Rapid and robust spin state amplification

Tom Close,[1,*] Femi Fadugba,[1] Simon C. Benjamin,[1,2] Joseph Fitzsimons,[2] and Brendon W. Lovett[1,3]

[1]Department of Materials, Oxford University, Oxford OX1 3PH, UK
[2]Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543
[3]School of Engineering and Physical Sciences, Heriot-Watt University, Edinburgh EH14 4AS

Electron and nuclear spins have been employed in many of the early demonstrations of quantum technology (QT). However applications in real world QT are limited by the difficulty of measuring single spins. Here we show that it is possible to rapidly and robustly amplify a spin state using a lattice of ancillary spins. The model we employ corresponds to an extremely simple experimental system: a homogenous Ising-coupled spin lattice in one, two or three dimensions, driven by a continuous microwave field. We establish that the process can operate at finite temperature (imperfect initial polarisation) and under the effects of various forms of decoherence.

PACS numbers:

The standard approach to implementing a quantum technology is to identify a physical system that can represent a qubit: it must exhibit two (or more) stable states, it should be manipulable through external fields and possess a long decoherence time. Provided that the system can controllably interact with other such systems, then it may be a strong candidate. Electron and nuclear spins, within suitable molecules or solid state structures, can meet these requirements. However the drawback with spin qubits is that they have not been directly measured through a detection of the magnetic field they produce. The magnetic moment of a single electron spin is orders of magnitude too weak to be detected by standard ESR techniques and even the most sensitive magnetometers still fall short of single spin detection [1] - meanwhile the situation with nuclear spins is worse still. In a few special systems it is possible to convert the spin information into another degree of freedom. For example, a spin-dependent optical transition allow spin to photon conversion in some crystal defects [2–4], self-assembled semiconductor quantum dots [5], and trapped atoms held in a vacuum [6]. Alternatively, spin to charge conversion is an established technology in lithographic quantum dots [7]. However, the majority of otherwise promising spin systems do not have such a convenient property [8] and therefore cannot be measured directly.

One suggested solution is to 'amplify' a single spin, by using a set of ancillary spins that are (ideally) initialised to $|0\rangle$. We would look for a transformation of the form

$$|0\rangle\,|0\rangle^{\otimes n} \rightarrow |0\rangle\,|0\rangle^{\otimes n} \qquad |1\rangle\,|0\rangle^{\otimes n} \rightarrow |1\rangle\,|1\rangle^{\otimes n}, \tag{1}$$

the idea being that the n ancillary spins constitute a large enough set that state of the art magnetic field sensing technologies can detect them. Note that the transformation need not be unitary or indeed even coherent: since the intention is to make a measurement of the primary spin, it is not necessary to preserve any superposition (that is, we need not limit ourselves to transformations that take $\alpha\,|0\rangle\,|0\rangle^{\otimes n} + \beta\,|1\rangle\,|0\rangle^{\otimes n}$ to a cat state like $\alpha\,|0\rangle^{\otimes n+1} + \beta\,|1\rangle^{\otimes n+1}$).

This is a rather broadly defined transformation and there are a number of ways that one might perform it. Clearly one would like to find the method that is the least demanding experimentally. Previous authors have proposed schemes using a strictly one-dimensional (1D) homogeneous lattice with continuous global driving [9], and an inhomogeneous three-dimensional (3D) lattice with alternating timed EM pulses [10]. The former result has the advantage of simplicity but the rate at which amplification occurs will inevitably be limited by the single dimension of the array; moreover such a system must be highly vulnerable to imperfect initialisation (i.e. finite temperature). Here we generalise to a homogeneous two-dimensional (2D) square lattice, showing that a continuous global EM field can drive an amplification process that succeeds at finite temperatures (imperfect initialisation of the ancilla spins) and in the presence of decoherence. By bringing the global EM field onto resonance with certain transitions, we are able to create a set of rules that govern locally how spins propagate over the lattice. We then look at the rate of increase in the total number of flipped spins as a measure of quality of the scheme. While our focus is on the 2D case, we are also able to predict the performance of the amplification protocol for a homogeneous 3D lattice with continuous driving.

The case of a 1D lattice has been studied in detail by Lee and Khitrin [9]. Before moving to the 2D spin lattice that will form the core of the paper, we first recall how to simplify the description of this (semi-infinite) 1D spin chain, with nearest neighbour Ising (ZZ) interactions. Under a microwave driving field of frequency ω, the Hamiltonian is given by

$$\mathcal{H} = \sum_{i=1}^{\infty} \epsilon_i \sigma_z^i + J_i \sigma_z^i \sigma_z^{i+1} + 2\Omega_i \sigma_x^i \cos(\omega t) \tag{2}$$

ϵ_i is the on-site Zeeman energy of spin i, and J_i is the magnitude of the coupling between spins i and $i+1$. Ω describes the coupling of spin i and the microwave field. In this case, spin $i = 1$ is the one whose state is supposed to be amplified. If we assume that the chain is uniform, such that $\Omega_i = \Omega$, $\epsilon_i = \epsilon$ and $J_i = J$, then moving into a frame rotating at frequency ω, making a rotating wave approximation and setting $\omega = \epsilon$ leads to

$$\mathcal{H} = \sum_{i=1}^{\infty} J \sigma_z^i \sigma_z^{i+1} + \Omega \sigma_x^i. \tag{3}$$

In order to understand the dynamics of the system, is it instructive to explicitly separate all terms that involve a particular spin k:

$$\mathcal{H} = J(\sigma_z^{k-1}+\sigma_z^{k+1})\sigma_z^k+\Omega\sigma_x^k+\sum_{i\neq\{k,k-1\}}\Omega\sigma_x^i+J\sigma_z^i\sigma_z^{i+1}+\Omega\sigma_x^{k-1} \tag{4}$$

Choosing a driving field such that $\Omega \ll J$ means that spin k will only undergo resonant oscillations when the first term in Eq. 4 goes to zero - i.e. when the two spins neighbouring spin k are oriented in opposite directions. In any other configuration the Ising coupling takes the spin k off resonance with the microwave and no appreciable dynamics are expected.

Let us now define a subset of states S that exist in the spin chain Hilbert space, $|n\rangle$, which have the first n spins of the chain in state $|\uparrow\rangle$ with the rest $|\downarrow\rangle$. If the rule we just derived holds exactly these states define a closed subspace. We may then write a very simple isolated Hamiltonian for this subspace:

$$\mathcal{H}_S = \Omega \sum_{n=1}^{\infty} |n\rangle\langle n+1| . \tag{5}$$

With this simplification of the 1D Hamiltonian in mind, we progress now to a semi-infinite square spin lattice with nearest-neighbour ZZ interactions. For this case we have

$$\mathcal{H} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \epsilon\sigma_z^{i,j}+J\sigma_z^{i,j}\sigma_z^{i+1,j}+J\sigma_z^{i,j}\sigma_z^{i,j+1}+2\Omega\sigma_x^{i,j}\cos(\omega t). \tag{6}$$

By again considering the terms affecting a particular spin in the main body of the lattice $(k(>1), l(>1)$ say) we find for $\omega = \epsilon$ and after moving to a rotating frame and making the rotating wave approximation:

$$\mathcal{H} = J\sigma_z^{k,l}(\sigma_z^{k+1,l} + \sigma_z^{k,l+1} + \sigma_z^{k-1,l} + \sigma_z^{k,l-1}) + ... \tag{7}$$

where we do not explicitly write out terms not involving spin (k, l). The microwave is now only resonant for spin (k, l) if it has two neighbour spins in each orientation. For a spin on the edge of the lattice there are an odd number of neighbours so resonance cannot be achieved in this case. However, applying a second microwave with $\omega = \epsilon - J$ allows resonant flips on the edge if two neighbours are down and one up - and this second field has no effect on the bulk spins.

The spin to be measured is the corner spin $(i = j = 1)$ and so would form part of a wider computational apparatus. We may therefore assume that it is a different species with a unique resonant frequency. The dynamics of the whole lattice may then be summarised by three rules (in order of precedence):

1. The corner (test) spin is fixed.
2. An edge spin can flip if it has one of its neighbours up and two down.
3. A body spin can flip if it has two of its neighbours up and two down.

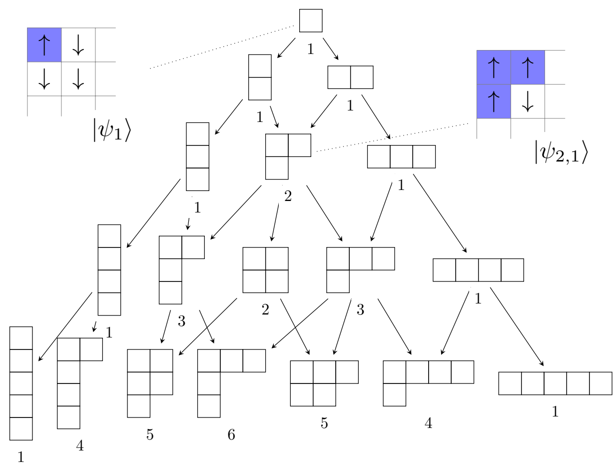

FIG. 1: Partition states arranged into a lattice. Edges represent a coupling through the Hamiltion of strength Ω. Weights represent the number of different paths through the lattice to a given state.

We begin by supposing all spins are initialised in the 'down' state apart from the test spin, which is located in the upper left hand corner of our lattice. We can describe this initial state by choosing two basis elements: $|0\rangle$ when the test spin is down, and $|1\rangle$ when the test spin is up. Using our heuristic rules we can see that these two states do not couple to each other - that $\langle 0| H |1\rangle = 0$. In fact $|0\rangle$ does not couple to any other state, so if we start in the $|0\rangle$ state no amplification occurs, as desired.

We will now seek to construct a basis for the subspace containing our system evolution, by looking at states connected by our Hamiltonian. It will be convenient to represent these states on the nodes of a graph, using the edges to represent non-zero elements of the Hamiltonian.

Our starting point is the state $|1\rangle$, with just the corner spin 'up'. From this position our rules allow two possibilities: either the spin to the right of the corner flips, or the spin below it flips (see Fig. 1). In each case the magnitude of the transition matrix element is Ω. As we continue this procedure, we notice that the states that arise for each excitation number can be characterised by a non-increasing sequence of integers that represent the number of 'up'-spins in each column of the lattice (see Fig. 1). Such sequences can also be used to define partitions of at integer: ways of splitting an integer up into a sum of other integers, e.g. $3 = 3 = 2 + 1 = 1 + 1 + 1$. In fact, the states that arise are in 1-to-1 correspondence with such partitions; we call these states 'partition states' and denote them with standard partition notation (see Fig. 1). The graph we have just described is depicted in Fig. 1 is known as 'Young's lattice' and arises in areas of pure mathematics, such as the representation theory of the symmetric group, and the theory of differential posets. We have drawn weights beneath each state, recording the number of ways the state can be constructed. We will now further reduce the dimension of this basis by eliminating combinations of states which are inaccessible.

Starting with $|1\rangle$ we see that $\langle 1| H (\alpha_{1,1} |\psi_{1,1}\rangle + \alpha_2 |\psi_2\rangle) = \Omega (\alpha_{1,1} + \alpha_2)$ so $|1\rangle$ does not couple to the two-excitation state $|\psi_{1,1}\rangle - |\psi_2\rangle$. We can eliminate this, leaving a single orthogonal, coupled state with two excitations: $|2\rangle := \frac{1}{\sqrt{2}} (|\psi_{1,1}\rangle + |\psi_2\rangle)$.

We may continue to build up coupled states with larger excitation numbers, and in fact we find that there is only a single coupled state in each case (i.e. we can always eliminate $k-1$ combinations of partition states with k excitations). To see this, first suppose we have the coupled state with k excitations, which by analogy with the 1D case we write as $|k\rangle$. We can write $|k\rangle = \frac{1}{N_k} \sum_{i \in P(k)} c_i |\psi_i\rangle$, where $P(k)$ is the set of partitions of the integer k and N_k a normalisation factor. We want to construct the state $|k+1\rangle$ by eliminating the k-dimensional subspace with $k+1$ excitations, to which $|k\rangle$ does not couple.

Let $|\psi\rangle = \sum_{j \in P(k+1)} \alpha_j |\psi_j\rangle$ and consider the states $|\psi\rangle$ such that

$$0 = \langle k| H |\psi\rangle = \sum_{i \in P(k)} \sum_{j \in P(k+1)} c_i^* \alpha_j \langle \psi_i | H |\psi_j\rangle$$

but $\langle \psi_i | H |\psi_j\rangle = \Omega$ if i is a *parent* of j (a state connect to j, in the lattice row above it), and 0 otherwise, so

$$0 = \langle k| H |\psi\rangle = \sum_{j \in P(k+1)} \alpha_j \sum_{i \in parents(j)} c_i^* .$$

This is the equation of a hyperplane in $|P(k+1)|$ dimensions, defining the states that are not coupled to $|k\rangle$ through the Hamiltonian. There is a unique single state orthogonal to this hyperplane, $\beta_j = \sum_{i \in parents(j)} c_i$, to which $|k\rangle$ couples. So the only state with $k+1$ 'up'-spins that $|k\rangle$ couples has coefficients proportional to β_j. After normalisation, we call this state $|k+1\rangle$.

Unfortunately there is no easy way to write down the partition states and weights for the nth row of the lattice. Fortunately, for our purposes, we only need to know that the states $|k\rangle$ exist and what the coupling between them is. To find this coupling, consider

$$\begin{aligned} g_{n-1,n} &= \langle n| H |n-1\rangle \\ &= \frac{1}{N_{n-1} N_n} \sum_{i \in P(n)} \sum_{j \in P(n-1)} c_i^* c_j \langle \psi_i | H |\psi_j\rangle \\ &= \frac{1}{N_{n-1} N_n} \Omega \sum_{i \in P(n)} c_i^* \sum_{j \in parents(i)} c_j \\ &= \frac{1}{N_{n-1} N_n} \Omega \sum_{i \in P(n)} |c_i|^2 = \Omega \frac{N_n}{N_{n-1}} \end{aligned} \tag{8}$$

To find the N_n we need the sum of the squares of the weights of partitions in a given row. A standard result about Young's lattice [11] immediately gives us this sum: $n!$. In deriving this result, it is crucial [15] that each partition state has one more child than it does parents, and also that every two states that share a parent also share precisely one child.

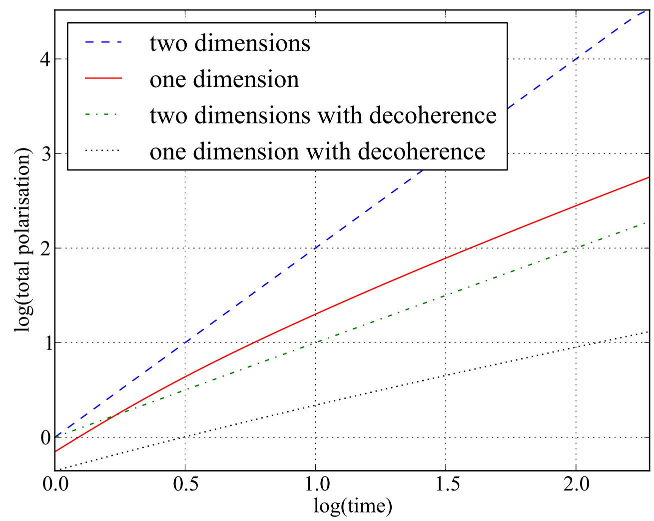

FIG. 2: Expected total polarisation against time. Time in units of $\frac{1}{\Omega}$, dephasing rate $\Gamma = 1$. The gradient of the 'one dimension with decoherence' line tends to $\frac{1}{2}$ asymptotically.

Referring back to Eq. (8), and using $N_i = \sqrt{i!}$, we see that

$$\mathcal{H} = \Omega \sum_n \sqrt{n} |n-1\rangle \langle n| . \tag{9}$$

In essence we have established a linear sequence of states, each coupled to the the next analogously to the states on a 1D chain 5. However, each of our states is in fact a superposition of many configurations of the 2D array, and crucially the coupling from each state to the next increases along the sequence.

It has been shown (e.g. [12]) that a quantum state released at the end of a semi-inifinite chain of states, with constant couplings, will travel ballistically: the average position of the state along the chain is proportional to the time passed, and inversely proportional to the coupling strength. Since, in the one-dimensional case, the position is proportional to the number of spins that have flipped, we have that the total polarisation will increase linearly with time.

We can establish the rate of propagation in the 2D case using the ansatz that the time taken to travel between two neighbouring nodes is inversely proportional to the strength of the coupling between them. The total time is then $t_{2D} \propto \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \simeq n^{\frac{1}{2}}$. As in the one-dimensional case, the position along the chain corresponds to the the number of spins that have flipped, and so we would expect the total polarisation to be proportional to t^2. This prediction of a quadratic speed-up of signal going from 1D to 2D is the central result of our paper, and was confirmed by simple numerical simulations of Eq. (9) (Fig. 2).

Unfortunately the mapping from 2D to 1D is not readily extendible to 3D. However, our results so far could have been anticipated using simple dimensional arguments; if one postulates that the rate of spin propagation is proportional to the boundary of the region, one can predict the correct scaling behaviour. In 1D the boundary size is independent of the region size; no matter how many spins have flipped, it still has size one. The coupling strength between states $|n\rangle$ is constant. In the 2D case, the boundary size scales with the square root of the area, and the coupling goes with $\sqrt{n}$. In 3D, the boundary scales like the cube root of the volume squared, and so we expect the coupling to scale as $n^{\frac{2}{3}}$. Following similar logic to that used in 2D case: $t_{3D} \propto \sum_{i=1}^{n} \frac{1}{i^{\frac{2}{3}}} \simeq n^{\frac{1}{3}}$, and so $n \sim t^3$.

We now consider the effect of decoherence. Much the early work on continuous time quantum random walks looked at the speedup they afforded over their classical counterparts [13], but didn't make any statement about the conditions under which we would expect the quantum walk to become classical. We would expect that, in a regime of suitably heavy dephasing, our quantum walk would be reduced to an 'equivalent' classical one.

We begin by considering a collective noise operator: $L = \sum_n n\,|n\rangle\,\langle n|$. This represents noise that applies uniformly to the whole lattice: global fluctuations in the magnetic field, for example. As the effect of this type of noise depends only on the number of 'up' spins, the system remains in the reduced basis of number states calculated earlier, with only the coherences between these states affected.

Our starting point is the Lindblad master equation

$$\dot{\rho} = i\,[\rho, H] + \frac{1}{2}\Gamma\left(2L\rho L^\dagger - L^\dagger L\rho - \rho L^\dagger L\right). \qquad (10)$$

We proceed by splitting up the equation into diagonal and off-diagonal terms:

$$\dot{\rho}_{ii} = i\sum_{k=\pm i}\left(\rho_{ik}g_{ki} - \rho_{ki}g_{ik}\right) = -2\sum_{k=\pm i} Re\left[\rho_{ik}g_{ki}\right] \quad (11)$$

$$\dot{\rho_{ij}} = i\left(\sum_{k=\pm j}\rho_{ik}g_{kj} - \sum_{k=\pm i}\rho_{kj}g_{ik}\right) - \Gamma\rho_{ij} \qquad (12)$$

where g_{ij} is the coupling between states i and j. In the limit of heavy dephasing ($\Gamma \gg g$), we have a process similar to adiabatic following, and we can make the approximation

$$\Gamma\rho_{ij} \approx i\left(\sum_{k=\pm j}\rho_{ik}g_{kj} - \sum_{k=\pm i}\rho_{kj}g_{ik}\right).$$

We consider the ρ_{ij} as a set of $\frac{n(n-1)}{2}$ variables and solve for them in terms of the ρ_{ii}. Neglecting terms that are second order in $\frac{g}{\Gamma}$, and substituting back into Eq. (11) gives

$$\dot{\rho}_{ii} = -\sum_{j=i\pm 1}\frac{2|g_{ij}|^2}{\Gamma}\left(\rho_{ii} - \rho_{jj}\right).$$

Our quantum chain reduces to a classical Markov chain on the same statespace, with transition rates proportional to the coupling squared.

Although states with more 'up' spins decohere more quickly, the decoherence rate Γ is not multiplied for higher states, as it is the *relative* decoherence rate between neighbouring states, which is of importance.

In one-dimension $g_{ij} = 1$ and we are reduced to a simple random walk on a semi-infinite line. By analogy with simple diffusion we expect that the resulting distribution is roughly Gaussian, with the expected number of flipped spins going with $\sqrt{t}$: the rate of spin propagation drops from t to $\sqrt{t}$. This result was confirmed numerically (Fig. 2).

In the two-dimensional case $g_{ij} = \sqrt{j}, j = i + 1$: We get a random walk with increasing transition rates. Numerically (Fig. 2), we find that the rate of spin propagation drops from t^2 to t - still an encouraging scaling.

The collective noise case is convenient to analyse for our system, as the system remains in the subspace covered by our basis of accessible states. However, a more realistic model involves treating the noise occurring at each site as independent. In this case we have Lindblad operators of the form

$$L_i = \sigma_z^i \qquad (13)$$

for lattice sites i. Following a similar procedure to before we find the equivalent classical chain to be

$$\dot{\rho}_{ii} = -\sum_{j\in P(i)}\frac{2|g_{ij}|^2}{\Gamma}\left(\rho_{ii} - \rho_{jj}\right) \qquad (14)$$

where, crucially, the index now runs over all the partition states, rather than our basis of accessible states. In fact, in the 1D case these states are one and the same, and so the spin propagation goes as $\sqrt{t}$, as found in the collective noise case. In the 2D case, we are now performing a continuous-time classical random walk on Young's lattice. We are able to use the property that each node always has one more child than parents, to predict that the rate of spin propagation will be proportional to t - the same as the collective noise case. Thus for both forms of decoherence we find that the amplification still functions; when the noise is severe then it will take longer to flip a given number of spins.

Finally we consider imperfect initial polarisation (i.e. finite temperature). Any real experimental system will have this property. To examine the behaviour of the scheme under imperfect initialisation, we consider initial states where a random subset of the lattice spins are in the 'up' state. A concern with any spin amplification scheme is that such imperfections in the initial state themselves become amplified, leading to false positives. Thanks to our spin propagation rules our scheme is highly robust against this sort of error; the fact that two neighbouring spins need to be 'up' for a spin to flip makes it difficult for imperfections to spread. Numerical simulations suggest that below an initialisation threshold of approximately 5% [16], is it extremely unlikely that a false positive occurs. In fact this threshold is a very loose lower bound, as it assumes deterministic growth of impurities, rather than the quantum oscillations that will occur. We anticipate that a real system could tolerate even higher levels of imperfection, and therefore our protocol should be well within experimental capabilities. For example for an array placed in a standard W-band electron spin resonance system (100 GHz) and cooled using liquid 4He to 1.4 degrees Kelvin, only 3.1% of electron spins will be in the 'up' state.

We thank Gerard Milburn and John Morton for useful discussion. This work was supported by the EPSRC, the National Research Foundation and Ministry of Education, Singapore, and the Royal Society.

* Electronic address: tom.close@materials.ox.ac.uk

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[15] It is instructive to look in some detail at the features of the lattice of states that lead to this result, which we do in the supplementary material, following the excellent explanation given in [14].
[16] Supplementary material: http://qunat.org/papers/amp/