.}} - W_{max}^{\text{non-sel.}}$ equals the work required to reset the memory of the measuring device, and that no such resetting is needed in the non-selective case. Consequently, a single-bath engine powered by either kind of measurement works at a net loss of $T_{Bath} \Delta S_{meas}^{\text{non-sel}}$ per cycle. By replacing the measurement by a reversible “premeasurement” and allowing a work source to couple to the system *and memory*, the cycle can be made completely reversible.

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The thermodynamics of (selective) measurements links information theory and physics: information on the micro-state of a system can be used to extract more work than would otherwise be possible, and the overall cost of acquiring, storing, deleting and using it needs to be included in a general statement of the second law[1–5]. In quantum mechanics, a non-trivial measurement necessarily disturbs the system, and can in general involve energy exchange between the measuring device and the measured system. Therefore, in stark contrast with the classical situation, even a generic ideal *non-selective* measurement, i.e., a measurement the outcome of which is not available (e.g., a measurement performed by a third party, such as the environment) will still have thermodynamic consequences. Whereas the work enabled by selective measurements is a consequences of the increase in information on the state (decrease of Von-Neumann entropy), a non-selective projective measurement is always entropy non-decreasing, and unless it leaves the state unchanged, strictly increasing[6]. Since a Gibbs state is the unique maximal Von-Neumann entropy state for a given mean energy[7], this also implies that the such a measurement performed on a system initially in thermal equilibrium must also increase its mean energy. Thus, a non-selective measurement of an observable which does not commute with the thermal state (i.e., one which does not commute with the Hamiltonian) will take the system out of thermal equilibrium with a bath, and into a higher-energy non-equilibrium state. This has been studied in great detail in the case where the system itself consists of a smaller “system” and a non-Markovian bath, where it was shown that even a purely dephasing non-selective measurement that affects only the “system”-bath correlation has thermodynamic consequences[8, 9]. Such a state can be harnessed to perform work on its way back to equilibrium with the bath. Can we determine

how much work? Thermodynamics tells us the maximal extractable work between two given states is that performed in a thermodynamically reversible process (i.e., a process without production of entropy, but possibly exchange of it). Were the post-measurement state of the system, ρ' , a thermal state (for some temperature), we could use textbook processes to “close the cycle”, but it is not. Nonetheless, for any temperature, there exists a Hamiltonian with respect to which ρ' is a Gibbs state, as is easy to check[19]. In particular, let H' be the Hamiltonian such that $\rho' = Z^{-1} e^{-\beta H'}$. Then performing a sudden change[20] of the actual Hamiltonian of the system $H \mapsto H'$ brings us to a new equilibrium state with the same bath! This non-standard (and rapid) step is reversible, even microscopically (unitary). The equally (macroscopically) reversible isothermal step of changing the Hamiltonian back $H' \mapsto H$ quasi-statically, always in contact with the bath will then return the system to the equilibrium state, closing the cycle. The whole cycle then consists of an irreversible “stroke” (the measurement) taking ρ to ρ' and a pair of reversible “strokes” bringing it back. The energy *cost* of the measurement is

$$\Delta E_{meas} = \langle H \rangle_{\rho'} - \langle H \rangle_{\rho}, \quad (1)$$

and ignorance of the outcome of the measurement (equivalently, the fact that no sorting into subensembles corresponding to the measurement outcomes is carried out) is responsible for the increase in the VN entropy:

$$\Delta S_{meas} = S(\rho') - S(\rho). \quad (2)$$

In the first return stroke, the change in Hamiltonian implies the work performed *by the system* on the constraints (may be negative[21]) is :

$$W_{sudden} = \langle H \rangle_{\rho'} - \langle H' \rangle_{\rho'}, \quad (3)$$

and the entropy is unchanged. For the last stroke we have:

$$\Delta E_{isotherm} = \langle H' \rangle_{\rho'} - \langle H \rangle_{\rho}, \quad (4)$$

$$\Delta S_{isotherm} = -\Delta S_{meas}. \quad (5)$$

Furthermore, the work performed during this last, isothermal, “stroke” is just the change in free energy[10]:

$$W_{isotherm} = \Delta E_{isotherm} - T\Delta S_{isotherm}. \quad (6)$$

Finally, combining these equations one gets an expression for $W_{extracted} = W_{sudden} + W_{isotherm}$ in terms of ΔE_{meas} and $T\Delta S_{meas}$:

$$W_{extracted} = \Delta E_{meas} - T\Delta S_{meas}. \quad (7)$$

Which is the work extracted in a maximally reversible process (reversible apart from the given measurement “stroke”), and thus the maximal possible. More generally, this proof implies the “generalized 2nd law for nonequilibrium initial state”[11]:

$$W_{ext} \geq -\Delta F_{system} \quad (8)$$

We have implicitly assumed that all quasi-static processes can be performed “adiabatically” (in the quantum, not thermodynamic sense!). However, an isothermal process involving a level crossing could introduce additional entropy production [12]. Such level crossings can be averted by physically exchanging the corresponding eigenstates of the initial state ρ , using an nearly instantaneous unitary right after the measurement.

Since for a non-selective measurement, $\Delta S_{meas} > 0$, as noted above, we see that the irreversibility of the non-selective measurement is responsible for a work deficit (compare[10, 13]) of:

$$\Delta W_{lost} = T\Delta S_{meas}. \quad (9)$$

For the selective case, let us denote the measurement basis by $\{|j\rangle\}_{j=1,\dots,n}$, the corresponding values of the energy, entropy and extractable work by: E_j, S_j, W_j^{ext} ; and define $p_j = \langle j|\rho|j\rangle$, $\bar{W} = \sum p_j W_j$. Then the maximal mean amount of extractable work will be \bar{W} :

$$\begin{aligned} W_{ext}^{sel} &= \sum p_j \Delta E_j - T \sum p_j \Delta S_j \\ &= \Delta E - T\Delta S^{non-sel} + T \left[\Delta S^{non-sel} - \sum p_j \Delta S_j \right] \end{aligned} \quad (10)$$

where $\Delta S^{non-sel} = S(\rho') - S(\rho)$, $\Delta S_j = S(|j\rangle\langle j|) - S(\rho) = -S(\rho)$. The last term on the RHS of eq.10 is equal to the Shannon entropy $H(\{p_j\})$, by the orthogonality of the $|j\rangle$ s [14], leading to:

$$W_{ext}^{sel} = W_{ext}^{non-sel} + T_{bath} H(\{p_j\}). \quad (12)$$

This is also the increase in the entropy of the auxiliary system serving as (faithful) memory register: if the memory is initially in a pure state, its entropy in the final state is $H(\{p_j\})$, and for an initial pure state the entropy increase similarly to that of the system. The minimal work required to reset the memory to its initial state, using only the single temperature bath and a reversible work source, is again given by the reduction in its free energy, $T_{bath} H(\{p_j\})$. This leads to the same work deficit for a complete cycle, as in the non-selective case:

$$\Delta W_{lost}^{sel} = T\Delta S_{meas}^{non-sel}. \quad (13)$$

This begs the question whether Eq. 9 needs to be corrected by deducting the work cost of resetting the memory. The answer is negative, and follows from a remarkable property of quantum measurements which may be called the “Bad apple theorem”: interaction with a single qubit can dephase any number of qubits. More precisely, consider a single “register” qubit, labeled by the index θ , and (non-degenerate) observables $\{O_i\}_{i=1,\dots,n}$ to be measured (non-selectively) on qubits $1, \dots, n$. If $n = 1$ we can just follow the standard (and essentially only) procedure for a *selective* measurement. Namely, take the register qubit to be in a known initial pure state, and then perform a conditional flip to the orthogonal state (conditioned on a projection onto a particular eigenstate of O_1, P_1). Choosing our bases such that the initial state of the register is $|0\rangle \equiv |\sigma_z^0 = -1\rangle$, and $[O_1, \sigma_z^\perp] = 0$, the desired measurement is given by the CNOT operation:

$$U_{meas}^{(10)} = P_+^1 \sigma_x^0 + P_-^1 I^0, \quad (14)$$

where $P_\pm = |\sigma_z = \pm\rangle\langle\sigma_z = \pm|$ (Note that $U^\dagger = U$). Viewed as a non-selective measurement, it corresponds to a dephasing operation (corresponding to the σ_z^\perp basis):

$$\rho_1 \mapsto \text{Tr}_0 \{U \rho_1 \otimes \rho_0 U\} = P_+ \rho_1 P_+ + P_- \rho_1 P_-. \quad (15)$$

The remarkable thing is, that the effect the CNOT operation has on the state ρ_1 is exactly the same if we replace the initial state of the register, ρ_0 by *any* state that commutes with σ_z (i.e, a “totally dephased” register), even the fully mixed state! On the other hand, it is not hard to verify that the CNOT operation transforms such states to others with this same property, so our register is never used up! In the extreme case where the register is totally mixed, $\rho_0 = \frac{1}{2}I$, the CNOT leaves the two qubits in a product state with the register qubit unchanged! The same register qubit can therefore be used to measure as many qubits and as often as we like. Thus, one “bad” (dephased) qubit can spoil the bunch. It is important to note that the register does not store information corresponding to n selective measurements, so it could not be used to circumvent Bennett’s argument regarding Szilard’s engine the Landauer principle. On the other hand, generically each dephasing operation corresponds to work

being performed on the system and register, and this is the full cost of performing the measurements.

If the measurement basis coincides with the eigenbasis of the initial state ρ , $\Delta S_{meas}^{non-sel} = 0$ and both cycles are reversible (the non-selective “cycle” consists of doing exactly nothing in this case). Such a measurement corresponds to the classical case. The reduced engine efficiency for other measurements is a quantum feature related to the constraint that except for using the classical result of the measurement (in the non-selective case) all operations are restricted to the system [13, 15], and to the fact that a measurement was implicitly defined to be irreversible. However, the scheme described above will have the same effect, even if we substitute a *reversible* “premeasurement” of the state of the system for the measurement (i.e., a Von Neumann-type measurement, the outcome of which is registered on a quantum detector). Although this does not satisfy Bohr’s requirement that a measurement must include an act of irreversible amplification ([16], cited in [17]), the effect on the reduced state of the system alone is identical. Such a measurement/premeasurement is effected by a unitary operation on the system to be measured and an auxiliary “quantum memory” system, which is reversible[22]. The whole process now consists of ostensibly reversible steps provided we include the quantum memory in our accounting. The latter does not return to its original state, though, so the cycle is incomplete. One may wonder whether after completing the whole scheme, one can use the information stored in the quantum memory to recoup the work due to its neglect (the work deficit) either directly, or by a “delayed choice quantum erasure”[18]. In fact, the neg-entropy residing in the unread measurement outcome will be gone by the end of the process, as far as macroscopic thermodynamics is concerned, because the “reversible” isothermal step will have entangled the state of the memory with that of the bath. In other words, the irreversibility of the measurement will now be attributed to the subsequent interaction with bath. Can one circumvent this “quantum penalty”? If we are allowed to jointly manipulate the system and memory register, this is possible. Trivially, the unitary premeasurement can be undone at short times (compared to the time of relaxation due to the bath). More interestingly, the same effect can be achieved by an appropriate (slowly changing) Hamiltonian for the system and memory, to protect their correlations from thermal degradation (i.e., H' now includes an interaction term with the memory).

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- [19] For any inverse temperature, $\beta = (kT)^{-1}$, and density operator, ρ , there is a Hamiltonian (i.e., a Hermitian operator with spectrum bound from below) such that ρ is the corresponding Gibbs state: $\rho = Z^{-1}e^{-\beta H}$ (with Z being the normalization to trace one). For positive temperatures (finite β) this H is unique up to an arbitrary additive constant and can be written explicitly in the eigenbasis of ρ : $H = \beta^{-1} \sum \log(p_i^{-1}) |i\rangle\langle i| + E_0$ (where $\rho = \sum p_i |i\rangle\langle i|$). For zero temperature, the restriction of H to the (possibly degenerate) ground state subspace is determined, and the rest is arbitrary, provided it commutes with this part and its spectrum lies higher than some finite gap above it.
- [20] For the present purpose, ‘sudden’ means much faster than the thermal relaxation time of the state due to its coupling to the bath, and to time scales associated with transitions due to the Hamiltonian (actually the family of Hamiltonians the system is subject to during the transition).
- [21] Clearly, the addition of a constant to H' leaves the corresponding Gibbs states unchanged, so the mean value of the energy, $\langle H' \rangle_{\rho'}$, can be made to have any desired value by appropriate choice of this constant. In particular, it may be convenient to choose it such that $\langle H' \rangle_{\rho'} = \langle H \rangle_{\rho'}$, making this “stroke” energy-conserving, as well as entropy-conserving. Alternately, a choice of $\langle H' \rangle_{\rho'} = \langle H \rangle_{\rho}$ may be useful in comparing with ref. [10]. In any event, we shall see that $\langle H' \rangle_{\rho'}$ will not appear in the final result.
- [22] The same memory register as before, the only difference being that no additional measurement of the state of the memory is assumed.