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Higher order Painlevé system of type $D^{(1)}_{2n+2}$ and monodromy preserving deformation

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Abstract

The higher order Painlevé system of type $D^{(1)}_{2n+2}$ is proposed by Y. Sasano. It is an extension of the sixth Painlevé equation ($P_{\rm VI}$) for the affine Weyl group symmetry and expressed as a Hamiltonian system of order $2n$. We give this system as the monodromy preserving deformation of a Fuchsian differential equation.

1 Introduction

The main object in this article is the higher order Painlevé system of type $D^{(1)}_{2n+2}$ proposed by Sasano [SY]. It is expressed as a Hamiltonian system on $\mathbb{P}^1(\mathbb{C})$

$$t(t-1)\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i},\quad t(t-1)\frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}\quad (i=1,\ldots,n), \tag{1.1}$$

with a *coupled* Hamiltonian

$$H=\sum_{i=1}^{n}H_{\rm VI}[\kappa_i^{(0)},\kappa_i^{(1)},\kappa_i^{(t)},\kappa_i^{(\infty)};q_i,p_i]+\sum_{1\le i<j\le n}2q_ip_i(q_j-1)\{(q_j-t)p_j+\alpha_{2j}\}, \tag{1.2}$$

where $H_{\rm VI}=H_{\rm VI}[\kappa_0,\kappa_1,\kappa_t,\kappa_\infty;q,p]$ is the Hamiltonian for $P_{\rm VI}$ defined by

$$\begin{aligned}H_{\rm VI}&=q(q-1)(q-t)p^2-\kappa_0(q-1)(q-t)p-\kappa_1q(q-t)p\\&\quad-(\kappa_t-1)q(q-1)p+\frac{1}{4}\{(1-\kappa_0-\kappa_1-\kappa_t)^2-\kappa_\infty^2\}q,\end{aligned}$$

and

$$\kappa_i^{(0)} = \alpha_1 + \sum_{j=1}^{i-1} \alpha_{2j+1}, \quad \kappa_i^{(1)} = \sum_{j=i}^{n-1} \alpha_{2j+1} + \alpha_{2n+2},$$

$$\kappa_i^{(t)} = \sum_{j=i}^{n-1} \alpha_{2j+1} + \sum_{j=i+1}^{n} 2\alpha_{2j} + \alpha_{2n+1}, \quad \kappa_i^{(\infty)} = \alpha_0 + \sum_{j=1}^{i-1} 2\alpha_{2j} + \sum_{j=1}^{i-1} \alpha_{2j+1}.$$

The fixed parameters $\alpha_0, \ldots, \alpha_{2n+2}$ satisfy a relation

$$\alpha_0 + \alpha_1 + \sum_{j=2}^{2n} 2\alpha_j + \alpha_{2n+1} + \alpha_{2n+2} = 1.$$

The Painlevé system (1.1) was proposed as an extension of P_{VI} for the affine Weyl group symmetry with the aid of algebraic geometry for initial value space. It also arises from the Drinfeld-Sokolov type integrable hierarchy by a similarity reduction [FS1]. But the relationship with monodromy preserving deformation has been not clarified. The aim of this article is to give the system (1.1) as the monodromy preserving deformation of a Fuchsian differential equation.

Recently, higher order generalizations of P_{VI} has been studied from a viewpoint of the monodromy preserving deformations of Fuchsian systems. The Fuchsian systems can be classified with the aid of algorithm proposed by Oshima [O]. According to it, Fuchsian systems with two accessary parameters are reduced to the one with the following spectral types:

4 singularities	$11,11,11,11$		
3 singularities	$111,111,111$	$22,1111,1111$	$33,222,111111$

The system with the spectral type $\{11,11,11,11\}$ gives P_{VI} as the monodromy preserving deformation. Note that the other three systems have no deformation parameters. Fuchsian systems with four accessary parameters are reduced to the one with the following spectral types:

5 singularities	$11,11,11,11,11$		
4 singularities	$21,21,111,111$	$31,22,22,1111$	$22,22,22,211$
3 singularities	$211,1111,1111$	$221,221,11111$	$32,11111,111111$
	$222,222,2211$	$33,2211,111111$	$44,2222,22211$
	$44,332,11111111$	$55,3331,22222$	$66,444,2222211$

The system with $\{11,11,11,11,11\}$ corresponds to the Garnier system in two variables [G]. And the systems with four singularities correspond to four-dimensional Painlevé equations, which are investigated by Sakai [Sak]. Among them, the system with $\{31, 22, 22, 1111\}$ corresponds to the Painlevé system (1.1) of the case $n = 2$. In this article, we consider its natural extension. Namely, we consider the Fuchsian system with the spectral type $\{(n, n), (n, n), (2n-1, 1), (1^{2n})\}$ and show that its monodromy preserving deformation gives the system (1.1).

Remark 1.1. *The system* (1.1) *is slightly different from the one given in* [SY]. *It is derived from* (1.1) *via a transformation of independent and dependent variables*

$$t \to 1 - \frac{1}{t}, \quad q_i \to t(1-q_i), \quad p_i \to -\frac{p_i}{t} \quad (i = 1, \dots, n).$$

Remark 1.2. *The Fuchsian system with the spectral type* $\{21, 21, 111, 111\}$ *corresponds the fourth order Painlevé system given in* [FS2]. *Furthermore the system with* $\{(n, 1), (n, 1), (1^{n+1}), (1^{n+1})\}$ *is systematically investigated by Tsuda. It corresponds the Schlesinger system* ${}_n\mathcal{H}_1$ *given in* [T], *or equivalently, the higher order Painlevé system given in* [Su].

2 Schlesinger system

2.1 General definition

We recall the Schlesinger system and its Poisson structure following [JMU]. Let $A_1, \dots, A_{N+2} \in M_L(\mathbb{C})$. Consider a Fuchsian system on $\mathbb{P}^1(\mathbb{C})$

$$\frac{\partial}{\partial x} Y(x) = \sum_{i=1}^{N+2} \frac{A_i}{x - t_i} Y(x), \tag{2.1}$$

with regular singularities $x = t_1, \dots, t_{N+2}, \infty$. We assume that each residue matrix A_i can be diagonalized and the residue matrix at $x = \infty$

$$A_\infty = -\sum_{i=1}^{N+2} A_i,$$

is a diagonal matrix. The monodromy preserving deformation of (2.1) is described as a system of partial differential equations

$$\frac{\partial A_j}{\partial t_i} = \frac{[A_i, A_j]}{t_i - t_j}, \quad \frac{\partial A_i}{\partial t_i} = -\sum_{j=1; j\neq i}^{N+2} \frac{[A_i, A_j]}{t_i - t_j}, \tag{2.2}$$

which is called *the Schlesinger system.*

The system (2.2) can be expressed as a Hamiltonian system

$$\frac{\partial A_j}{\partial t_i} = \{K_i, A_j\}, \quad K_i = \sum_{j=1; j\neq i}^{N+2} \frac{\mathrm{tr} A_i A_j}{t_i - t_j}, \tag{2.3}$$

where the Poisson bracket is given by

$$\{(A_i)_{k,l}, (A_j)_{r,s}\} = \delta_{i,j}\delta_{r,l}(A_i)_{k,s} - \delta_{i,j}\delta_{k,s}(A_i)_{r,l}. \tag{2.4}$$

In order to derive the canonical Hamiltonian system from (2.3), we use the method established in [JMMS]. Let $A_i = B_i C_i$. Then the Poisson bracket (2.4) coincides with the one over the space of matrices B_i and C_i defined by

$$\{(B_i)_{k,l}, (C_j)_{r,s}\} = -\delta_{i,j}\delta_{k,s}\delta_{r,l}.$$

The number of the canonical variables for (2.3) is generally less than the dimension of the space of matrices B_i and C_i. In fact, it is equivalent to the number of accessory parameters of the Fuchsian system (2.1). We denote the multiplicity data of eigenvalues of $A_1, \dots, A_{N+2}, A_\infty$, called *spectral type*, by

$$\{(m_{1,1}, \dots, m_{1,l_1}), \dots, (m_{N+2,1}, \dots, m_{N+2,l_{N+2}}), (m_{\infty,1}, \dots, m_{\infty,l_\infty})\}.$$

Then the number of accessory parameters of (2.1) is given by

$$(N+1)L^2 - \sum_{i=1}^{N+2}\sum_{j=1}^{l_i} m_{i,j}^2 - \sum_{j=1}^{l_\infty} m_{\infty,j}^2 + 2.$$

2.2 Spectral type $\{(n,n),(n,n),(2n-1,1),(1^{2n})\}$

In order to derive the Painlevé system (1.1), we consider a Fuchsian system

$$\frac{\partial}{\partial x} Y(x) = \left(\frac{A_t}{x-t} + \frac{A_1}{x-1} + \frac{A_0}{x} \right) Y(x). \tag{2.5}$$

with a spectral type $\{(n,n),(n,n),(2n-1,1),(1^{2n})\}$. The data of eigenvalues of residue matrices is given by

$$\begin{array}{llllllllll}
\theta_t, & \theta_t, & \dots, & \theta_t, & 0, & \dots, & 0 & \text{at} & x = t, \\
\theta_1, & \theta_1, & \dots, & \theta_1, & 0, & \dots, & 0 & \text{at} & x = 1, \\
\theta_0, & 0, & \dots, & 0, & 0, & \dots, & 0 & \text{at} & x = 0, \\
\kappa_1, & \kappa_2, & \dots, & \kappa_n, & \kappa_{n+1}, & \dots, & \kappa_{2n} & \text{at} & x = \infty.
\end{array}$$

Note that the eigenvalues satisfy *the Fuchsian relation*

$$n\theta_t + n\theta_1 + \theta_0 + \sum_{i=1}^{2n} \kappa_i = 0.$$

The monodromy preserving deformation of (2.5) is described as the Schlesinger system

$$\frac{\partial A_\xi}{\partial t} = \{K, A_\xi\} \quad (\xi = t, 1, 0), \quad K = \frac{\mathrm{tr} A_t A_1}{t-1} + \frac{\mathrm{tr} A_t A_0}{t}. \tag{2.6}$$

We consider a gauge transformation

$$\widetilde{A}_\xi = G^{-1} A_\xi G \quad (\xi = t, 1, 0),$$

such that $\widetilde{A}_0$ and $\widetilde{A}_\infty = -\widetilde{A}_t - \widetilde{A}_1 - \widetilde{A}_0$ are lower and upper triangle matrices respectively. Then the system (2.6) is transformed into the Hamiltonian system

$$\frac{\partial \widetilde{A}_\xi}{\partial t} = \{K, \widetilde{A}_\xi\} \quad (\xi = t, 1, 0), \quad K = \frac{\mathrm{tr} \widetilde{A}_t \widetilde{A}_1}{t-1} + \frac{\mathrm{tr} \widetilde{A}_t \widetilde{A}_0}{t}, \tag{2.7}$$

with $2n \times 2n$ matrices

$$\widetilde{A}_\xi = \begin{bmatrix} I_n \\ B_\xi \end{bmatrix} \begin{bmatrix} \theta_\xi I_n - C_\xi B_\xi & C_\xi \end{bmatrix} \quad (\xi = t, 1),$$

where $B_\xi, C_\xi \in M_n(\mathbb{C})$,

$$\widetilde{A}_0 = \begin{bmatrix} \theta_0 & a_2^{(0)} & \dots & a_{2n}^{(0)} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad \widetilde{A}_\infty = \begin{bmatrix} \kappa_1 & & & \boldsymbol{O} \\ a_2^{(\infty)} & \kappa_2 & & \\ \vdots & & \ddots & \\ a_{2n}^{(\infty)} & \boldsymbol{O} & & \kappa_{2n} \end{bmatrix},$$

and the relation

$$\widetilde{A}_t + \widetilde{A}_1 + \widetilde{A}_0 + \widetilde{A}_\infty = 0. \tag{2.8}$$

The Poisson bracket is given by

$$\{(B_\xi)_{i,j}, (C_\xi)_{j,i}\} = -1 \quad (i, j = 1, \dots, n; \xi = t, 1), \quad \{\text{otherwise}\} = 0.$$

The number of accessary parameters of the Fuchsian system (2.5) is given by

$$2(2n)^2 - 2n^2 - 2n^2 - \{(2n-1)^2 + 1^2\} - 2n \cdot 1^2 + 2 = 2n.$$

Therefore the system (2.7) with (2.8) can be rewritten into the Hamiltonian system of order $2n$, which is just equivalent to (1.1).

3 Main Theorem

Let

$$B_\xi = \left[b_{i,j}^{(\xi)} \right]_{i,j=1}^{n}, \quad C_\xi = \left[c_{i,j}^{(\xi)} \right]_{i,j=1}^{n} \quad (\xi = t, 1).$$

Denote by

$$\Delta_{j_1,\dots,j_r}^{i_1,\dots,i_r}(\xi) = \begin{vmatrix} c_{i_1,j_1}^{(\xi)} & \cdots & c_{i_1,j_r}^{(\xi)} \\ \vdots & \ddots & \vdots \\ c_{i_r,j_1}^{(\xi)} & \cdots & c_{i_r,j_r}^{(\xi)} \end{vmatrix}.$$

Then we have

Theorem 3.1. *Under the system* (2.7)*, we take*

$$\begin{aligned} p_i &= (-1)^{n-i} t^{-1} \frac{\Delta_{i,i+1,\dots,n}^{1,i+1,\dots,n}(1)}{\Delta_{i+1,\dots,n}^{i+1,\dots,n}(t)} \sum_{k=1}^{i} \frac{\Delta_{k,i+1,\dots,n}^{i,i+1,\dots,n}(t)}{\Delta_{i,\dots,n}^{i,\dots,n}(t)} b_{k,1}^{(t)}, \\ q_i &= (-1)^{n-i+1} t \frac{\Delta_{i,i+1,\dots,n}^{1,i+1,\dots,n}(t)}{\Delta_{i,i+1,\dots,n}^{1,i+1,\dots,n}(1)} \quad (i = 1, \dots, n), \end{aligned} \tag{3.1}$$

as canonical coordinates of a $2n$*-dimensional system with a Poisson structure*

$$\{p_i, q_j\} = \delta_{i,j}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0 \quad (i, j = 1, \dots, n).$$

We also set

$$\begin{aligned} &\alpha_1 = -\theta_t, \quad \alpha_2 = -\kappa_{n+1}, \quad \alpha_{2i-1} = \theta_t + \theta_1 + \kappa_i + \kappa_{n+i-1}, \\ &\alpha_{2i} = -\theta_t - \theta_1 - \kappa_i - \kappa_{n+i} \quad (i = 2, \dots, n), \\ &\alpha_{2n+1} = \theta_t + \theta_1 + \kappa_1 + \kappa_{2n}, \quad \alpha_{2n+2} = -\kappa_1 + \kappa_{2n} + 1. \end{aligned}$$

Then they satisfy the Painlevé system (1.1)*.*

3.1 Proof of Theorem 3.1

Under the system (2.7), the dependent variables p_i, q_i given in (3.1) satisfy

$$\frac{\partial p_i}{\partial t} = \{\widetilde{K}, p_i\}, \quad \frac{\partial q_i}{\partial t} = \{\widetilde{K}, q_i\} \quad (i = 1, \dots, n),$$

where

$$\widetilde{K} = \frac{\mathrm{tr} \widetilde{A}_t \widetilde{A}_1}{t-1} + \frac{\mathrm{tr} \widetilde{A}_t \widetilde{A}_0}{t} + \sum_{i=1}^{n} \frac{p_i q_i}{t}.$$

Hence it is enough to verify that the Hamiltonian $\widetilde{K}$ is equivalent to H given by (1.2).

First we consider a partition of residue matrix

$$\widetilde{A}_\xi = \begin{bmatrix} A_{11}^{(\xi)} & A_{12}^{(\xi)} & A_{13}^{(\xi)} \\ A_{21}^{(\xi)} & A_{22}^{(\xi)} & A_{23}^{(\xi)} \\ A_{31}^{(\xi)} & A_{32}^{(\xi)} & A_{33}^{(\xi)} \end{bmatrix} \quad (\xi = t, 1, 0, \infty),$$

where each block $A_{ij}^{(\xi)}$ is an $n_i \times n_j$ matrix with $(n_1, n_2, n_3) = (1, n-1, n)$. With this block form, the relation (2.8) is described as

$$A_{ij}^{(t)} + A_{ij}^{(1)} + A_{ij}^{(0)} + A_{ij}^{(\infty)} \quad (i, j = 1, 2, 3). \tag{3.2}$$

The Hamiltonian $\widetilde{K}$ is given by

$$\mathrm{tr}\widetilde{A}_t\widetilde{A}_1 = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathrm{tr}A_{ij}^{(t)}A_{ji}^{(1)}, \tag{3.3}$$

and

$$\mathrm{tr}\widetilde{A}_t\widetilde{A}_0 = \theta_0 A_{11}^{(t)} - \mathrm{tr}A_{21}^{(t)}(A_{12}^{(t)} + A_{12}^{(1)}) - \mathrm{tr}A_{31}^{(t)}(A_{13}^{(t)} + A_{13}^{(1)}). \tag{3.4}$$

Note that

$$A_{ij}^{(0)} = 0 \quad (i = 2, 3), \quad A_{12}^{(\infty)} = A_{13}^{(\infty)} = A_{23}^{(\infty)} = A_{32}^{(\infty)} = 0.$$

Next we rewrite the Hamiltonian given by (3.3) and (3.4) into the one expressed in terms of the matrices B_t, C_t, B_1, C_1. Let

$$E_1 = \mathrm{diag}[1, 0, \ldots, 0], \quad E_{2n} = \mathrm{diag}[0, 1, \ldots, 1], \quad E_1 + E_{2n} = I_n.$$

Then the relation (3.2) implies

$$E_1(C_tB_t + C_1B_1)E_1 - (\theta_t + \theta_1 + \theta_0 + \kappa_1)E_1 = 0, \tag{3.5}$$

for $(i, j) = (1, 1)$,

$$E_{2n}(C_tB_t + C_1B_1)E_{2n} - \mathrm{diag}[0, \theta_t + \theta_1 + \kappa_2, \ldots, \theta_t + \theta_1 + \kappa_n] = 0, \tag{3.6}$$

for $(i, j) = (3, 3)$,

$$E_{2n}(C_t + C_1) = 0, \tag{3.7}$$

for $(i, j) = (2, 3)$ and

$$B_tC_t + B_1C_1 + \mathrm{diag}[\kappa_{n+1}, \ldots, \kappa_{2n}] = 0, \tag{3.8}$$

for $(i,j)=(3,3)$. From the relations (3.5), (3.6), (3.7) and (3.8), we obtain

$$
\begin{aligned}
\mathrm{tr}\widetilde{A}_t\widetilde{A}_1 = &(\mathrm{tr}E_1C_tB_1)(\mathrm{tr}E_1C_tB_t) - (\theta_t+\theta_0+\kappa_1)\mathrm{tr}E_1C_tB_1 \\
&- 2(\mathrm{tr}E_1C_tB_t)^2 - (\mathrm{tr}E_1C_tB_t)(\mathrm{tr}E_1C_1B_t) \\
&+ (3\theta_t+\theta_1+2\theta_0+2\kappa_1)\mathrm{tr}E_1C_tB_t + \theta_t\mathrm{tr}E_1C_1B_t \\
&- \mathrm{tr}E_1C_t(B_t-B_1)E_{2n}C_tB_1 - \mathrm{tr}E_1C_1(B_t-B_1)E_{2n}C_tB_t \\
&+ n\theta_t\theta_1 - \frac{1}{2}\sum_{i=2}^{n}(\theta_t+\theta_1-\kappa_i)(\theta_t+\theta_1+\kappa_i) + \frac{1}{2}\sum_{i=1}^{n}\kappa_{n+i}^2 \\
&- \frac{1}{2}(\theta_t+\theta_1+\theta_0+\kappa_1)^2 - \theta^t(\theta_t+\theta_1+\theta_0+\kappa_1),
\end{aligned} \tag{3.9}
$$

and

$$
\begin{aligned}
\mathrm{tr}\widetilde{A}_t\widetilde{A}_0 = &(\mathrm{tr}E_1C_tB_t)^2 + (\mathrm{tr}E_1C_tB_t)(\mathrm{tr}E_1C_1B_t) - (\theta_t+\theta_0)\mathrm{tr}E_1C_tB_t \\
&- \theta_t\mathrm{tr}E_1C_1B_t + \mathrm{tr}E_1C_1(B_t-B_1)E_{2n}C_tB_t + \theta_t\theta_0.
\end{aligned} \tag{3.10}
$$

In order to derive the Hamiltonian H given by (1.2), we introduce the following three lemmas. Here we denote by

$$\beta_1 = -\kappa_{n+1}, \quad \beta_i = -\theta_t - \theta_1 - \kappa_i - \kappa_{n+i} \quad (i=2,\ldots,n).$$

Note that

$$\sum_{i=1}^{n}\beta_i = \theta_t+\theta_1+\theta_0+\kappa_1.$$

Lemma 3.2. *We have relations*

$$\mathrm{tr}E_1C_tB_t = -\sum_{i=1}^{n}q_ip_i, \tag{3.11}$$

$$\mathrm{tr}E_1C_1B_t = t\sum_{i=1}^{n}p_i. \tag{3.12}$$

Lemma 3.3. *We have a relation*

$$\mathrm{tr}E_1C_tB_1 = -\frac{1}{t}\sum_{i=1}^{n}q_i(q_ip_i+\beta_i). \tag{3.13}$$

Lemma 3.4. *We have relations*

$$\begin{aligned}
&\mathrm{tr} E_1 C_1 (B_t - B_1) E_{2n} C_t B_t \\
&= t \sum_{i=1}^{n} p_i \left\{ -\sum_{j=1}^{i-1} (q_j p_j + \beta_j) - \beta_i - \kappa_{n+i} + \sum_{j=i+1}^{n} q_j p_j \right\}, \qquad (3.14) \\
&\mathrm{tr} E_1 C_t (B_t - B_1) E_{2n} C_t B_1 \\
&= -\frac{1}{t} \sum_{i=1}^{n} q_i (q_i p_i + \beta_i) \left\{ \sum_{j=1}^{i-1} q_j p_j - \beta_i - \kappa_{n+i} - \sum_{j=i+1}^{n} (q_j p_j + \beta_j) \right\}. \qquad (3.15)
\end{aligned}$$

Applying Lemma 3.2, 3.3 and 3.4 to (3.9) and (3.10), we obtain the Hamiltonian H by a direct computation. The proofs of these lemmas are given in Section 3.2, 3.3 and 3.4.

3.2 Proof of Lemma 3.2

The first relation of (3.1) is rewritten into

$$P \begin{bmatrix} b_{1,1}^{(t)} \\ \vdots \\ b_{n,1}^{(t)} \end{bmatrix} = t \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}, \qquad (3.16)$$

with an $n \times n$ matrix

$$P = \begin{bmatrix} f_{1,1} & & & O \\ f_{2,1} & f_{2,2} & & \\ \vdots & \vdots & \ddots & \\ f_{n,1} & f_{n,2} & \cdots & f_{n,n} \end{bmatrix},$$

where

$$f_{i,k} = (-1)^{n-i} \frac{\Delta_{i,i+1,\dots,n}^{1,i+1,\dots,n}(1) \Delta_{k,i+1,\dots,n}^{i,i+1,\dots,n}(t)}{\Delta_{i+1,\dots,n}^{i+1,\dots,n}(t) \Delta_{i,\dots,n}^{i,\dots,n}(t)}.$$

We derive the relations (3.11) and (3.12) via an adjoint action of P on the matrices $E_1 C_t B_t$ and $E_1 C_1 B_t$.

The inverse matrix of P is given by

$$P^{-1} = \begin{bmatrix} g_{1,1} & & & O \\ g_{2,1} & g_{2,2} & & \\ \vdots & \vdots & \ddots & \\ g_{n,1} & g_{n,2} & \cdots & g_{n,n} \end{bmatrix},$$

where

$$g_{k,i} = (-1)^{n-k} \frac{\Delta^{i+1,\dots,n}_{i,\dots,\widehat{k},\dots,n}(t)}{\Delta^{1,i+1,\dots,n}_{i,i+1,\dots,n}(1)},$$

and

$$\Delta^{i+1,\dots,n}_{i,\dots,\widehat{k},\dots,n}(\xi) = \begin{vmatrix} c^{(\xi)}_{i+1,i} & \dots & c^{(\xi)}_{i+1,k-1} & c^{(\xi)}_{i+1,k+1} & \dots & c^{(\xi)}_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c^{(\xi)}_{k,i} & \dots & c^{(\xi)}_{k,k-1} & c^{(\xi)}_{k,k+1} & \dots & c^{(\xi)}_{k,n} \\ c^{(\xi)}_{k+1,i} & \dots & c^{(\xi)}_{k+1,k-1} & c^{(\xi)}_{k+1,k+1} & \dots & c^{(\xi)}_{k+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c^{(\xi)}_{n,i} & \dots & c^{(\xi)}_{n,k-1} & c^{(\xi)}_{n,k+1} & \dots & c^{(\xi)}_{n,n} \end{vmatrix}.$$

In fact, (i,j)-component of the matrix $P^{-1}P$ is given by

$$\sum_{k=j}^{i} g_{i,k} f_{k,j} = \begin{cases} 0 & (i<j) \\ 1 & (i=j) \\ \sum_{k=i}^{j} (-1)^{j-k} \frac{\Delta^{k,k+1,\dots,n}_{j,k+1,\dots,n}(t)\Delta^{k+1,\dots,n}_{k,\dots,\widehat{i},\dots,n}(t)}{\Delta^{k+1,\dots,n}_{k+1,\dots,n}(t)\Delta^{k,\dots,n}_{k,\dots,n}(t)} & (i>j) \end{cases}.$$

And we obtain

$$\sum_{k=j}^{i} (-1)^{j-k} \frac{\Delta^{k,k+1,\dots,n}_{j,k+1,\dots,n}(t)\Delta^{k+1,\dots,n}_{k,\dots,\widehat{i},\dots,n}(t)}{\Delta^{k+1,\dots,n}_{k+1,\dots,n}(t)\Delta^{k,\dots,n}_{k,\dots,n}(t)} = 0,$$

by using the Plucker relation

$$\begin{aligned} &\Delta^{k,\dots,n}_{k,\dots,n}(t)\Delta^{k+1,\dots,n}_{j,k+1,\dots,\widehat{i},\dots,n}(t) \\ &= \Delta^{k,\dots,n}_{j,k+1,\dots,n}(t)\Delta^{k+1,\dots,n}_{k,\dots,\widehat{i},\dots,n}(t) - \Delta^{k+1,\dots,n}_{k+1,\dots,n}(t)\Delta^{k,\dots,n}_{j,k,\dots,\widehat{i},\dots,n}(t). \end{aligned} \tag{3.17}$$

The trace of the matrix $E_1 C_t B_t$ is invariant under an adjoint action of P, namely

$$\begin{aligned} \mathrm{tr} E_1 C_t B_t &= \mathrm{tr} B_t E_1 C_t \\ &= \mathrm{tr} P B_t E_1 C_t P^{-1} \\ &= \mathrm{tr} P \begin{bmatrix} b^{(t)}_{1,1} \\ \vdots \\ b^{(t)}_{n,1} \end{bmatrix} \begin{bmatrix} c^{(t)}_{1,1} & \dots & c^{(t)}_{n,1} \end{bmatrix} P^{-1}, \end{aligned} \tag{3.18}$$

On the other hand, we obtain from the second relation of (3.1)

$$\begin{bmatrix} c^{(t)}_{1,1} & \dots & c^{(t)}_{n,1} \end{bmatrix} P^{-1} = -\frac{1}{t} \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix}. \tag{3.19}$$

Combining (3.16), (3.18) and (3.19), we derive the relation (3.11).

In a similar way, we can derive (3.12) by using

$$\begin{bmatrix} c_{1,1}^{(1)} & \dots & c_{n,1}^{(1)} \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}.$$

3.3 Proof of Lemma 3.3

The proof of Lemma 3.3 requires the relation

$$q_i p_i + \beta_i = (-1)^{n-i} \frac{\Delta_{i,i+1,\dots,n}^{1,i+1,\dots,n}(1)}{\Delta_{i+1,\dots,n}^{i+1,\dots,n}(t)} \sum_{k=1}^{i} \frac{\Delta_{k,i+1,\dots,n}^{i,i+1,\dots,n}(t)}{\Delta_{i,\dots,n}^{i,\dots,n}(t)} b_{k,1}^{(1)}, \tag{3.20}$$

or equivalently

$$P \begin{bmatrix} b_{1,1}^{(1)} \\ \vdots \\ b_{n,1}^{(1)} \end{bmatrix} = \begin{bmatrix} q_1 p_1 + \beta_1 \\ \vdots \\ q_n p_n + \beta_n \end{bmatrix}, \tag{3.21}$$

where the matrix P is given in the previous subsection. Recall that the relation (3.7) is explicitly described as

$$c_{i,j}^{(1)} = -c_{i,j}^{(t)} \quad (i = 2, \dots, n).$$

First we show the relation (3.20) for $i = 1$. The relations (3.1) and (3.8) imply

$$q_1 p_1 + \beta_1 = -\frac{\Delta_{1,\dots,n}^{1,\dots,n}(t)}{\Delta_{2,\dots,n}^{2,\dots,n}(t)} b_{1,1}^{(t)} + \sum_{k=1}^{n} (c_{k,1}^{(t)} b_{1,k}^{(t)} + c_{k,1}^{(1)} b_{1,k}^{(1)}).$$

Therefore it is enough to verify that

$$\frac{\Delta_{1,\dots,n}^{1,\dots,n}(t)}{\Delta_{2,\dots,n}^{2,\dots,n}(t)} b_{1,1}^{(t)} - \sum_{k=1}^{n} c_{k,1}^{(t)} b_{1,k}^{(t)} = (-1)^n \frac{\Delta_{1,\dots,n}^{1,\dots,n}(1)}{\Delta_{2,\dots,n}^{2,\dots,n}(t)} b_{1,1}^{(1)} + \sum_{k=1}^{n} c_{k,1}^{(1)} b_{1,k}^{(1)}. \tag{3.22}$$

By using the upper triangle part of (3.8)

$$\sum_{k=1}^{n} (c_{k,j}^{(t)} b_{i,k}^{(t)} + c_{k,j}^{(1)} b_{i,k}^{(1)}) = 0 \quad (i, j = 1, \dots, n; i < j), \tag{3.23}$$

we obtain

$$
\begin{aligned}
(\text{LHS of } (3.22)) &= \sum_{l=2}^{n}(-1)^{l+1}\frac{\Delta^{2,\ldots,n}_{1,\ldots,\widehat{l},\ldots,n}(t)}{\Delta^{2,\ldots,n}_{2,\ldots,n}(t)}c^{(t)}_{1,l}b^{(t)}_{1,1} \\
&\quad+\sum_{k=2}^{n}\sum_{l=2}^{n}(-1)^{l+1}\frac{\Delta^{2,\ldots,n}_{1,\ldots,\widehat{l},\ldots,n}(t)}{\Delta^{2,\ldots,n}_{2,\ldots,n}(t)}c^{(t)}_{k,l}b^{(t)}_{1,k} \\
&= -\sum_{k=1}^{n}\sum_{l=2}^{n}(-1)^{l+1}\frac{\Delta^{2,\ldots,n}_{1,\ldots,\widehat{l},\ldots,n}(t)}{\Delta^{2,\ldots,n}_{2,\ldots,n}(t)}c^{(1)}_{k,l}b^{(1)}_{1,k}.
\end{aligned}
$$

On the other hand, we obtain

$$
\begin{aligned}
(\text{RHS of } (3.22)) &= \sum_{k=1}^{n}(-1)^{n}\frac{\Delta^{k,2,\ldots,n}_{1,2,\ldots,n}(1)}{\Delta^{2,\ldots,n}_{2,\ldots,n}(t)}b^{(1)}_{1,k}+\sum_{k=1}^{n}c^{(1)}_{k,1}b^{(1)}_{1,k} \\
&= -\sum_{k=1}^{n}\sum_{l=2}^{n}(-1)^{l+1}\frac{\Delta^{2,\ldots,n}_{1,\ldots,\widehat{l},\ldots,n}(t)}{\Delta^{2,\ldots,n}_{2,\ldots,n}(t)}c^{(1)}_{k,l}b^{(1)}_{1,k}.
\end{aligned}
$$

Hence the relation (3.22) is derived.

Next we show the relation (3.20) for $i \geq 2$. The relations (3.1), (3.6) and (3.8) imply

$$
\begin{aligned}
q_ip_i+\beta_i &= -\frac{\Delta^{1,i+1,\ldots,n}_{i,i+1,\ldots,n}(t)}{\Delta^{i+1,\ldots,n}_{i+1,\ldots,n}(t)}\sum_{k=1}^{i}\frac{\Delta^{i,i+1,\ldots,n}_{k,i+1,\ldots,n}(t)}{\Delta^{i,\ldots,n}_{i,\ldots,n}(t)}b^{(t)}_{k,1} \\
&\quad-\sum_{j=1}^{n}(c^{(t)}_{i,j}b^{(t)}_{j,i}+c^{(1)}_{i,j}b^{(1)}_{j,i}-c^{(t)}_{j,i}b^{(t)}_{i,j}-c^{(1)}_{j,i}b^{(1)}_{i,j}).
\end{aligned}
$$

Therefore it is enough to verify that

$$
\begin{aligned}
&\sum_{k=1}^{i}\sum_{l=i}^{n}(-1)^{l-i}\frac{\Delta^{i+1,\ldots,n}_{i,\ldots,\widehat{l},\ldots,n}(t)\Delta^{i,i+1,\ldots,n}_{k,i+1,\ldots,n}(t)}{\Delta^{i+1,\ldots,n}_{i+1,\ldots,n}(t)\Delta^{i,\ldots,n}_{i,\ldots,n}(t)}(c^{(t)}_{1,l}b^{(t)}_{k,1}+c^{(1)}_{1,l}b^{(1)}_{k,1}) \\
&\quad+\sum_{j=1}^{n}(c^{(t)}_{i,j}b^{(t)}_{j,i}+c^{(1)}_{i,j}b^{(1)}_{j,i}-c^{(t)}_{j,i}b^{(t)}_{i,j}-c^{(1)}_{j,i}b^{(1)}_{i,j})=0.
\end{aligned}
\tag{3.24}
$$

By using (3.23), we obtain

$$\begin{aligned}
&\sum_{l=i}^{n}(-1)^{l-i}\frac{\Delta^{i+1,\ldots,n}_{i,\ldots,\widehat{l},\ldots,n}(t)\Delta^{i,i+1,\ldots,n}_{k,i+1,\ldots,n}(t)}{\Delta^{i+1,\ldots,n}_{i+1,\ldots,n}(t)\Delta^{i,\ldots,n}_{i,\ldots,n}(t)}(c^{(t)}_{1,l}b^{(t)}_{k,1}+c^{(1)}_{1,l}b^{(1)}_{k,1})\\
&=\sum_{l=i}^{n}(-1)^{l-i+1}\frac{\Delta^{i+1,\ldots,n}_{i,\ldots,\widehat{l},\ldots,n}(t)\Delta^{i,i+1,\ldots,n}_{k,i+1,\ldots,n}(t)}{\Delta^{i+1,\ldots,n}_{i+1,\ldots,n}(t)\Delta^{i,\ldots,n}_{i,\ldots,n}(t)}\sum_{j=2}^{n}(c^{(t)}_{j,l}b^{(t)}_{k,j}+c^{(1)}_{j,l}b^{(1)}_{k,j}) \qquad (3.25)\\
&=-\sum_{j=2}^{i}\frac{\Delta^{j,i+1,\ldots,n}_{i,i+1,\ldots,n}(t)\Delta^{i,i+1,\ldots,n}_{k,i+1,\ldots,n}(t)}{\Delta^{i+1,\ldots,n}_{i+1,\ldots,n}(t)\Delta^{i,\ldots,n}_{i,\ldots,n}(t)}(b^{(t)}_{k,j}-b^{(1)}_{k,j}),
\end{aligned}$$

for $k=1,\ldots,i-1$ and

$$\begin{aligned}
&\sum_{l=i}^{n}(-1)^{l-i}\frac{\Delta^{i+1,\ldots,n}_{i,\ldots,\widehat{l},\ldots,n}(t)}{\Delta^{i+1,\ldots,n}_{i+1,\ldots,n}(t)}(c^{(t)}_{1,l}b^{(t)}_{i,1}+c^{(1)}_{1,l}b^{(1)}_{i,1})-\sum_{j=1;j\neq i}^{n}(c^{(t)}_{j,i}b^{(t)}_{i,j}+c^{(1)}_{j,i}b^{(1)}_{i,j})\\
&=\sum_{l=i}^{n}(-1)^{l-i+1}\frac{\Delta^{i+1,\ldots,n}_{i,\ldots,\widehat{l},\ldots,n}(t)}{\Delta^{i+1,\ldots,n}_{i+1,\ldots,n}(t)}\sum_{j=2}^{n}(c^{(t)}_{j,l}b^{(t)}_{i,j}+c^{(1)}_{j,l}b^{(1)}_{i,j})+c^{(t)}_{i,i}b^{(t)}_{i,i}+c^{(1)}_{i,i}b^{(1)}_{i,i} \qquad (3.26)\\
&=-\sum_{j=2}^{i}\frac{\Delta^{j,i+1,\ldots,n}_{i,i+1,\ldots,n}(t)}{\Delta^{i+1,\ldots,n}_{i+1,\ldots,n}(t)}(b^{(t)}_{i,j}-b^{(1)}_{i,j})+c^{(t)}_{i,i}b^{(t)}_{i,i}+c^{(1)}_{i,i}b^{(1)}_{i,i}.
\end{aligned}$$

The relations (3.25) and (3.26) imply

$$\begin{aligned}
(\text{LHS of (3.24)})=&-\sum_{k=1}^{i}\sum_{j=2}^{i}\frac{\Delta^{j,i+1,\ldots,n}_{i,i+1,\ldots,n}(t)\Delta^{i,i+1,\ldots,n}_{k,i+1,\ldots,n}(t)}{\Delta^{i+1,\ldots,n}_{i+1,\ldots,n}(t)\Delta^{i,\ldots,n}_{i,\ldots,n}(t)}(b^{(t)}_{k,j}-b^{(1)}_{k,j})\\
&+\sum_{k=1}^{n}c^{(t)}_{i,k}(b^{(t)}_{k,i}-b^{(1)}_{k,i}).
\end{aligned} \qquad (3.27)$$

Furthermore, by using the lower triangle part of (3.6)

$$\sum_{k=1}^{n}(c^{(t)}_{i,k}b^{(t)}_{k,j}+c^{(1)}_{i,k}b^{(1)}_{k,j})=0 \quad (i,j=2,\ldots,n;i>j),$$

we obtain

$$
\begin{aligned}
&-\sum_{k=1}^{i} \frac{\Delta^{j,i+1,\dots,n}_{i,i+1,\dots,n}(t)\Delta^{i,i+1,\dots,n}_{k,i+1,\dots,n}(t)}{\Delta^{i+1,\dots,n}_{i+1,\dots,n}(t)\Delta^{i,\dots,n}_{i,\dots,n}(t)}(b^{(t)}_{k,j}-b^{(1)}_{k,j})\\
&=-\sum_{k=1}^{i}\sum_{l=i}^{n}(-1)^{l-i}\frac{\Delta^{j,i+1,\dots,n}_{i,i+1,\dots,n}(t)\Delta^{i,\dots,\widehat{l},\dots,n}_{i+1,\dots,n}(t)}{\Delta^{i+1,\dots,n}_{i+1,\dots,n}(t)\Delta^{i,\dots,n}_{i,\dots,n}(t)}c^{(t)}_{l,k}(b^{(t)}_{k,j}-b^{(1)}_{k,j})\\
&=\sum_{k=i+1}^{n}\sum_{l=i}^{n}(-1)^{l-i}\frac{\Delta^{j,i+1,\dots,n}_{i,i+1,\dots,n}(t)\Delta^{i,\dots,\widehat{l},\dots,n}_{i+1,\dots,n}(t)}{\Delta^{i+1,\dots,n}_{i+1,\dots,n}(t)\Delta^{i,\dots,n}_{i,\dots,n}(t)}c^{(t)}_{l,k}(b^{(t)}_{k,j}-b^{(1)}_{k,j})\\
&=0,
\end{aligned}
$$

for $j=2,\dots,i-1$. It follows that

$$
\begin{aligned}
(\text{RHS of } (3.27)) &= -\sum_{k=1}^{i}\frac{\Delta^{i,i+1,\dots,n}_{k,i+1,\dots,n}(t)}{\Delta^{i+1,\dots,n}_{i+1,\dots,n}(t)}(b^{(t)}_{k,i}-b^{(1)}_{k,i})+\sum_{k=1}^{n}c^{(t)}_{i,k}(b^{(t)}_{k,i}-b^{(1)}_{k,i})\\
&=\sum_{k=1}^{n}\left(c^{(t)}_{i,k}-\frac{\Delta^{i,i+1,\dots,n}_{k,i+1,\dots,n}(t)}{\Delta^{i+1,\dots,n}_{i+1,\dots,n}(t)}\right)(b^{(t)}_{k,i}-b^{(1)}_{k,i})\\
&=-\sum_{k=1}^{n}\sum_{l=i+1}^{n}(-1)^{l-i}\frac{\Delta^{i,\dots,\widehat{l},\dots,n}_{i+1,\dots,n}(t)}{\Delta^{i+1,\dots,n}_{i+1,\dots,n}(t)}(c^{(t)}_{l,k}b^{(t)}_{k,i}+c^{(1)}_{l,k}b^{(1)}_{k,i})\\
&=0.
\end{aligned}
\tag{3.28}
$$

Thanks to (3.21), we can prove Lemma 3.3 in a similar manner as Section 3.2.

3.4 Proof of Lemma 3.4

In this subsection, we derive the relation (3.14). The relation (3.15) is derived in a similar way.

The relation (3.8) rewrite (3.14) into

$$
\begin{aligned}
&\operatorname{tr}\begin{bmatrix}\kappa_{n+1} & & O\\ & \ddots & \\ O & & \kappa_{2n}\end{bmatrix}\begin{bmatrix}b^{(t)}_{1,1}\\ \vdots\\ b^{(t)}_{n,1}\end{bmatrix}\begin{bmatrix}c^{(t)}_{1,1} & \dots & c^{(t)}_{n,1}\end{bmatrix}\\
&=t\sum_{i=1}^{n}p_i\left\{\sum_{j=1}^{i-1}q_jp_j-\kappa_{n+i}-\sum_{j=i+1}^{n}(q_jp_j+\beta_j)\right\}.
\end{aligned}
\tag{3.29}
$$

Here we have

$$(\text{LHS of } (3.29)) = \text{tr} P \begin{bmatrix} \kappa_{n+1} & & O \\ & \ddots & \\ O & & \kappa_{2n} \end{bmatrix} P^{-1} P \begin{bmatrix} b_{1,1}^{(t)} \\ \vdots \\ b_{n,1}^{(t)} \end{bmatrix} \begin{bmatrix} c_{1,1}^{(t)} & \dots & c_{n,1}^{(t)} \end{bmatrix} P^{-1}$$
$$= \text{tr} P \begin{bmatrix} \kappa_{n+1} & & O \\ & \ddots & \\ O & & \kappa_{2n} \end{bmatrix} P^{-1} \begin{bmatrix} tp_1 \\ \vdots \\ tp_n \end{bmatrix} \begin{bmatrix} 1 & \dots & 1 \end{bmatrix},$$

and

$$(\text{RHS of } (3.29)) = \text{tr} \begin{bmatrix} \kappa_{n+1} & & & & O \\ \varphi_{2,1} & \kappa_{n+2} & & & \\ \varphi_{3,1} & \varphi_{3,2} & \kappa_{n+3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \varphi_{n,1} & \varphi_{n,2} & \varphi_{n,3} & \dots & \kappa_{2n} \end{bmatrix} \begin{bmatrix} tp_1 \\ \vdots \\ tp_n \end{bmatrix} \begin{bmatrix} 1 & \dots & 1 \end{bmatrix},$$

where

$$\varphi_{j,i} = q_i p_j - q_j p_j - \beta_j.$$

Therefore it is enough to verify that

$$\begin{bmatrix} \kappa_{n+1} & & O \\ & \ddots & \\ O & & \kappa_{2n} \end{bmatrix} P^{-1} = P^{-1} \begin{bmatrix} \kappa_{n+1} & & & & O \\ \varphi_{2,1} & \kappa_{n+2} & & & \\ \varphi_{3,1} & \varphi_{3,2} & \kappa_{n+3} & & \\ \vdots & \vdots & \vdots & \ddots & \\ \varphi_{n,1} & \varphi_{n,2} & \varphi_{n,3} & \dots & \kappa_{2n} \end{bmatrix},$$

or equivalently,

$$(\kappa_{n+k} - \kappa_{n+i}) g_{k,i} = \sum_{j=i+1}^{k} g_{k,j} \varphi_{j,i} \quad (i, k = 1, \dots, n; i < k). \tag{3.30}$$

We recall that

$$g_{k,i} = (-1)^{n-k} \frac{\Delta_{i,\dots,\widehat{k},\dots,n}^{i+1,\dots,n}(t)}{\Delta_{i,i+1,\dots,n}^{1,i+1,\dots,n}(1)}.$$

In the following, the indices $i < k$ are fixed. By using the Plucker relation (3.17), we obtain

$$b_{k,1}^{(t)} = \sum_{j=1}^{k} (-1)^{k-j} \frac{\Delta_{j,\dots,\widehat{k},\dots,n}^{j+1,\dots,n}(t)}{\Delta_{j+1,\dots,n}^{j+1,\dots,n}(t)} \sum_{l=1}^{j} \frac{\Delta_{l,j+1,\dots,n}^{j,j+1,\dots,n}(t)}{\Delta_{j,\dots,n}^{j,\dots,n}(t)} b_{l,1}^{(t)},$$
$$b_{k,1}^{(1)} = \sum_{j=1}^{k} (-1)^{k-j} \frac{\Delta_{j,\dots,\widehat{k},\dots,n}^{j+1,\dots,n}(t)}{\Delta_{j+1,\dots,n}^{j+1,\dots,n}(t)} \sum_{l=1}^{j} \frac{\Delta_{l,j+1,\dots,n}^{j,j+1,\dots,n}(t)}{\Delta_{j,\dots,n}^{j,\dots,n}(t)} b_{l,1}^{(1)},$$

and

$$\sum_{j=l}^{i}(-1)^{k-j}\frac{\Delta^{j+1,\dots,n}_{j,\dots,\widehat{k},\dots,n}(t)\Delta^{j,j+1,\dots,n}_{l,j+1,\dots,n}(t)}{\Delta^{j+1,\dots,n}_{j+1,\dots,n}(t)\Delta^{j,\dots,n}_{j,\dots,n}(t)} = (-1)^{k-i}\frac{\Delta^{i+1,\dots,n}_{l,i+1,\dots,\widehat{k},\dots,n}(t)}{\Delta^{i+1,\dots,n}_{i+1,\dots,n}(t)}.$$

They imply

$$\begin{aligned}
&(\text{RHS of (3.30)}) - (\text{LHS of (3.30)}) \\
&= (-1)^{n-k}\frac{\Delta^{i+1,\dots,n}_{i,\dots,\widehat{k},\dots,n}(t)}{\Delta^{1,i+1,\dots,n}_{i,i+1,\dots,n}(1)}\sum_{j=1}^{n}(c^{(t)}_{j,i}b^{(t)}_{i,j} + c^{(1)}_{j,i}b^{(1)}_{i,j} - c^{(t)}_{j,k}b^{(t)}_{k,j} - c^{(1)}_{j,k}b^{(1)}_{k,j}) \\
&\quad + (-1)^{n-i}\frac{\Delta^{1,i+1,\dots,n}_{i,i+1,\dots,n}(t)}{\Delta^{1,i+1,\dots,n}_{i,i+1,\dots,n}(1)}b^{(t)}_{k,1} + b^{(1)}_{k,1} \\
&\quad - \sum_{l=1}^{i}(-1)^{k-i}\frac{\Delta^{i+1,\dots,n}_{l,i+1,\dots,\widehat{k},\dots,n}(t)}{\Delta^{i+1,\dots,n}_{i+1,\dots,n}(t)}\left\{(-1)^{n-i}\frac{\Delta^{1,i+1,\dots,n}_{i,i+1,\dots,n}(t)}{\Delta^{1,i+1,\dots,n}_{i,i+1,\dots,n}(1)}b^{(t)}_{l,1} + b^{(1)}_{l,1}\right\}.
\end{aligned} \tag{3.31}$$

Furthermore, by using (3.7) and (3.8), we obtain

$$\begin{aligned}
&(-1)^{n-i}\Delta^{1,i+1,\dots,n}_{i,i+1,\dots,n}(t)b^{(t)}_{k,1} + \Delta^{1,i+1,\dots,n}_{i,i+1,\dots,n}(1)b^{(1)}_{k,1} \\
&= (-1)^{n-k}\Delta^{i+1,\dots,n}_{i,\dots,\widehat{k},\dots,n}(t)(c^{(t)}_{1,k}b^{(t)}_{k,1} + c^{(1)}_{1,k}b^{(1)}_{k,1}) \\
&\quad - \sum_{l=i;l\neq k}^{n}(-1)^{n-l}\Delta^{i+1,\dots,n}_{i,\dots,\widehat{l},\dots,n}(t)\sum_{j=2}^{n}(c^{(t)}_{j,l}b^{(t)}_{k,j} + c^{(1)}_{j,l}b^{(1)}_{k,j}) \\
&= (-1)^{n-k}\Delta^{i+1,\dots,n}_{i,\dots,\widehat{k},\dots,n}(t)\sum_{j=1}^{n}(c^{(t)}_{j,k}b^{(t)}_{k,j} + c^{(1)}_{j,k}b^{(1)}_{k,j}) \\
&\quad - \sum_{j=2}^{i}(-1)^{n-i}\Delta^{j,i+1,\dots,n}_{i,\dots,n}(t)(b^{(t)}_{k,j} - b^{(1)}_{k,j}),
\end{aligned}$$

and

$$\sum_{l=1}^{i}(-1)^{k-i}\Delta^{i+1,\dots,n}_{l,i+1,\dots,\widehat{k},\dots,n}(t)\left\{(-1)^{n-i}\Delta^{1,i+1,\dots,n}_{i,i+1,\dots,n}(t)b^{(t)}_{l,1}+\Delta^{1,i+1,\dots,n}_{i,i+1,\dots,n}(1)b^{(1)}_{l,1}\right\}$$
$$=(-1)^{n-k}\Delta^{i+1,\dots,n}_{i,\dots,\widehat{k},\dots,n}(t)\Delta^{i+1,\dots,n}_{i+1,\dots,n}(t)(c^{(t)}_{1,i}b^{(t)}_{i,1}+c^{(1)}_{1,i}b^{(1)}_{i,1})$$
$$+\sum_{l=1}^{i-1}(-1)^{n-k}\Delta^{i+1,\dots,n}_{l,i+1,\dots,\widehat{k},\dots,n}(t)\Delta^{i+1,\dots,n}_{i+1,\dots,n}(t)(c^{(t)}_{1,i}b^{(t)}_{l,1}+c^{(1)}_{1,i}b^{(1)}_{l,1})$$
$$+\sum_{l=1}^{i}(-1)^{n-k}\Delta^{i+1,\dots,n}_{l,i+1,\dots,\widehat{k},\dots,n}(t)\sum_{r=i+1}^{n}(-1)^{r-i}\Delta^{i+1,\dots,n}_{i,\dots,\widehat{r},\dots,n}(t)(c^{(t)}_{1,r}b^{(t)}_{l,1}+c^{(1)}_{1,r}b^{(1)}_{l,1})$$
$$=(-1)^{n-k}\Delta^{i+1,\dots,n}_{i,\dots,\widehat{k},\dots,n}(t)\Delta^{i+1,\dots,n}_{i+1,\dots,n}(t)\sum_{j=1}^{n}(c^{(t)}_{j,i}b^{(t)}_{i,j}+c^{(1)}_{j,i}b^{(1)}_{i,j})$$
$$-\sum_{l=1}^{i}(-1)^{n-k}\Delta^{i+1,\dots,n}_{l,i+1,\dots,\widehat{k},\dots,n}(t)\sum_{j=2}^{n}\Delta^{j,i+1,\dots,n}_{i,\dots,n}(t)(b^{(t)}_{l,j}-b^{(1)}_{l,j}).$$

They imply

$$\text{(RHS of (3.31))}=-\sum_{j=2}^{i}\frac{\Delta^{j,i+1,\dots,n}_{i,\dots,n}(t)}{\Delta^{1,i+1,\dots,n}_{i,i+1,\dots,n}(1)}(-1)^{n-i}(b^{(t)}_{k,j}-b^{(1)}_{k,j})$$
$$+\sum_{j=2}^{i}\frac{\Delta^{j,i+1,\dots,n}_{i,\dots,n}(t)}{\Delta^{1,i+1,\dots,n}_{i,i+1,\dots,n}(1)}\sum_{l=1}^{i}(-1)^{n-k}\frac{\Delta^{i+1,\dots,n}_{l,i+1,\dots,\widehat{k},\dots,n}(t)}{\Delta^{i+1,\dots,n}_{i+1,\dots,n}(t)}(b^{(t)}_{l,j}-b^{(1)}_{l,j}). \tag{3.32}$$

The right-hand side of (3.32) is equivalent to zero. In fact, we have

$$\sum_{l=1}^{i}\Delta^{i+1,\dots,n}_{l,i+1,\dots,\widehat{k},\dots,n}(t)(b^{(t)}_{l,j}-b^{(1)}_{l,j})$$
$$=\sum_{l=1}^{i}\sum_{r=i+1}^{n}(-1)^{r-i-1}\Delta^{i+1,\dots,\widehat{r},\dots,n}_{i+1,\dots,\widehat{k},\dots,n}(t)c^{(t)}_{r,l}(b^{(t)}_{l,j}-b^{(1)}_{l,j})$$
$$=-\sum_{l=i+1}^{n}\sum_{r=i+1}^{n}(-1)^{r-i-1}\Delta^{i+1,\dots,\widehat{r},\dots,n}_{i+1,\dots,\widehat{k},\dots,n}(t)c^{(t)}_{r,l}(b^{(t)}_{l,j}-b^{(1)}_{l,j})$$
$$=(-1)^{k-i}\Delta^{i+1,\dots,n}_{i+1,\dots,n}(t)(b^{(t)}_{l,j}-b^{(1)}_{l,j}).$$

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References

[FS1] K. Fuji and T. Suzuki, Higher order Painlevé system of type $D^{(1)}_{2n+2}$ arising from integrable hierarchy, Int. Math. Res. Not. **1** (2008), 1-21.

[FS2] K. Fuji and T. Suzuki, Drinfeld-Sokolov hierarchies of type A and fourth order Painlevé systems, Funkcial. Ekvac. **53** (2010), 143-167.

[G] R. Garnier, Sur des équations différentielles du troisiéme ordre dont l'intégrale est uniform et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses point critiques fixés, Ann. Sci. École Norm. Sup. **29** (1912), 1-126.

[JMMS] M. Jimbo, T. Miwa, Y. Mori and M. Sato, Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent, Physica **1D** (1980), 80-158.

[JMU] M. Jimbo, T.Miwa and K.Ueno, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients I, Physica **2D** (1981), 306-352.

[O] T. Oshima, Classification of Fuchsian systems and their connection problem, preprint (arXiv:0811.2916).

[Su] T. Suzuki, A class of higher order Painlevé systems arising from integrable hierarchies of type A, preprint (arXiv:1002.2685).

[Sak] H. Sakai, Monodromy preserving deformation and four-dimensional Painlevé equation, RIMS Kokyuroku **1662** (2009), 65-72 (in Japanese).

[SY] Y. Sasano and Y.Yamada, Symmetry and holomorphy of Painlevé type systems, RIMS Kokyuroku Bessatsu **B2** (2007), 215-225.

[T] T. Tsuda, UC hierarchy and monodromy preserving deformation, MI Preprint Series **7** (Kyushu Univ., 2010), 1-31.