Graphs with few matching roots

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Abstract

We determine all graphs whose matching polynomials have at most five distinct zeros. As a consequence, we find new families of graphs which are determined by their matching polynomial. In particular, we show that for any positive integer $n \neq 2$, the friendship graph F_n (the graph consisting of n triangles intersecting in a single vertex) is determined by its matching polynomial.

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1 Introduction

All the graphs that we consider in this paper are finite, simple and undirected. Let G be a graph. Throughout this paper the *order* of G is the number of vertices of G. A k*-matching* in G is a set of k pairwise nonincident edges and the number of k-matchings in G is denoted by $m(G,k)$. If G is of order n, the *matching polynomial* $\mu(G,x)$ is defined by

$$\mu(G,x) = \sum_{k\geq 0} (-1)^k m(G,k) x^{n-2k},$$

where $m(G,0)$ is considered to be 1. The roots of matching polynomial of any graph are all real numbers. (This was first proved independently in [14] and [17].) The matching polynomial is related to the characteristic polynomial of G, which is defined to be the characteristic polynomial of the adjacency matrix of G. In particular these two coincide if and only if G is a forest [13]. Also the matching polynomial of any connected graph is a factor of the characteristic polynomial of some tree (see [11, Theorem 6.1.1]). This is another way to see that the roots of matching polynomial are real numbers because the adjacency matrix of any graph is a symmetric matrix and so the roots of its characteristic polynomial are real numbers. The roots of $\mu(G, x)$ are called the *matching roots* of G. Two nonisomorphic graphs with the same matching polynomials are said to be *comatching*. A graph G is said to be *matching unique* if it has no comatching graph. We also denote the multiset of the roots of the matching polynomial of G by $R(G)$. We use exponent symbol to show the multiplicities of the elements of $R(G)$.

The determination of graphs with few distinct roots of characteristic polynomials of matrices associated to graphs (i.e. graphs with few distinct eigenvalues) have been the subject of many researches. Graphs with three adjacency eigenvalues have been studied by Bridges and Mena [3], Klin and Muzychuk [16], and van Dam [5, 6]. Connected regular graphs with four distinct adjacency eigenvalues have been studied by Doob [9, 10], van Dam [5], and van Dam and Spence [8]. Graphs with three Laplacian eigenvalues have been treated by van Dam and Haemers [7]. Ayoobi, Omidi and Tayfeh-Rezaie [1] investigated nonregular graphs whose signless Laplacian matrix has three distinct eigenvalues. For a complete survey on this subject see Chapter 14 of Brouwer and Haemers [4].

So far, few families of graphs have been shown to be matching unique; these include unique cages (regular graphs with minimum number of vertices and given degree and girth), 2-regular graphs, $mK_{r,r}$, mL, where L is a unique Moore graph with given degree and odd girth, and the regular complete multipartite graphs. It is also known that if a graph is matching unique, then its complement is also matching unique (see [2]).

In this paper, we determine all graphs with at most five distinct matching roots. As a result, we find new families of matching unique graphs. In particular, we show that for any positive integer $n \neq 2$, the friendship graph F_n (the graph consisting of n triangles intersecting in a single vertex) is matching unique.

2 Graphs with few matching roots

We denote the complete graph of order n by K_n and the complete bipartite graph with parts of sizes r and s by $K_{r,s}$. The graph $K_{1,s}$ is called a *star*.

The roots of the matching polynomial of any graph, like those of characteristic polynomial, have the "interlacing" property ([15], see also [11, Corollary 6.1.3]):

Lemma 1. *Let G be a graph and u be a vertex of that. Then the roots of $\mu(G-u, x)$ interlace those of $\mu(G, x)$, i.e. if $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_{n-1}$ are the matching roots of G*

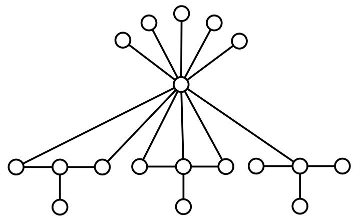

Figure 1: The graph $\mathcal{S}(3,3,5;9,2)$

and $G-u$, respectively, then

$$\theta_1 \geq \eta_1 \geq \theta_2 \geq \eta_2 \geq \cdots \geq \theta_{n-1} \geq \eta_{n-1} \geq \theta_n.$$

Let $\mathrm{mult}(\theta, G)$ denote the multiplicity of θ as a root of $\mu(G, x)$.

Lemma 2. (Godsil [12]) *For a connected graph G, if $\mathrm{mult}(\theta, G) \geq 2$, then there is a vertex u of G such that $\mathrm{mult}(\theta, G-u) \geq \mathrm{mult}(\theta, G)$.*

Remark 3. Any graph with an odd number of vertices has a zero matching root and if θ is a matching root of a graph, then so is $-\theta$.

Lemma 4. *Let G be connected graph. If the roots of $\mu(G, x)$ are ≥ -1, then $G \simeq K_2$.*

Proof. Let G be of order n. Since the roots of $\mu(K_{1,2}, x)$ are $\{0, \pm\sqrt{2}\}$, by interlacing, G has no $K_{1,2}$ as an induced subgraph. Thus G must be the complete graph K_n. If $n \geq 3$, then, by interlacing, $\mu(K_n, x)$ has a root $\leq -\sqrt{3}$ because $\mu(K_3, x) = x^3 - 3x$. This implies $G \simeq K_2$. $\square$

We define a family of graphs which will be used frequently. Add a single vertex u to the graph $rK_{1,k} \cup tK_1$ and join u to the other vertices by $p+q$ edges so that the resulting graph is connected and u is adjacent with q centers of the stars (for $K_{1,1}$ either of the vertices may be considered as center). We denote the resulting graph by $\mathcal{S}(r,k,t;p,q)$. (The graph $\mathcal{S}(3,3,5;9,2)$ is depicted in Figure 1.) Clearly $r+t \leq p+q \leq r(k+1)+t$ and $0 \leq q \leq r$.

Lemma 5. $\mu(\mathcal{S}(r,k,t;p,q), x) = x^{r(k-1)+t-1}\left(x^4 - (p+q+k)x^2 + t + p(k-1)\right)(x^2-k)^{r-1}$.

Proof. The graph $\mathcal{S}(r,k,t;p,q)$ has a vertex u such that $\mathcal{S}(r,k,t;p,q) - u = rK_{1,k} \cup tK_1$. Since $\mu(K_{1,k}, x) = x^{k-1}(x^2-k)$, by interlacing, $\{(\pm\sqrt{k})^{r-1}, 0^{r(k-1)+t-1}\} \subseteq R(G)$. Let $\pm\alpha$ and $\pm\beta$ be the remaining elements of $R(G)$. Since the squares of the roots of matching polynomial of a graph sum to its number of edges, we have $\alpha^2+\beta^2+(r-1)k = rk+p+q$. We note that the multiplicity of zero is equal to the vertices missed by a fix maximum matching. Since the maximum matching of $\mathcal{S}(r,k,t;p,q)$ has $r+1$ edges, the multiplicity of zero is $r(k-1)+t-1$. Thus α and β are nonzero. As the product of the squares of the nonzero roots of matching polynomial of a graph is equal to its number of maximum matchings,

$$\alpha^2\beta^2k^{r-1} = m(\mathcal{S}(r,k,t;p,q),\, r+1) = tk^r + (p-t)(k-1)k^{r-1}.$$

Therefore,

$$\alpha^2+\beta^2 = p+q+k, \quad \text{and} \quad \alpha^2\beta^2 = t+p(k-1).$$

The result now follows. □

We distinguish some special cases which are important for our purpose. We denote the graph $\mathcal{S}(r,1,0;p,q)$ by $\mathcal{S}(r;s)$ where $s=p+q$. Its matching polynomial is

$$\mu(\mathcal{S}(r;s),x) = x(x^2-s-1)(x^2-1)^{r-1}.$$

We also denote $\mathcal{S}(1,1,t;t+1,0)$ and $\mathcal{S}(1,1,t;t+1,1)$ by $\mathcal{S}(t)$ and $\mathcal{S}'(t)$, respectively. In fact, $\mathcal{S}(t)$ is the graph obtained by joining a new vertex to one of the pendant vertices of $K_{1,t+1}$ and $\mathcal{S}'(t)$ is the resulting graph of joining two pendant vertices of $K_{1,t+2}$. Their matching polynomials are

$$\mu(\mathcal{S}(t),x) = x^{t-1}\left(x^4-(t+2)x^2+t\right),$$
$$\mu(\mathcal{S}'(t),x) = x^{t-1}\left(x^4-(t+3)x^2+t\right).$$

Moreover, we denote $\mathcal{S}(1,k,0;s,0)$ and $\mathcal{S}(1,k,0;s,1)$ by $\mathcal{K}(k;s)$ and $\mathcal{K}'(k;s)$, respectively. We have

$$\mu(\mathcal{K}(k;s),x) = x^{k-2}\left(x^4-(k+s)x^2+s(k-1)\right),$$
$$\mu(\mathcal{K}'(k;s),x) = x^{k-2}\left(x^4-(k+s+1)x^2+s(k-1)\right).$$

Theorem 6. *Let G be a connected graph and $z(G)$ be the number of its distinct matching roots.*

(i) *If $z(G)=2$, then $G\simeq K_2$.*

(ii) *If $z(G)=3$, then G is either a star or K_3.*

(iii) *If $z(G)=4$, then G is a non-star graph with 4 vertices.*

(iv) *If $z(G)=5$, then G is one of the graphs $\mathcal{K}(k;s)$, $\mathcal{K}'(k;s)$, $\mathcal{S}(t)$, $\mathcal{S}'(t)$, $\mathcal{S}(r;s)$, for some integers k,r,s,t, or a connected non-star graph with 5 vertices.*

Proof. (i) If $z(G) = 2$, then $R(G) = \{(\pm\alpha)^r\}$, for some $\alpha \neq 0$. Thus, from Lemma 2 it follows that $r = 1$, and so $G \simeq K_2$.

(ii) If $z(G) = 3$, then $R(G) = \{(\pm\alpha)^r, 0^s\}$, for some $\alpha \neq 0$. If $r \geq 2$, then for some vertex u, $\{0^{s-1}, (\pm\alpha)^r\} \subseteq R(G - u)$. Considering the coefficients of the matching polynomials of G and $G - u$, it turns out that G and $G - u$ have the same number of edges, a contradiction. So $r = 1$ and we have $m(G, 2) = 0$. Hence, G is either the star $K_{1,s+1}$ or K_3.

(iii) If $z(G) = 4$, then $R(G) = \{(\pm\alpha)^r, (\pm\beta)^s\}$, for some nonzero α, β. By Lemma 2, it is impossible to have both r and s greater than 1. So we may let $s = 1$. If $r \geq 2$, then, there is some vertex u such that $R(G - u) = \{0, (\pm\alpha)^r\}$. Since $r \geq 2$, $G - u$ cannot be connected by part (ii). Hence, $G - u$ is disconnected, and by part (i), $\alpha = 1$ and $G - u \simeq rK_2 \cup K_1$. Thus $G \simeq \mathcal{S}(r, 1, 1; p, q)$, for some p, q. But this graph has more than five nonzero distinct matching roots by Lemma 5, a contradiction. Thus $r = 1$ and so G has four vertices. It turns out that G can be any connected non-star graph on four vertices as such a graph has a matching of size 2.

(iv) If $z(G) = 5$, then $R(G) = \{(\pm\alpha)^r, (\pm\beta)^s, 0^t\}$, for some nonzero α, β. Again either of r or s is necessarily equal to 1. Let s be so.

If $t = r = 1$, then G is a graph on 5 vertices with $m(G, 2) > 0$. So G can be any connected non-star graph on 5 vertices.

If $t = 1$ and $r \geq 2$, then $R(G - u)$ is either $\{(\pm\alpha)^r, 0^2\}$ or $\{(\pm\alpha)^r, \pm\eta\}$, for some $\eta \neq 0$. If $R(G - u) = \{(\pm\alpha)^r, 0^2\}$, then $G - u \simeq rK_2 \cup 2K_1$ or $G - u \simeq 2K_{1,2}$. This implies that $G \simeq \mathcal{S}(r, 1, 2; p, q)$ or $G \simeq \mathcal{S}(2, 2, 0; p, q)$, for some p, q. By Lemma 5, these two graphs have more than five distinct matching roots, a contradiction. Therefore, $R(G - u) = \{(\pm\alpha)^r, \pm\eta\}$. If $\eta \neq \alpha$, then $\alpha = 1$ and $G - u$ must be a graph of the form $G_1 \cup (r - 1)K_2$ where G_1 is a non-star graph of order 4 with $\pm 1 \in R(G_1)$. But there is no graph of order 4 with ± 1 as root of its matching polynomial (see Appendix). Thus $\eta = \alpha$. It follows that $\alpha = 1$ and $G - u \simeq (r + 1)K_2$. Thus, for some s, $G \simeq \mathcal{S}(r + 1; s)$. By Lemma 5, $s = \beta^2 - 1$. Therefore $G \simeq \mathcal{S}(r + 1; \beta^2 - 1)$.

If $t \geq 2$ and $r = 1$, then there is a vertex u such that $R(G - u) = \{\pm\eta, 0^{t+1}\}$, for some $\eta \neq 0$. Therefore, $G - u$ is either $K_{1,t+2}$ or $K_2 \cup (t + 1)K_1$. If $G - u \simeq K_{1,t+2}$, then either $G \simeq \mathcal{K}(t + 2; s)$ or $G \simeq \mathcal{K}'(t+2; s)$, for some s. If $G - u \simeq K_2 \cup (t+1)K_1$, then either $G \simeq \mathcal{S}(t+1)$ or $G \simeq \mathcal{S}'(t+1)$.

If $t \geq 2$ and $r \geq 2$, then, for some vertex u, $R(G-u) = \{(\pm\alpha)^r, 0^{t+1}\}$. It turns out that $\alpha = \sqrt{k}$, for some integer k, and $G - u = rK_{1,k} \cup t'K_1$, where $t' = t + 1 - r(k - 1)$. So $G \simeq \mathcal{S}(r, k, t'; p, q)$, for some p, q. Note that the polynomial $x^4 - (p + q + k)x^2 + t + p(k - 1)$, with the conditions on k, p, q, t arisen by the definition of $\mathcal{S}(r, k, t; p, q)$, has no $\pm\sqrt{k}$ root and it has zero root if and only if $k = 1$ and $t = 0$. So $G \simeq \mathcal{S}(r, k, t'; p, q)$ has five nonzero distinct matching roots if and only if $k = 1$ and $t' = 0$. But if $k = 1$, then $t' \geq 3$ and this graph has more than five distinct matching roots, a contradiction. □

3 Characterization by matching polynomial

In this section we characterize the graphs $\mathcal{S}(r;s)$, $\mathcal{K}(k;s)$, $\mathcal{K}'(k;s)$, $\mathcal{S}(t)$, and $\mathcal{S}'(t)$ by their matching polynomials.

We remark that if $r \le r' \le s \le 2r$, then the graphs $\mathcal{S}(r';s)$ and $\mathcal{S}(r;s) \cup (r'-r)K_2$ have the same matching polynomials. So in general, for a given s, the graphs $\mathcal{S}(r;s)$ are not matching unique unless for the smallest value of r such that $\mathcal{S}(r;s)$ can be defined (these include the friendship graph $F_n \simeq \mathcal{S}(n;2n)$). This is shown below.

Theorem 7. *For any positive integer s, the graph $\mathcal{S}(\lceil \frac{s}{2} \rceil; s)$ is matching unique except for $s \in \{3,4,5\}$. In particular, for any positive integer $n \neq 2$, the friendship graph F_n is matching unique.*

Proof. For $s = 1, 2$, the graphs $\mathcal{S}(\lceil \frac{s}{2} \rceil; s)$ are isomorphic to $K_{1,2}$ and K_3, respectively, which are obviously matching unique. So let $s \ge 3$. Let G be a graph with $\mu(G,x) = \mu(\mathcal{S}(\lceil \frac{s}{2} \rceil; s), x)$. From Theorem 6, it is seen that G consists of a connected component H and, probably, some copies of K_2 and K_1, such that $\{\pm 1, \pm\sqrt{s+1}\} \subseteq R(H)$. By Theorem 6, no graph H' with $z(H') = 4$ has ± 1 as roots of its matching polynomial. Hence $z(H) = 5$, and

$$R(H) = \{0, \pm 1, \pm\sqrt{s+1}\}, \quad \text{for some } s \ge 3. \tag{1}$$

Thus, H is one of the graphs of Theorem 6 (iv).

We first assume that H has 5 vertices. From the Appendix table we see that only graphs of order 5 satisfying (1) are the graphs 5.5, 5.6, 5.10, 5.11, 5.16, and 5.17 of that table. The graph 5.16 is isomorphic to $\mathcal{S}(2;3)$, and 5.11 is isomorphic to $\mathcal{S}(2;4)$. Therefore, it turns out that the graph $\mathcal{S}(2;4)$ is comatching with 5.10, $\mathcal{S}(2;3)$ is comatching with 5.17, and $\mathcal{S}(3;5)$ is comatching with the union of a K_2 with either 5.5 or 5.6.

Clearly, for any t, neither $R(\mathcal{S}(t))$ nor $R(\mathcal{S}'(t))$ contains ± 1.

In order to have $\pm 1 \in R(\mathcal{K}(k;s'))$, k and s' must satisfy $k + s' = s'(k-1) + 1$. The only feasible solution of this equation is $k = 3, s' = 2$. So $\mathcal{S}(2;3)$ and $\mathcal{K}(3;2)$ have the same matching polynomial. Similarly, $\pm 1 \in R(\mathcal{K}'(k;s'))$ if and only if $k + s' + 1 = s'(k-1) + 1$. The only feasible solutions of this equation are $k = s' = 3$ and $k = 4, s' = 2$. As $\mathrm{mult}(0,H) = 1$, H cannot be $\mathcal{K}'(4;2)$. It follows that $\mathcal{S}(3;5)$ and $\mathcal{K}'(3;3) \cup K_2$ are comatching. □

Theorem 8. *For any integer $t \ge 0$, the graph $\mathcal{S}(t)$ is matching unique unless $t \in \{2,3,4\}$; and $\mathcal{S}'(t)$ is matching unique unless $t \in \{2,3\}$.*

Proof. For $t = 0, 1$, we have $\mathcal{S}(0) \simeq K_{1,2}$ and $\mathcal{S}(1)$ isomorphic to the path on 4 vertices which are matching unique. Let $t \ge 2$ and G be a graph with $\mu(G,x) = \mu(\mathcal{S}(t),x)$. Since $\pm 1 \notin R(\mathcal{S}(t))$, in view of Theorem 6, G consists of a connected component H and probably some isolated vertices such that $R(H)$ and $R(\mathcal{S}(t))$ have the same nonzero elements. Whence $z(H) = 4$ or 5 and so H is one of the graphs described in Theorem 6 (iii),(iv). Further, we have

$$m(H,1) = m(H,2) + 2, \quad \text{and} \quad t = m(H,2) \geq 2. \tag{2}$$

First, let H be of order 4. From the Appendix table, we see that the only graph of order 4 satisfying (2) is the graph 4.4, i.e. $K_{2,2}$, which corresponds to $t = 2$. So it follows that $\mathcal{S}(2)$ and $K_{2,2} \cup K_1$ are comatching.

Now let H have 5 vertices. Among the connected graphs of order 5, from the Appendix table we see that the graphs 5.9 (corresponding to $t = 4$), 5.15 (corresponding to $t = 3$), and 5.20 satisfy (2). (The graph 5.20 is actually isomorphic to $\mathcal{S}(2)$.) So we have that $\mathcal{S}(4)$ and 5.9 with two extra isolated vertices are comatching and so are $\mathcal{S}(3)$ and 5.15 with an extra isolated vertex.

No graph $\mathcal{S}(r;s)$ satisfy (2), and so H cannot be such a graph.

Now we examine the possibility of H being a $\mathcal{K}(k;s)$ or $\mathcal{K}'(k;s)$. The only graphs $\mathcal{K}(k;s)$ satisfy (2) have the parameters $s = 1$ and k any positive integer with $t = m(\mathcal{K}(k;1),2) = k-1$. But for these parameters we have $\mathcal{K}(k;1) \simeq \mathcal{S}(k-1)$. Among the graphs $\mathcal{K}'(k;s)$, only $\mathcal{K}'(3;2)$ satisfies (2), that is the graph 5.9 which is already considered.

This completes the proof for $\mathcal{S}(t)$. The proof for $\mathcal{S}'(t)$ is similar. $\square$

Theorem 9. *For any integer $k \geq s \geq 1$, the graph $\mathcal{K}(k;s)$ is matching unique unless $s = k-2$ or*

$$(k,s) \in \{(7,2),(6,2),(5,2),(5,1),(4,2),(4,1),(3,3),(3,2),(3,1)\};$$

and $\mathcal{K}'(k;s)$ is matching unique unless $s = k-2$ or

$$(k,s) \in \{(6,3),(5,3),(4,4),(4,3),(4,2),(4,1),(3,1)\}.$$

Proof. For $k = 1,2$, form the Appendix table it is easily seen the graphs $\mathcal{K}(k;s)$ are matching unique. Let $k \geq 3$ and G be a graph with $\mu(G,x) = \mu(\mathcal{K}(k;s),x)$. By the proof of Theorem 7, $\pm 1 \in R(\mathcal{K}(k;s))$ if and only if $k = 3, s = 2$, and the graph $\mathcal{K}(3;2)$ is comatching with $\mathcal{S}(2;3)$. So we may assume that $\pm 1 \notin R(\mathcal{K}(k;s))$ and so G consists of a connected component H and probably some isolated vertices such that $R(H)$ and $R(\mathcal{K}(k;s))$ have the same nonzero elements. Whence $z(H) = 4$ or 5 and so H is one of the graphs described in Theorem 6 (iii),(iv). Further, for some integers $k \geq 2$, H satisfies

$$m(H,1) = k+s, \quad m(H,2) = s(k-1), \quad \text{and} \quad 1 \leq s \leq k. \tag{3}$$

We first examine the possibility of H being a $\mathcal{K}(k';s')$. So we must have

$$s'(k'-1) = s(k-1), \quad \text{and} \quad k' + s' = k + s.$$

graph	comatching
$\mathcal{K}(3;1)$	$4.4 \cup K_1$
$\mathcal{K}(3;2)$	5.16
$\mathcal{K}(3;3)$	5.12
$\mathcal{K}(4;1)$	$5.15 \cup K_1$
$\mathcal{K}(4;2)$	$5.12 \cup K_1$, $5.13 \cup K_1$
$\mathcal{K}(5;1)$	$5.9 \cup 2K_1$
$\mathcal{K}(5;2)$	$5.8 \cup 2K_1$
$\mathcal{K}(6;2)$	$5.4 \cup 3K_1$
$\mathcal{K}(7;2)$	$5.2 \cup 4K_1$

graph	comatching
$\mathcal{K}'(3;1)$	$4.2 \cup K_1$
$\mathcal{K}'(4;1)$	$K_4 \cup 2K_1$
$\mathcal{K}'(4;2)$	$5.6 \cup K_1$
$\mathcal{K}'(4;3)$	$5.3 \cup K_1$
$\mathcal{K}'(4;4)$	$5.2 \cup K_1$
$\mathcal{K}'(5;3)$	$5.2 \cup 2K_1$
$\mathcal{K}'(6;3)$	$K_5 \cup 3K_1$

Table 1: The graphs $\mathcal{K}(k;s)$ and $\mathcal{K}'(k;s)$ with a comatching containing a component of order 4 or 5.

The only non-trivial solution to this system is $s' = k-1, k' = s+1$. Taking into account the condition $k \geq s$ and $k > k' \geq s'$, the feasible solution exists only when $s = k-2$ which is $s' = k' = k-1$. It turns oust that the graphs $\mathcal{K}(k;k-2)$ and $\mathcal{K}(k-1;k-1) \cup K_1$ have the same matching polynomial.

Now, if we consider the connected graphs of order 4 or 5 satisfying (3) for some integers k, s, we come up with the left list of Table 1.

The only r, s' such that $\mathcal{S}(r;s')$ satisfies (3), for some k, s, are $r = 2, s' = 3$ which corresponds with $k = 3, s = 2$; but the two graphs $\mathcal{K}(3;2)$ and $\mathcal{S}(2;3)$ are comatching.

Except for the case $\mathcal{K}(k;1) \simeq \mathcal{S}(k-1)$, the graph H cannot be a $\mathcal{S}(t)$ as seen in the proof of Theorem 8. Also, no $\mathcal{S}'(t)$ satisfies (3) for any k, s.

This completes the proof for $\mathcal{K}(k;s)$. The proof for $\mathcal{K}'(k;s)$ is similar. □

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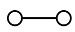
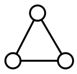
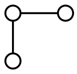
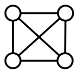
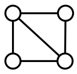
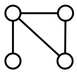
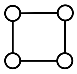
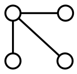
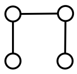
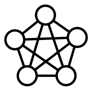
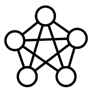
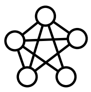
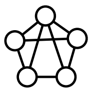
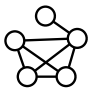
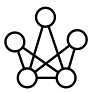
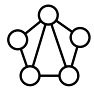
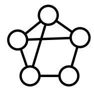
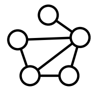
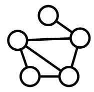
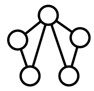
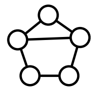
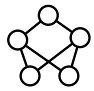
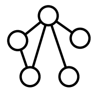
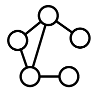
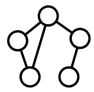
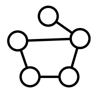
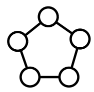
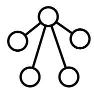
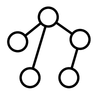
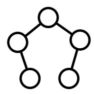
Appendix. Connected graphs up to five vertices and their matching polynomial

label	graph	matching polynomial
2.1		$x^2 - 1$
3.1		$x^3 - 3x$
3.2		$x^3 - 2x$
4.1		$x^4 - 6x^2 + 3$
4.2		$x^4 - 5x^2 + 2$
4.3		$x^4 - 4x^2 + 1$
4.4		$x^4 - 4x^2 + 2$
4.5		$x^4 - 3x^2$
4.6		$x^4 - 3x^2 + 1$
5.1		$x^5 - 10x^3 + 15x$
5.2		$x^5 - 9x^3 + 12x$
5.3		$x^5 - 8x^3 + 9x$
5.4		$x^5 - 8x^3 + 10x$
5.5		$x^5 - 7x^3 + 6x$
5.6		$x^5 - 7x^3 + 6x$

label	graph	matching polynomial
5.7		$x^5 - 7x^3 + 7x$
5.8		$x^5 - 7x^3 + 8x$
5.9		$x^5 - 6x^3 + 4x$
5.10		$x^5 - 6x^3 + 5x$
5.11		$x^5 - 6x^3 + 5x$
5.12		$x^5 - 6x^3 + 6x$
5.13		$x^5 - 6x^3 + 6x$
5.14		$x^5 - 5x^3 + 2x$
5.15		$x^5 - 5x^3 + 3x$
5.16		$x^5 - 5x^3 + 4x$
5.17		$x^5 - 5x^3 + 4x$
5.18		$x^5 - 5x^3 + 5x$
5.19		$x^5 - 4x^3$
5.20		$x^5 - 4x^3 + 2x$
5.21		$x^5 - 4x^3 + 3x$