A characterization of compact operators via the non-connectedness of the attractors of a family of IFSs

by

ALEXANDRU MIHAIL (Bucharest) and
RADU MICULESCU (Bucharest)

Abstract. In this paper we present a result which establishes a connection between the theory of compact operators and the theory of iterated function systems. For a Banach space X, S and T bounded linear operators from X to X such that $\|S\|, \|T\| < 1$ and $w \in X$, let us consider the IFS $S_w = (X, f_1, f_2)$, where $f_1, f_2 : X \to X$ are given by $f_1(x) = S(x)$ and $f_2(x) = T(x) + w$, for all $x \in X$. On one hand we prove that if the operator S is compact, then there exists a family $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X such that $A_{\mathcal{S}_w}$ is not connected, for all $w \in H - \underset{n\in\mathbb{N}}{\cup} K_n$. One the other hand we prove that if H is an infinite dimensional Hilbert space, then a bounded linear operator $S : H \to H$ having the property that $\|S\| < 1$ is compact provided that for every bounded linear operator $T : H \to H$ such that $\|T\| < 1$ there exists a sequence $(K_{T,n})_n$ of compact subsets of H such that $A_{\mathcal{S}_w}$ is not connected for all $w \in H - \underset{n}{\cup} K_{T,n}$. Consequently, given an infinite dimensional Hilbert space H, there exists a complete characterization of the compactness of an operator $S : H \to H$ by means of the non-connectedness of the attractors of a family of IFSs related to the given operator.

1. Introduction. IFSs were introduced in their present form by John Hutchinson (see [9]) and popularized by Michael Barnsley (see [2]). They are one of the most common and most general ways to generate fractals. Although the fractals sets are defined by means of measure theory concepts (see [7]), they have very interesting topological properties. The connectivity of the attractor of an iterated function system has been studied, for example, in [14] (for the case of an iterated multifunction system) and in [6] (for the case of an infinite iterated function system).

It is well known the role of the compact operators theory in functional

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analysis and, in particular, in the theory of the integral equations. In this frame, a natural question is to provide equivalent characterizations for compact operators. Let us mention some results on this direction. A bounded operator T on a separable Hilbert space H is compact if and only if $\lim_{n\to\infty} < Te_n, e_n >= 0$ (or equivalently $\lim_{n\to\infty} Te_n = 0$), for each orthonormal basis $\{e_n\}$ for H (see [1], [8], [16] and [17]) if and only if every orthonormal basis $\{e_n\}$ for H has a rearrangement $\{e_{\sigma(n)}\}$ such that $\sum \frac{1}{n} \left\| Te_{\sigma(n)} \right\| < \infty$ (see [18]). In a more general framework, in [10] a characterization of the compact operators on a fixed Banach space in terms of a construction due to J.J.M. Chadwick and A.W. Wickstead (see [3]) is presented and in [11] a purely structural characterization of compact elements in a C^* algebra is given.

In contrast to the above mentioned characterizations of the compact operators which are confined to the framework of the functional analysis, in this paper we present such a characterization by means of the non-connectedness of the attractors of a family of IFSs related to the considered operator.

In this way we establish an unexpected connection between the theory of compact operators and the theory of iterated function systems.

2. Preliminary results. In this paper, for a function f and $n \in \mathbb{N}$, by $f^{[n]}$ we mean the composition of f by itself n times.

DEFINITION 2.1. Let (X, d) be a metric space. A function $f : X \to X$ is called a *contraction* in case there exists $\lambda \in (0, 1)$ such that

$$d(f(x), f(y)) \leq \lambda d(x, y),$$

for all $x, y \in X$.

THEOREM 2.2 (The Banach-Cacciopoli-Picard contraction principle). *If X is a complete metric space, then for each contraction $f : X \to X$ there exists a unique fixed point x^* of f.*

Moreover

$$x^* = \lim_{n\to\infty} f^{[n]}(x_0),$$

for each $x_0 \in X$.

NOTATION. Given a metric space (X, d), by $K(X)$ we denote the set of non-empty compact subsets of X.

DEFINITION 2.3. For a metric space (X, d), the function $h : K(X) \times K(X) \to [0, +\infty)$ defined by

$$h(A, B) = \max(d(A, B), d(B, A)) =$$

$$= \inf\{r \in [0, \infty) : A \subseteq B(B, r) \text{ and } B \subseteq B(A, r)\},$$

where

$$B(A, r) = \{x \in X : d(x, A) < r\}$$

and

$$d(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A}(\inf_{y \in B} d(x, y)),$$

turns out to be a metric which is called the *Hausdorff-Pompeiu metric*.

REMARK 2.4. The metric space $(K(X), h)$ is complete, provided that (X, d) is a complete metric space.

DEFINITION 2.5. Let (X, d) be a complete metric space. An *iterated function system* (for short an IFS) on X, denoted by $S = (X, (f_k)_{k \in \{1,2,...,n\}})$, consists of a finite family of contractions $(f_k)_{k \in \{1,2,...,n\}}$, $f_k : X \to X$.

THEOREM 2.6. *Given $\mathcal{S} = (X, (f_k)_{k \in \{1,2,...,n\}})$ an iterated function system on X, the function $F_{\mathcal{S}} : K(X) \to K(X)$ defined by*

$$F_{\mathcal{S}}(C) = \bigcup_{k=1}^{n} f_k(C),$$

for all $C \in K(X)$, which is called the set function associated to $\mathcal{S}$, turns out to be a contraction and its unique fixed point, denoted by $A_{\mathcal{S}}$, is called the attractor of the IFS $\mathcal{S}$.

REMARK 2.7. For each $i \in \{1, 2, ..., n\}$, the fixed point of f_i is an element of $A_{\mathcal{S}}$.

REMARK 2.8. If $A \in K(X)$ has the property that $F_{\mathcal{S}}(A) \subseteq A$, then $A_{\mathcal{S}} \subseteq A$.

`Proof`. The proof is similar to the one of Lemma 3.6 from [13]. □

DEFINITION 2.9. Let (X,d) be a metric space and $(A_i)_{i\in I}$ a family of nonempty subsets of X. The *family* $(A_i)_{i\in I}$ is said to be *connected* if for every $i,j\in I$, there exist $n\in\mathbb{N}$ and $\{i_1,i_2,...,i_n\}\subseteq I$ such that $i_1=i$, $i_n=j$ and $A_{i_k}\cap A_{i_{k+1}}\neq\emptyset$ for every $k\in\{1,2,..,n-1\}$.

THEOREM 2.10 (see [12], Theorem 1.6.2, page 33). *Given an IFS $\mathcal{S}=(X,(f_k)_{k\in\{1,2,...,n\}})$, where (X,d) is a complete metric space, the following statements are equivalent:*

1) the family $(f_i(A_{\mathcal{S}}))_{i\in\{1,2,...,n\}}$ is connected;

2) $A_{\mathcal{S}}$ is arcwise connected.

3) $A_{\mathcal{S}}$ is connected.

PROPOSITION 2.11. *For a given complete metric space (X,d), let us consider the IFSs $\mathcal{S}=(X,f_1,f_2)$ and $\mathcal{S}'=(X,f_1^{[m]},f_2)$, where $m\in\mathbb{N}$.*

If $A_{\mathcal{S}'}$ is connected, then $A_{\mathcal{S}}$ is connected.

`Proof`. Since $F_{\mathcal{S}'}(A_{\mathcal{S}})=f_1^{[m]}(A_{\mathcal{S}})\cup f_2(A_{\mathcal{S}})\subseteq A_{\mathcal{S}}$, we get (using Remark 2.8) $A_{\mathcal{S}'}\subseteq A_{\mathcal{S}}$ and hence $f_2(A_{\mathcal{S}'})\subseteq f_2(A_{\mathcal{S}})$. Because $f_1^{[m]}(A_{\mathcal{S}'})\subseteq f_1(A_{\mathcal{S}})$, it follows that $f_1^{[m]}(A_{\mathcal{S}'})\cap f_2(A_{\mathcal{S}'})\subseteq f_1(A_{\mathcal{S}})\cap f_2(A_{\mathcal{S}})$ (*). Since $A_{\mathcal{S}'}$ is connected, taking into account Theorem 2.10, we deduce that $f_1^{[m]}(A_{\mathcal{S}'})\cap f_2(A_{\mathcal{S}'})\neq\varnothing$, which, using $(*)$, implies that $f_1(A_{\mathcal{S}})\cap f_2(A_{\mathcal{S}})\neq\varnothing$. Then, using again Theorem 2.10, we infer that $A_{\mathcal{S}}$ is connected. □

PROPOSITION 2.12 (see [5], page 238, lines 11-12). *Assume that H is a Hilbert space. Let us consider a self-adjoint operator $N:H\to H$ and E its spectral decomposition. Then for each $\lambda\in\mathbb{R}$ we have*

$$NE((-\infty,\lambda))\leq\lambda E((-\infty,\lambda))$$

and

$$\lambda E((\lambda,\infty))\leq NE((\lambda,\infty)),$$

for all $\lambda\in\mathbb{R}$.

PROPOSITION 2.13 (see [5], page 226, Observation 7). *Assume that H is a Hilbert space. Let us consider two self-adjoint operators $N_1,N_2:H\to H$.*

If

$$0\leq N_1\leq N_2,$$

then

$$\|N_1\| \leq \|N_2\|.$$

PROPOSITION 2.14 (see [19], ex. 25, page 344). *Assume that H is a Hilbert space. Let us consider a normal operator $N : H \to H$, g a bounded Borel function on $\sigma(N)$ and $S = g(T)$. If E_N and E_S are the spectral decomposition of N and S, then*

$$E_S(\omega) = E_N(g^{-1}(\omega)),$$

for every Borel set $\omega \subseteq \sigma(S)$.

PROPOSITION 2.15 (see [4], Proposition 4.1, page 278). *Assume that H is a Hilbert space. Let us consider a normal operator $N : H \to H$ and E its spectral decomposition. Then N is compact if and only if $E(\{z \mid |z| > \varepsilon\})$ has finite rank, for every $\varepsilon > 0$.*

PROPOSITION 2.16. *Assume that H is a Hilbert space. Let us consider a bounded linear operator $A : H \to H$ which is invertible. Then $Id_H - A^*A$ is compact if and only if $Id_H - AA^*$ is compact.*

`Proof`. According to the well known polar decomposition theorem there exists an unitary operator $U : H \to H$ and a positive operator $P : H \to H$ such that $P^2 = A^*A$ and $A = UP$. Then

$$Id_H - AA^* = Id_H - UP(UP)^* = Id_H - UPP^*U^* = Id_H - UP^2U^* =$$

$$= UU^* - UP^2U^* = U(Id_H - P^2)U^* = U(Id_H - A^*A)U^*.$$

Hence $Id_H - AA^* = U(Id_H - A^*A)U^*$ and $Id_H - A^*A = U^*(Id_H - AA^*)U$. From the last two relations we obtain the conclusion. □

COROLLARY 2.17. *Assume that H is a Hilbert space. Let us consider a bounded linear operator $S : H \to H$ such that $\|S\| < 1$. Then $S + S^* - SS^*$ is compact if and only if $S + S^* - S^*S$ is compact.*

`Proof`. The operator $A = Id_H - S$ is invertible since $\|S\| < 1$. According to Proposition 2.16 $Id_H - A^*A$ is compact if and only if $Id_H - AA^*$ is compact i.e. $S + S^* - SS^*$ is compact if and only if $S + S^* - S^*S$ is compact. □

PROPOSITION 2.18 (see [19], ex. 14, page 324). *Assume that H is a Hilbert space and let us consider a bounded linear operator $S : H \to H$. If S^*S is a compact operator, then S is compact.*

3. A sufficient condition for the compactness of an operator. In this section, H is an infinite-dimensional Hilbert space. We shall use the notation Id_H for the function $Id_H : H \to H$, given by $Id_H(x) = x$, for all $x \in H$. If S and T are bounded linear operators from H to H such that $\|S\|, \|T\| < 1$, then S and T are contractions. For $w \in X$, we consider the IFS $S_w = (X, f_1, f_2)$, where $f_1, f_2 : X \to X$ are given by $f_1(x) = S(x)$ and $f_2(x) = T(x) + w$, for all $x \in X$.

THEOREM 3.1. *In the preceding framework, let us consider a bounded linear operator $S : H \to H$ satisfying the condition $\|S\| < 1$. If for every bounded linear operator $T : H \to H$ such that $\|T\| < 1$ there exists a sequence $(K_{T,n})_n$ of compact subsets of H having the property that $A_{\mathcal{S}_w}$ is not connected for all $w \in H - \underset{n}{\cup} K_{T,n}$, then the operator S is compact.*

`Proof`. For each $m \in \mathbb{N}$ let us consider the bounded linear operator $U = S^{[m]}$. Obviously $\|U\| < 1$. Let us consider $P_\varepsilon = E((-\infty, 1-\varepsilon))$ and $\widetilde{P}_\varepsilon = E((1+\varepsilon, \infty))$, where E is the spectral decomposition of the positive (so self-adjoint, so normal) bounded linear operator

$$N = (Id_H - U)^*(Id_H - U) = Id_H - U - U^* + U^*U.$$

We claim *that P_ε has finite rank for every $\varepsilon > 0$.*

Indeed, if there is to be an $\varepsilon_0 > 0$ such that P_{ε_0} has infinite rank, then let us consider the operator $T = (Id_H - U)P_{\varepsilon_0}$ and remark that

$$NP_{\varepsilon_0} = NP_{\varepsilon_0}^2 = NP_{\varepsilon_0}^* P_{\varepsilon_0} = P_{\varepsilon_0}^* N P_{\varepsilon_0} = P_{\varepsilon_0}^*((Id_H - U)^*(Id_H - U))P_{\varepsilon_0} =$$

$$= ((Id_H - U)P_{\varepsilon_0})^*((Id_H - U)P_{\varepsilon_0}) \geq 0.$$

Hence, according to Proposition 2.12, we have $0 \leq NP_{\varepsilon_0} \leq (1-\varepsilon_0)P_{\varepsilon_0}$ and therefore, using Proposition 2.13, it follows that $\|NP_{\varepsilon_0}\| \leq 1 - \varepsilon_0$. Consequently we obtain

$$\|T\|^2 = \|T^*T\| = \|(Id_H - U)P_{\varepsilon_0})^*(Id_H - U)P_{\varepsilon_0}\| =$$

$$= \left\|P_{\varepsilon_0}^*(Id_H - U)^*(Id_H - U)P_{\varepsilon_0}\right\| = \|P_{\varepsilon_0} N P_{\varepsilon_0}\| \leq$$

$$\leq \|P_{\varepsilon_0}\| \, \|NP_{\varepsilon_0}\| = \|NP_{\varepsilon_0}\| \leq 1 - \varepsilon_0$$

and thus

$$\|T\| \leq \sqrt{1-\varepsilon_0} < 1.$$

For $w \in H$, let us consider, besides $\mathcal{S}_w$, the IFS $\mathcal{S}_w^{'} = (H, f, f_2)$, where $f : H \to H$ is given by $f(x) = U(x)$, for all $x \in H$.

Now let us choose an arbitrary $w \in (Id_H - T)P_{\varepsilon_0}(H)$. On one hand, since 0 is the fixed point of f, using Remark 2.7, we infer that $0 \in A_{\mathcal{S}_w}$. On the other hand, using the same argument, we get that e, the fixed point of f_2, belongs to $A_{\mathcal{S}_w}$, that is $e = U^{-1}(w) = (Id_H - T)^{-1}(w) \in A_{\mathcal{S}'_w}$. Since $f(e) = f_2(0) = w$, we obtain $w \in f(A_{\mathcal{S}'_w}) \cap f_2(A_{\mathcal{S}'_w})$, which implies $f(A_{\mathcal{S}'_w}) \cap f_2(A_{\mathcal{S}'_w}) \neq \emptyset$, and therefore, according to Theorem 2.10, $A_{\mathcal{S}'_w}$ is connected. We conclude (using Proposition 2.11) that $A_{\mathcal{S}_w}$ is connected.

Consequently there exists a bounded linear operator $T : H \to H$ having $\|T\| < 1$ such that $A_{\mathcal{S}_w}$ is connected for every $w \in (Id_H - T)P_{\varepsilon_0}(H)$.

According to the hypothesis there exists a sequence $(K_{T,n})_n$ of compact subsets of H having the property that $A_{\mathcal{S}_w}$ is not connected, for all $w \in H - \underset{n}{\cup} K_{T,n} m$.

Therefore we obtain $(Id_H - T)P_{\varepsilon_0}(H) \subseteq \underset{n}{\cup} K_{T,n}$ which (taking into account the fact that $(Id_H - T)P_{\varepsilon_0}(H)$ is infinite dimensional, that the closed unit ball in a normed linear space X is compact if and only if X is infinite dimensional and Baire's theorem) generates a contradiction.

We assert *that* $\widetilde{P}_{\varepsilon}$ *has finite rank for every* $\varepsilon > 0$.

Indeed, if by contrary we suppose that there exists $\varepsilon_0 > 0$ such that $\widetilde{P}_{\varepsilon_0}$ has infinite rank, let R_{ε_0} designates the orthogonal projection of H onto $(Id_H - U)\widetilde{P}_{\varepsilon_0}(H)$ and let us consider the bounded linear operator $T = (Id_H - U)^{-1} R_{\varepsilon_0}$. Based upon Proposition 2.12, we have

$$N\widetilde{P}_{\varepsilon_0} = (Id_H - U)^*(Id_H - U)\widetilde{P}_{\varepsilon_0} \geq (1+\varepsilon_0)\widetilde{P}_{\varepsilon_0},$$

which implies that

$$\left\|(Id_H - U)\widetilde{P}_{\varepsilon_0}(x)\right\|^2 =< N\widetilde{P}_{\varepsilon_0}(x), \widetilde{P}_{\varepsilon_0}(x) > \geq (1+\varepsilon_0) \left\|\widetilde{P}_{\varepsilon_0}(x)\right\|^2,$$

i.e.

$$\sqrt{1+\varepsilon_0} \left\|\widetilde{P}_{\varepsilon_0}(x)\right\| \leq \left\|(Id_H - U)\widetilde{P}_{\varepsilon_0}(x)\right\|, \qquad (0)$$

for each $x \in H$. So, as for each $u \in H$ there exists $x_u \in H$ such that $R_{\varepsilon_0}(u) = (Id_H - U)\widetilde{P_{\varepsilon_0}}(x_u)$, we infer that

$$\|T(u)\| = \left\|(Id_H - U)^{-1}R_{\varepsilon_0}(u)\right\| = \left\|(Id_H - U)^{-1}(Id_H - U)\widetilde{P_{\varepsilon_0}}(x_u)\right\| =$$

$$= \left\|\widetilde{P_{\varepsilon_0}}(x_u)\right\| \overset{(0)}{\leq} \frac{1}{\sqrt{1+\varepsilon_0}} \left\|(Id_H - U)\widetilde{P_{\varepsilon_0}}(x)\right\| =$$

$$= \frac{1}{\sqrt{1+\varepsilon_0}} \|R_{\varepsilon_0}(u)\| \leq \frac{1}{\sqrt{1+\varepsilon_0}} \|R_{\varepsilon_0}\| \, \|u\| = \frac{1}{\sqrt{1+\varepsilon_0}} \|u\|$$

i.e. $\|T(u)\| \leq \frac{1}{\sqrt{1+\varepsilon_0}} \|u\|$, for each $u \in H$, which takes on the form

$$\|T\| \leq \frac{1}{\sqrt{1+\varepsilon_0}} < 1.$$

For $w \in H$, let us consider, besides $\mathcal{S}_w$, the IFS $\mathcal{S}_w^{'} = (H, f, f_2)$, where $f : H \to H$ is given by $f(x) = U(x)$, for all $x \in H$.

Now let us choose an arbitrary $w \in (Id_H - T)\widetilde{P_{\varepsilon_0}}(H)$. Then there exists $u \in H$ such that $w = (Id_H - T)\widetilde{P_{\varepsilon_0}}(u)$. On one hand, since 0 is the fixed point of f, using Remark 2.7, we infer that $0 \in A_{\mathcal{S}_w^{'}}$. On the other hand, using the same argument, we get that e (the fixed point of f_2) belongs to $A_{\mathcal{S}_w^{'}}$, that is $e = U^{-1}(w) = (Id_H - T)^{-1}(w) \in A_{\mathcal{S}_w^{'}}$, and therefore $f(e) \in A_{\mathcal{S}_w^{'}}$. Since $f(0) = 0$, on one hand we infer that

$$0 \in f(A_{\mathcal{S}_w^{'}}). \tag{1}$$

On the other hand we have

$$f_2(f(e)) = TU(e) + w = TU(Id_H - T)^{-1}(w) + (Id_H - T)(Id_H - T)^{-1}(w) =$$

$$= (Id_H - T(Id_H - U))(Id_H - T)^{-1}(w) =$$

$$= (Id_H - T(Id_H - U))(Id_H - T)^{-1}(Id_H - T)\widetilde{P_{\varepsilon_0}}(u) =$$

$$= (Id_H - T(Id_H - U))\widetilde{P_{\varepsilon_0}}(u) = \widetilde{P_{\varepsilon_0}}(u) - (Id_H - U)^{-1}R_{\varepsilon_0}(Id_H - U))\widetilde{P_{\varepsilon_0}}(u) =$$

$$= \widetilde{P_{\varepsilon_0}}(u) - (Id_H - U)^{-1}(Id_H - U))\widetilde{P_{\varepsilon_0}}(u) = 0,$$

so

$$0 \in f_2(A_{\mathcal{S}_w^{'}}). \tag{2}$$

From (1) and (2) we obtain $0 \in f(A_{\mathcal{S}'_w}) \cap f_2(A_{\mathcal{S}'_w})$, i.e. $f(A_{\mathcal{S}'_w}) \cap f_2(A_{\mathcal{S}'_w}) \neq \varnothing$, so, relying on Theorem 2.10, $A_{S'_w}$ is connected. We appeal to Proposition 2.11 to deduce that $A_{\mathcal{S}_w}$is connected.

Consequently there exists a bounded linear operator $T : H \to H$ having $\|T\| < 1$ such that $A_{\mathcal{S}_w}$is connected for every $w \in (Id_H - T)\widetilde{P_{\varepsilon_0}}(H)$.

Taking into account the hypothesis there exists a sequence $(K_{T,n})_n$ of compact subsets of H having the property that A_{S_w} is not connected for all $w \in H - \underset{n}{\cup} K_{T,n} m$.

Thus we obtain the inclusion $(Id_H - T)\widetilde{P_{\varepsilon_0}}(H) \subseteq \underset{n}{\cup} K_{T,n}$ which generates a contradiction by invoking the same arguments that we used in the final part of the previous claim's proof.

Now *we state that* $Id_H - (Id_H - U)^*(Id_H - U)$ *is compact.*

If $\mathcal{E}$ is the spectral decomposition of $Id_H - N$, using Proposition 2.14, we obtain $E((-\infty, 1-\varepsilon) \cup (1+\varepsilon, \infty)) = E(g^{-1}((-\infty, -\varepsilon) \cup (\varepsilon, \infty))) = \mathcal{E}((-\infty, -\varepsilon) \cup (\varepsilon, \infty)) = \mathcal{E}((-\infty, -\varepsilon) \cup (\varepsilon, \infty))$, where $g(x) = 1 - x$. Since from the above two claims we infer that the operator $E(((-\infty, 1-\varepsilon) \cup (1+\varepsilon, \infty))) = E((-\infty, 1-\varepsilon)) + E((1+\varepsilon, \infty))$ has finite rank, we get that $\mathcal{E}((-\infty, -\varepsilon) \cup (\varepsilon, \infty))$ has finite rank, for every $\varepsilon > 0$. Proposition 2.15 assures us that $Id_H - N$ is compact, i.e. $Id_H - (Id_H - U)^*(Id_H - U) = U + U^* - U^*U$ is compact.

Hence

$$S^{[m]} + (S^{[m]})^* - S^{[m]}(S^{[m]})^*$$

is compact, for every $m \in \mathbb{N}$.

For $m = 1$, we get that $S + S^* - S^*S$ is compact. Note that, by Corollary 2.17, $S + S^* - SS^*$ is compact and hence $SS^* - S^*S$ is compact (3).

Consequently $S^*(S^*S - SS^*)S = (S^*)^{[2]}S^{[2]} - S^*SS^*S$ is compact. (4)

Moreover, for $m = 2$, we obtain that $S^{[2]} + (S^*)^{[2]} - (S^*)^{[2]}S^{[2]}$ is compact. (5)

But

$$(S + S^* - S^*S)(S + S^* - S^*S) =$$

$$= (S + S^*)^{[2]} - (S + S^* - S^*S)S^*S - S^*S(S + S^* - S^*S) - S^*SS^*S$$

is compact.

Since $S + S^* - S^*S$ is compact, we infer that

$$(S + S^*)^{[2]} - S^*SS^*S =$$

$$= S^{[2]} + (S^*)^{[2]} + SS^* + S^*S - S^*SS^*S =$$

$$= S^{[2]} + (S^*)^{[2]} - (S^*)^{[2]}S^{[2]} + SS^* + S^*S + (S^*)^{[2]}S^{[2]} - S^*SS^*S$$

is compact. (6)

Then, from (4), (5) and (6), we get that $SS^* + S^*S$ is compact. (7)

From (3) and (7) we deduce that S^*S is a compact operator and, using Proposition 2.18, we conclude that S is compact. □

4. A necessary condition for the compactness of an operator. In this section X is a Banach space. We shall designate by Id_X the function $Id_X : X \to X$, given by $Id_X(x) = x$, for all $x \in X$. If S and T be bounded linear operator from X to X such that $\|S\|, \|T\| < 1$, then S and T are contractions and $T^{[n]} - Id_X$ is invertible, for each $n \in \mathbb{N}$. For $w \in X$, we consider the IFS $S_w = (X, f_1, f_2)$, where $f_1, f_2 : X \to X$ are given by $f_1(x) = S(x)$ and $f_2(x) = T(x) + w$, for all $x \in X$.

THEOREM 4.1. *In the above mentioned setting, if the operator S is compact, then there exists a family $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X such that $A_{\mathcal{S}_w}$ is not connected, for all $w \in H - \underset{n\in\mathbb{N}}{\cup} K_n$.*

Proof. The proof given in Theorem 5, from [15], applies with little change. More precisely let C_0 be the compact set $\overline{S(B(0,1))}$. Let $X', X_1, X_2, ..., X_n, ...$ be given by

$$X' = S(X) = \underset{k\in\mathbb{N}}{\cup} kC_0$$

and

$$X_n = (T - Id_X)(T^{[n]} - Id_X)^{-1}(X' - T^{[n]}(X')),$$

for each $n \in \mathbb{N}$. We have

$$X_n = (T - Id_X)(T^{[n]} - Id_X)^{-1}(\underset{k\in\mathbb{N}}{\cup} kC_0 - T^{[n]}(\underset{l\in\mathbb{N}}{\cup} lC_0)) =$$

$$= (T - Id_X)(T^{[n]} - Id_X)^{-1}(\underset{k\in\mathbb{N}}{\cup} kC_0 - \underset{l\in\mathbb{N}}{\cup} lT^{[n]}(C_0)) =$$

$$= (T - Id_X)(T^{[n]} - Id_X)^{-1}(\underset{k,l\in\mathbb{N}}{\cup} (kC_0 - lT^{[n]}(C_0)),$$

for each $n \in \mathbb{N}$ and since $kC_0 - lT^{[n]}(C_0)$ is compact for all $k, l \in \mathbb{N}$, we infer that X_n is a countable union of compact subsets of X. Therefore there exists a family $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X such that $\underset{n\in\mathbb{N}}{\cup} X_n = \underset{n\in\mathbb{N}}{\cup} K_n$. The rest of the proof of the Theorem mentioned above does not require any modification.

Hence $A_{\mathcal{S}_w}$ is disconnected, for each $w \in X \setminus \underset{n\in\mathbb{N}}{\cup} X_n = X \setminus \underset{n\in\mathbb{N}}{\cup} K_n$. $\square$

REMARK 4.2. If X is infinite dimensional, then $W \overset{not}{=} X \setminus \underset{n\in\mathbb{N}}{\cup} X_n = X \setminus \underset{n\in\mathbb{N}}{\cup} K_n$ is dense in X.

Proof. Indeed, let us note that K_n is a closed set. Moreover $\overset{\circ}{K_n} = \varnothing$ since if this is not the case, then the closure of the unit ball of the infinite-dimensional space X is compact which is a contradiction. Consequently X_n is nowhere dense, for each $n \in \mathbb{N}$, and therefore W is dense in X.

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Department of Mathematics
Faculty of Mathematics and Informatics
University of Bucharest
Academiei Street, no. 14
010014 Bucharest, Romania
E-mail: mihail_alex@yahoo.com
miculesc@yahoo.com