Essential parabolic structures and their infinitesimal automorphisms

Jesse Alt

November 2, 2010

Abstract

Using the theory of Weyl structures, we give a natural generalization of the notion of essential conformal structures and conformal Killing fields to arbitrary parabolic geometries. We show that a parabolic structure is inessential whenever the automorphism group acts properly on the base space. As a corollary of the generalized Ferrand-Obata Theorem proved by C. Frances, this proves a generalization of the "Lichnérowicz Conjecture" for conformal Riemannian, strictly pseudo-convex CR, and quaternionic/octonionic contact manifolds in positive-definite signature. For an infinitesimal automorphism with a singularity, we give a generalization of the dictionary introduced by Frances for conformal Killing fields, which characterizes (locally) essential singularites via their holonomy.

1 Introduction

1.1 Motivation from conformal geometry

Let (M, c) be a smooth, n-dimensional semi-Riemannian conformal manifold. For any choice of semi-Riemannian metric g from the equivalence class c defining the conformal structure, we have the obvious inclusion of the group of isometric diffeomorphisms of (M, g) in the group of conformal diffeomorphisms of (M, c), $\mathrm{Isom}(M, g) \subseteq \mathrm{Conf}(M, c)$. At the infinitesimal level of vector fields, we have the corresponding inclusion of Killing fields in the conformal vector fields, $\mathrm{KVF}(M, g) \subseteq \mathrm{CVF}(M, c)$, which is obvious from the definitions: $\mathrm{KVF}(M, g) := \{X \in \mathfrak{X}(M) \,|\, \mathcal{L}_X g = 0\}$; and $\mathrm{CVF}(M, c) := \{X \in \mathfrak{X}(M) \,|\, \mathcal{L}_X g = \lambda g, \ \exists\, \lambda \in C^\infty(M)\}$.

A conformal diffeomorphism $\varphi \in \mathrm{Conf}(M, c)$ is called essential if φ is not an isometry of any metric $g \in c$, and the conformal structure (M, c) is essential if $\mathrm{Isom}(M, g)$ is a proper subgroup of $\mathrm{Conf}(M, c)$ for all representatives $g \in c$. Similarly, a conformal vector field $X \in \mathrm{CVF}(M, c)$ is called essential if there is no representative $g \in c$ for which $X \in \mathrm{KVF}(M, g)$. It is a fact – although not necessarily obvious from the preceding definitions – that there are compact and non-compact essential conformal structures in all dimensions $n \geq 2$ and all signatures (p, q), which moreover admit essential conformal vector fields. The standard compact example is given by the conformal "Möbius sphere" $(S^{p,q}, c)$ of any signature (p, q) (also called the Einstein universe – these are the conformally flat homogeneous models of conformal geometry, which in Riemannian signature are just the standard n-spheres equipped with the conformal class of the round metric), while the standard non-compact example is $\mathbb{R}^{p+q}$ equipped with the conformal class of the flat metric of signature (p, q). In fact, as a result of the following well-known theorem, giving a positive answering to the so-called Lichnérowicz Conjecture, we know that in Riemannian signature these two examples are the only essential structures:

Theorem A. (Ferrand-Obata) If (M, c) is an essential Riemannian conformal structure of dimension $n \geq 2$, then it is conformally diffeomorphic to the n-dimensional sphere with the round metric, or to n-dimensional Euclidean space.

For compact manifolds, this Theorem was proven by M. Obata and J. Lelong-Ferrand in the late 1960's and early 1970's. A proof for the non-compact case, announced in 1972 by Alekseevski, was later discovered to be incomplete, and a complete proof was first given in 1994 by Ferrand (cf. [6], [8] and references therein). Recently, a corresponding result was proven at the infinitesimal level by C. Frances [9] (note that this theorem does not simply follow from an application of the Ferrand-Obata Theorem, because the conformal vector fields are not assumed to be complete):

Theorem B. (Frances) Let (M, c) be a conformal Riemannian manifold of dimension $n \geq 3$, endowed with a conformal vector field X which vanishes at $x_0 \in M$. Then either: (1) There exists a neighborhood U of x_0 on which X is complete, generates a relatively compact flow in $\mathrm{Conf}(U, c)$, and is inessential on U, i.e. $X \in \mathrm{KVF}(U, g)$ for some $g \in c_{|U}$; or (2) There is a conformally flat neighborhood U of x_0, and X is essential on each neighborhood of x_0.

One direction of research into how these results do (or do not) generalize to other settings is to consider the analogous questions for *pseudo*-Riemannian metrics, where essential conformal structures turn out to be much more prevalent (cf. [8] for a survey). The aim of the present text is to introduce generalizations of the notion of essential structure, and the corresponding notion at the infinitesimal level, to the class of all parabolic geometries. Furthermore, we establish a generalization of Theorem A to a class of geometries which have been called "rank one parabolic geometries": conformal Riemannian structures; strictly pseudo-convex CR structures of hypersurface type; positive-definite quaternionic contact structures; and octonionic contact structures. (In fact, once our general definitions have been introduced and some basic properties established, we only have to prove the easy part of this generalized Theorem A, the difficult part having been taken care of in [7].) Finally, we establish some local properties of essential infinitesimal automorphisms which generalize some of the tools used in [9] to prove Theorem B for conformal vector fields on Riemannian manifolds.

1.2 Background and main definitions

Let us begin by recalling the definitions of parabolic geometries and their Weyl structures (the latter, introduced by A. Čap and J. Slovák in [3], will be central to our notion of essential parabolic structures, etc.). Parabolic geometries are certain types of Cartan geometries, which are very general: given a closed subgroup P of a Lie group G, a *Cartan geometry of type* (G, P) (or *modelled on the homogeneous space* G/P) is given by a principal P bundle $\pi : \mathcal{G} \to M$, equipped with a *Cartan connection* ω. That is, $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ satisfies:

$$R_p^*(\omega) = \mathrm{Ad}(p^{-1}) \circ \omega, \text{ for all } p \in P; \tag{1}$$

$$\omega(\tilde{X}) = X, \text{ for any } X \in \mathfrak{p}, \tilde{X} \text{ its fundamental vector field on } \mathcal{G}; \tag{2}$$

$$\omega(u) : T_u\mathcal{G} \to \mathfrak{g} \text{ is a linear isomorphism for all } u \in \mathcal{G}. \tag{3}$$

A Cartan geometry of type (G, P) is a *parabolic geometry* if G is a real or complex semi-simple Lie group, and $P \subset G$ is a parabolic subgroup as in representation theory. (For a more detailed discussion of the basic properties of parabolic subgroups and parabolic geometries, the reader is referred to Section 2 of [3] and the references therein. Here we only attempt to cite some of the key facts which are germane to the subsequent text.) In particular, in the parabolic setting the Lie algebra $\mathfrak{g}$ of G has an induced $|k|$-grading for some natural number k, so $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_k$ with $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ and the subalgebra $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$. The Lie algebra of the parabolic subgroup P is the parabolic subalgebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \ldots \oplus \mathfrak{g}_k$, which has Levi decomposition $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ with $\mathfrak{g}_0$ reductive and $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ the nilradical of $\mathfrak{p}$. At the group level, we have a reductive subgroup $G_0 \subset P$ whose Lie algebra is $\mathfrak{g}_0$, and $P \cong G_0 \ltimes P_+$ where $P_+ = \exp(\mathfrak{p}_+)$ is a normal, nilpotent subgroup of P globally diffeomorphic to $\mathfrak{p}_+$ via the exponential map.

The above-stated properties of the parabolic pair (G, P) and their Lie algebras, are used to identify the following important geometric structures associated to a parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P). The orbit space $\mathcal{G}_0 := \mathcal{G}/P_+$ of the P_+-action on $\mathcal{G}$ defines a G_0-principal bundle $\pi_0 : \mathcal{G}_0 \to M$, while by definition we also have a P_+-principal bundle $\pi_+ : \mathcal{G} \to \mathcal{G}_0$. The filtration of $\mathfrak{g}$ by $\mathrm{Ad}(P)$-invariant submodules $\mathfrak{g}^i = \mathfrak{g}_i \oplus \ldots \oplus \mathfrak{g}_k$ descends to a filtration of $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_-$ which is invariant under the quotient representation $\overline{\mathrm{Ad}} : P \to Gl(\mathfrak{g}/\mathfrak{p})$, and thus determines a filtration of the tangent bundle on the base space, $TM = T^{-k}M \supset \ldots \supset T^{-1}M$, via the isomorphism

$$TM \cong_\omega \mathcal{G} \times_{\overline{\mathrm{Ad}}(P)} \mathfrak{g}/\mathfrak{p},$$

(which holds in general for Cartan geometries) and setting $T^iM \cong_\omega \mathcal{G} \times_{\overline{\mathrm{Ad}}(P)} \mathfrak{g}^i/\mathfrak{p}$. Furthermore, the Cartan connection ω descends to $\mathcal{G}_0$ to identify it as a reduction to G_0 of the structure group of the *associated graded tangent bundle* $\mathrm{gr}(TM) = \mathrm{gr}_{-k}(TM) \oplus \ldots \oplus \mathrm{gr}_{-1}(TM)$, where $\mathrm{gr}_i(TM) = T^iM/T^{i+1}M$.

The data $(M, \{T^iM\}, \mathcal{G}_0)$ – consisting of a smooth manifold M, a filtration $\{T^iM\}$ of its tangent bundle which satisfies $\mathrm{rk}(T^iM) = \dim(\mathfrak{g}^i/\mathfrak{p})$, and a reduction $\mathcal{G}_0$ of the structure group of $\mathrm{gr}(TM)$ to G_0 –, is called an *infinitesimal flag structure of type* $(\mathfrak{g}, P)$. The flag structure is *regular* if the Lie derivative of vector fields on M respects the filtration, i.e. $[\Gamma(T^iM), \Gamma(T^jM)] \subset \Gamma(T^{i+j}M)$, and if the alternating bilinear form thus induced on $\mathrm{gr}(TM)$ gives it a point-wise Lie algebra structure isomorphic to $\mathfrak{g}_-$. When the flag structure is induced by a parabolic geometry of type (G, P), this regularity assumption can be related to an equivalent regularity condition that the curvature of the Cartan connection ω have strictly positive homogeneity (cf. discussion in Sections 2.6 and 2.7 of [3], as well as references cited there). A fundamental theorem of parabolic geometry states that, for any regular infinitesimal flag structure of type $(\mathfrak{g}, P)$, there exists a regular parabolic geometry of type (G, P) which induces it. This parabolic geometry is uniquely determined up to isomorphism by a normalisation condition on the curvature of the Cartan connection in all but an exceptional family of parabolic types (G, P). We will make use of this fundamental fact to identify the geometric structure of an infinitesimal flag structure with the regular, normal parabolic geometry inducing it, noting that this is a somewhat loose use of terminology since in the case of the exceptional types a choice of "first prolongation" must also be fixed for the latter to be uniquely determined.

In [3], Čap and Slovák define a *Weyl structure* for any parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G, P), to be a G_0-equivariant section $\sigma : \mathcal{G}_0 \to \mathcal{G}$ of the P_+-principal bundle $\pi_+ : \mathcal{G} \to \mathcal{G}_0$. By Proposition 3.2 of [3], global Weyl structures always exist for parabolic geometries in the real (smooth) category, and they exist locally in the holomorphic category. Considering the pull-back of the Cartan connection, $\sigma^*\omega$, the $|k|$-grading of $\mathfrak{g}$ gives a decomposition into G_0-invariant components, $\sigma^*\omega = \sigma^*\omega_{-k} + \ldots + \sigma^*\omega_k$, and by the observation that σ commutes with fundamental vector fields and the defining properties of the Cartan connection (cf. 3.3 of [3]), it follows that $\sigma^*\omega_i$ is horizontal for all $i \neq 0$, and that $\sigma^*\omega_0$ defines a principal G_0 connection on $\mathcal{G}_0 \to M$. In particular, we see that the pair $(\mathcal{G}_0 \to M, \sigma^*\omega_{\leq})$ defines a Cartan geometry of type (P^*, G_0), where $P^* \cong \exp(\mathfrak{g}_-) \rtimes G_0$ is the subgroup of G containing G_0 with Lie algebra $\mathfrak{p}^* = \mathfrak{g}_- \oplus \mathfrak{g}_0$, and the Cartan connection is given by

$$\sigma^*\omega_{\leq} = \sigma^*\omega_{-k} + \ldots + \sigma^*\omega_0 \in \Omega^1(\mathcal{G}_0, \mathfrak{p}^*). \tag{4}$$

One reason Weyl structures are very useful for studying a parabolic geometry, is that they are in fact determined by very simple induced geometric objects, namely by the $\mathbb{R}^+$-principal connections they induce on certain ray bundles associated to $\mathcal{G}_0$. Fix an element E_λ in the center of the reductive Lie algebra $\mathfrak{g}_0$ such that $\mathrm{ad}(E_\lambda)$ acts by scalar multiplication on each grading component $\mathfrak{g}_i$ of $\mathfrak{g}$ (for example the grading element E, which always exists and satisfies $\mathrm{ad}(E)_{|\mathfrak{g}_i} = i\cdot$). Then there is a unique representation $\lambda : G_0 \to \mathbb{R}^+$ satisfying $\lambda'(A) = B(E_\lambda, A)$ for all $A \in \mathfrak{g}_0$, B the Killing form, and hence an associated $\mathbb{R}^+$-principal bundle $\mathcal{L}^\lambda \to M$. For any Weyl structure σ, the 1-form $\lambda' \circ \sigma^*\omega_0 \in \Omega^1(\mathcal{G}_0)$ induces a $\mathbb{R}^+$-principal connection σ^λ on $\mathcal{L}^\lambda \cong \mathcal{G}_0/\mathrm{Ker}(\lambda)$. After introducing these objects and studying their properties in Section 3 of [3], Čap and Slovák prove the fundamental result that the correspondence $\sigma \mapsto \sigma^\lambda$ defines a bijective correspondence between the set of Weyl structures and the set of principal connections on $\mathcal{L}^\lambda$ (cf. Theorem 3.12 of [3]).

In particular, this fact makes it possible to defined certain distinguished classes of Weyl structures: A Weyl structure σ is *closed* if the induced $\mathbb{R}^+$-principal connection σ^λ has vanishing curvature; it is *exact* is σ^λ is a trivial connection induced by a global trivialisation of the scale bundle $\mathcal{L}^\lambda \to M$. Čap and Slovák prove that closed and exact Weyl structures always exist (in the smooth category), and the spaces of closed and exact Weyl structures are affine spaces over the closed, respectively over the exact, 1-forms on M. Equivalently, an exact Weyl structure σ is characterized by the existence of a holonomy reduction of the G_0-principal connection $\sigma^*\omega_0$ to the subgroup $\mathrm{Ker}(\lambda) \subset G_0$ (cf. Sections 3.13-3.14 of [3]). We will denote this reduction by $r : \overline{\mathcal{G}}_0 \hookrightarrow \mathcal{G}_0$, and the corresponding reduction of $\mathcal{G}$ to the structure group $\mathrm{Ker}(\lambda)$ by

$$\overline{\sigma} := \sigma \circ r : \overline{\mathcal{G}}_0 \to \mathcal{G}.$$

Thus an exact Weyl structure determines a Cartan geometry $(\overline{\mathcal{G}}_0 \to M, \overline{\sigma}^*\omega_{\leq})$ of type $(\overline{P^*}, \mathrm{Ker}(\lambda))$ for $\overline{P^*} \cong \exp(\mathfrak{g}_-) \rtimes \mathrm{Ker}(\lambda)$ the subgroup of G containing $\mathrm{Ker}(\lambda)$ with Lie algebra $\overline{\mathfrak{p}^*} := \mathfrak{g}_- \oplus \mathrm{Ker}(\lambda')$.

Now we are ready to define essential parabolic structures and essential infinitesimal automorphisms. For now, let us take the following definitions for automorphisms, respectively infinitesimal automorphisms, of a Cartan geometry. For a Cartan geometry $(\mathcal{G} \to M, \omega)$ of arbitrary type (G, P), an *automorphism* $\Phi \in \mathrm{Aut}(\mathcal{G}, \omega)$ is a P-principal bundle morphism of $\mathcal{G}$ such that $\Phi^*\omega = \omega$. An *infinitesimal automorphism* $\mathbf{X} \in \inf(\mathcal{G}, \omega)$ is given by $\mathbf{X} \in \mathfrak{X}(\mathcal{G})$, such that $(R_p)_*\mathbf{X} = \mathbf{X}$, and the Lie derivative satisfies $\mathcal{L}_{\mathbf{X}}\omega = 0$. Note that since $\Phi : \mathcal{G} \to \mathcal{G}$ is a P-bundle morphism, it naturally induces both a G_0-bundle morphism $\Phi_0 : \mathcal{G}_0 \to \mathcal{G}_0$ and a diffeomorphism $\varphi : M \to M$ which are determined, respectively, by the relations $\Phi_0 \circ \pi_+ = \pi_+ \circ \Phi$ and $\varphi \circ \pi = \pi \circ \Phi$. If the morphism Φ_0 preserves the sub-bundle $r(\overline{\mathcal{G}}_0) \subset \mathcal{G}_0$ determined by some exact Weyl structure, then we also get a $\mathrm{Ker}(\lambda)$-bundle morphism $\overline{\Phi}_0 : \overline{\mathcal{G}}_0 \to \overline{\mathcal{G}}_0$ by restriction. Similar statements hold for an infinitesimal automorphism $\mathbf{X} \in \mathfrak{X}(\mathcal{G})$, and we carry over the notation in an obvious way, i.e. $\mathbf{X}_0 \in \mathfrak{X}(\mathcal{G}_0)$, $X \in \mathfrak{X}(M)$ and $\overline{\mathbf{X}}_0 \in \mathfrak{X}(\overline{\mathcal{G}}_0)$.

Definition 1.1. Let $(\mathcal{G} \to M, \omega)$ be a parabolic geometry of type (G, P), and $\sigma : \mathcal{G}_0 \to \mathcal{G}$ a Weyl structure. The automorphism group of σ is the subgroup

$$\mathrm{Aut}(\sigma) := \{\Phi \in \mathrm{Aut}(\mathcal{G}, \omega) \,|\, \Phi_0 \in \mathrm{Aut}(\mathcal{G}_0, \sigma^*\omega_{\leq})\}.$$

If σ is exact, we define the subgroup of *exact automorphisms of* σ to be:

$$\mathrm{Aut}(\overline{\sigma}) := \{\Phi \in \mathrm{Aut}(\mathcal{G}, \omega) \,|\, \Phi_0(\overline{\mathcal{G}}_0) \subset \overline{\mathcal{G}}_0 \,\text{and}\, \overline{\Phi}_0 \in \mathrm{Aut}(\overline{\mathcal{G}}_0, \overline{\sigma}^*\omega_{\leq})\} \tag{5}$$

$$= \{\Phi \in \mathrm{Aut}(\sigma) \,|\, \Phi_0(\overline{\mathcal{G}}_0) \subset \overline{\mathcal{G}}_0\}. \tag{6}$$

An automorphism $\Phi \in \mathrm{Aut}(\mathcal{G}, \omega)$ is *essential* if it is not an exact automorphism of any exact Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$. We call $(\mathcal{G}, \omega)$ an *essential parabolic structure* if $\mathrm{Aut}(\overline{\sigma})$ is a proper subgroup of $\mathrm{Aut}(\mathcal{G}, \omega)$ for every exact Weyl structure σ. We call a regular infinitesimal flag structure $\mathcal{M} = (M, \{T^iM\}, \mathcal{G}_0)$ of type $(\mathfrak{g}, P)$ an *essential structure* if the regular, normal parabolic geometry inducing it is essential.

Similarly, we define the subalgebra of *infinitesimal automorphisms of* σ to be

$$\inf(\sigma) := \{\mathbf{X} \in \inf(\mathcal{G}, \omega) \,|\, \mathbf{X}_0 \in \inf(\mathcal{G}_0, \sigma^*\omega_{\leq})\}$$

and if σ is exact, the *exact infinitesimal automorphisms of* σ are:

$$\inf(\overline{\sigma}) := \{\mathbf{X} \in \inf(\mathcal{G}, \omega) \,|\, \overline{\mathbf{X}}_0 := (\mathbf{X}_0)_{|\overline{\mathcal{G}}_0} \in \mathfrak{X}(\overline{\mathcal{G}}_0) \,\text{and}\, \overline{\mathbf{X}}_0 \in \inf(\overline{\mathcal{G}}_0, \overline{\sigma}^*\omega_{\leq})\} \tag{7}$$

$$= \{\mathbf{X} \in \inf(\sigma) \,|\, (\mathbf{X}_0)_{|\overline{\mathcal{G}}_0} \in \mathfrak{X}(\overline{\mathcal{G}}_0)\} \tag{8}$$

An infinitesimal automorphism of the parabolic geometry $\mathbf{X} \in \inf(\mathcal{G}, \omega)$ is called essential if it is not an exact infinitesimal automorphism for any exact Weyl structure. For $\mathcal{M}$ a regular infinitesimal flag structure as above, we say a vector field $X \in \mathfrak{X}(M)$ is an *essential infinitesimal automorphism* of $\mathcal{M}$ if it lifts to an essential infinitesimal automorphism of the regular, normal parabolic geometry inducing $\mathcal{M}$.

Remark 1.2. As Charles Frances has pointed out to us, this definition turns out to be equivalent to the one given in Section 2.2 of [8] which does not make explicit use of Weyl structures.

Remark 1.3. The equivalence of (5) and (6) (respectively of (7) and (8)) follows immediately from the definitions, noting that the morphisms Φ_0 of $\mathcal{G}_0$ and $\overline{\Phi}_0$ of $\overline{\mathcal{G}}_0$ are related, whenever $\Phi_0(\overline{\mathcal{G}}_0) \subset \overline{\mathcal{G}}_0$, by $r \circ \overline{\Phi}_0 = \Phi_0 \circ r$.

1.3 Organisation of the text and summary of main results

Given Definition 1.1, we must first show that this is indeed a generalization of the notion of essential conformal structures. This is done in Section 2.1 via a Lemma giving equivalent characterisations of when an automorphism $\Phi \in \mathrm{Aut}(\mathcal{G}, \omega)$ lies in $\mathrm{Aut}(\sigma)$ (resp. in $\mathrm{Aut}(\overline{\sigma})$), given a fixed (exact) Weyl structure σ. This is then used in Section 2.2 to establish the general global result, that a parabolic structure is essential only if the action of the automorphism group $\mathrm{Aut}(\mathcal{G}, \omega)$ on M is non-proper.

With this, we may apply the main theorem of [7] to prove a Lichnérowicz Theorem for rank one parabolic geometries, confirming the conjecture formulated in Section 2.2 of [8] for these geometries. At the level of infinitesimal flag structures, the rank one parabolic geometries are: conformal Riemannian structures; strictly pseudo-convex, partially-integrable CR structures of codimension one; positive-definite quaternionic contact structures; and octonionic contact structures. The homogeneous models of these parabolic geometries are, respectively: $G/P \approx S^n$ with $G = SO(n+1,1)$; $G/P \approx S^{2n+1}$ with $G = SU(n+1,1)$; $G/P \approx S^{4n+3}$ with $G = Sp(n+1,1)$; and $G/P \approx S^{15}$ with $G = F_4^{-20}$. (These homogeneous models may be viewed as the "conformal infinities" of the rank one Riemannian symmetric spaces, a view emphasized in [1] where quaternionic and octonionic contact structures were defined.) The Lichnérowicz Theorem for rank one parabolic geometries then states that the only essential parabolic structures of these types are the compact homogeneous models G/P, and the non-compact spaces $G/P \backslash \{eP\}$.

Section 3 develops local properties of essential infinitesimal automorphisms. This amounts to studying infinitesimal automorphisms near a singularity x_0, since using the same methods needed to establish the Lichnérowicz Theorem shows that any infinitesimal automorphism of a parabolic geometry is inessential in some neighborhood of any point x such that $X(x) \neq 0$. Section 3.1 establishes a result characterizing infinitesimal automorphisms of arbitrary Cartan geometries $(\mathcal{G} \to M, \omega)$ via an identity involving the curvature of ω. This generalises an identity established in [2] for parabolic geometries, which is necessary because we require the identity for the Cartan geometry $(\mathcal{G}_0, \sigma^*\omega_\leq)$ (respectively, for $(\overline{\mathcal{G}}_0, \overline{\sigma}^*\omega_\leq)$) induced by a Weyl structure σ. In Section 3.2, we apply this to prove a generalization of results of [9], which give a "dictionary" relating essentiality of an infinitesimal automorphism near a singularity x_0 to properties of its holonomy h^t, a one-parameter subgroup of P which is determined up to conjugacy (cf. Definition 3.2). Already in conformal geometry, this result is of some interest because it can be used to determine whether a conformal vector field is locally essential from looking at the adjoint tractor it determines. We expect that our generalization of this result to arbitrary parabolic geometries, will yield a generalization of Theorem B, characterizing the local properties of essential conformal vector fields on Riemannian manifolds, to the other rank one parabolic geometries.

Acknowledgements: I am grateful to Charles Frances for discussions about the methods used in [7] and [9], and to Felipe Leitner for useful comments on an earlier version of the text.

2 Proof of global results

Here and in the sequel, we let $\mathcal{M} = (M, \{T^iM\}, \mathcal{G}_0)$ denote a regular infinitesimal flag structure of some parabolic type $(\mathfrak{g}, P)$, which we will generally take to be of non-exceptional type, so that the regular, normal parabolic geometry of type (G, P) inducing it, is unique up to isomorphism. If we need to distinguish this parabolic geometry from others of the same type, we will use the notation $(\mathcal{G}, \omega^{nc})$ to signify the canonical (**n**ormal **C**artan) geometry.

In this setting, we can define an automorphism of the structure in terms of $\mathcal{M}$: An automorphism of the regular infinitesimal flag structure, $\varphi \in \mathrm{Aut}(\mathcal{M})$, is a diffeomorphism $\varphi \in \mathrm{Diff}(M)$ which satisfies: (i) $\varphi_*(T^i_xM) \subseteq T^i_{\varphi(x)}M$ for all $x \in M$ and all $-k \leq i \leq -1$; and (ii) the induced bundle map $\mathrm{gr}(\varphi)$ (which as a consequence of (i) is a lift of φ defined on the bundle $\mathcal{F}(\mathrm{gr}(TM))$ of frames of the associated graded tangent bundle) preserves $\mathcal{G}_0$ as a subbundle of $\mathcal{F}(\mathrm{gr}(TM))$ (and hence $\mathrm{gr}(\varphi)$ restricts to a G_0-bundle morphism Φ_0 of $\mathcal{G}_0$). We can identify $\mathrm{Aut}(\mathcal{M})$ with $\mathrm{Aut}(\mathcal{G}, \omega^{nc})$ since by uniqueness of $(\mathcal{G}, \omega^{nc})$ up to isomoprhism, φ (and Φ_0) lift to a unique P-bundle morphism Φ of $\mathcal{G}$ preserving ω^{nc} under pullback. We will do this in the sequel, e.g. thinking of an automorphism $\varphi \in \mathrm{Aut}(\mathcal{M})$ as including as well the automorphism $\Phi \in \mathrm{Aut}(\mathcal{G}, \omega^{nc})$ and the induced G_0-bundle morphism $\Phi_0 = \mathrm{gr}(\varphi)_{|\mathcal{G}_0}$ of $\mathcal{G}_0$.

2.1 General results on essential automorphisms

Lemma 2.1. *Let $\Phi \in \mathrm{Aut}(\mathcal{G}, \omega)$ be an automorphism of a parabolic geometry, let σ be a Weyl structure, and let a scale bundle $\mathcal{L}^\lambda \to M$ be fixed. The following are equivalent:*

(i) $\Phi \in \mathrm{Aut}(\sigma)$*;*
(ii) Φ *preserves the scale bundle connection* $\sigma^\lambda \in \Omega^1(\mathcal{L}^\lambda)$*:* $\Phi_\lambda^* \sigma^\lambda = \sigma^\lambda$*;*
(iii) $\Phi_0^* \sigma^* \omega = \sigma^* \omega$*;*
(iv) $(\Phi \circ \sigma)^* \omega = (\sigma \circ \Phi_0)^* \omega$*.*

If σ *is exact, then* $\Phi \in \mathrm{Aut}(\overline{\sigma})$ *if and only if the associated global scale* $s_\sigma \in \Gamma(\mathcal{L}^\lambda)$ *is* Φ*-invariant, i.e.* $s_\sigma \circ \varphi = \Phi_\lambda \circ s_\sigma$*.*

Proof. To begin with, since we may identify $\mathcal{L}^\lambda \cong \mathcal{G}_0/\mathrm{Ker}(\lambda)$ and denote the resulting projection map $\pi_\lambda : \mathcal{G}_0 \to \mathcal{L}^\lambda$, then we have a naturally induced $\mathbb{R}^+$-bundle morphism Φ_λ of $\mathcal{L}^\lambda$ which is defined by the relation $\Phi_\lambda \circ \pi_\lambda = \pi_\lambda \circ \Phi_0$. Now we see that (i) $\Rightarrow$ (ii), since $\Phi_0^* \sigma^* \omega_\leq = \sigma^* \omega_\leq$ implies that $\Phi_0^* \sigma^* \omega_0 = \sigma^* \omega_0$. In particular, $\Phi_0^*(\lambda' \circ \sigma^* \omega_0) = \lambda' \circ \Phi_0^* \sigma^* \omega_0 = \lambda' \circ \sigma^* \omega_0$. Hence, the $\mathbb{R}^+$-bundle morphism Φ_λ and the $\mathbb{R}^+$-principal connection σ^λ, induced on $\mathcal{L}^\lambda$ by Φ_0 and $\lambda' \circ \sigma^* \omega_0$, respectively, satisfy: $\Phi_\lambda^* \sigma^\lambda = \sigma^\lambda$.

Next, consider the Weyl structure,

$$\Phi^* \sigma := \Phi^{-1} \circ \sigma \circ \Phi_0 : \mathcal{G}_0 \to \mathcal{G}.$$

This is a Weyl structure: It is a section of $\pi_+ : \mathcal{G} \to \mathcal{G}_0$ by the calculation,

$$\begin{aligned} \pi_+ \circ \Phi^* \sigma &:= \pi_+ \circ \Phi^{-1} \circ \sigma \circ \Phi_0 \\ &= (\Phi_0)^{-1} \circ \pi_+ \circ \sigma \circ \Phi_0 = \mathrm{Id}_{\mathcal{G}_0}. \end{aligned}$$

Also, by the G_0- and P-equivariance of the bundle maps Φ_0 and Φ, respectively, we see that $\Phi^* \sigma$ is G_0-equivariant whenever σ is.

We now show that (ii) $\Rightarrow$ (iii): Consider the $\mathbb{R}^+$-principal connection $(\Phi^* \sigma)^\lambda \in \Omega^1(\mathcal{L}^\lambda)$. This is induced by:

$$\begin{aligned} \lambda' \circ (\Phi^* \sigma)^* \omega_0 &= \lambda' \circ ((\Phi^{-1} \circ \sigma \circ \Phi_0)^* \omega_0) \\ &= \lambda' \circ (\Phi_0^* \sigma^* (\Phi^{-1})^* \omega_0) \\ &= \lambda' \circ (\Phi_0^* \sigma^* \omega_0) = \Phi_0^* (\lambda' \circ \sigma^* \omega_0). \end{aligned}$$

So $(\Phi^* \sigma)^\lambda = \Phi_\lambda^* \sigma^\lambda$, which equals σ^λ by assumption (ii). Thus, by Theorem 3.12 of [3] (cf. discussion in Section 1.2), the Weyl structures $\Phi^* \sigma$ and σ are equal. In particular, $\sigma^* \omega = (\Phi^* \sigma)^* \omega$. But a calculation similar to the ones above, plugging in the definition of $\Phi^* \sigma$ and using the fact that $(\Phi^{-1})^* \omega = \omega$, one calculates $(\Phi^* \sigma)^* \omega = \Phi_0^* \sigma^* \omega$, so we have (iii).

Clearly, (iii) $\Rightarrow$ (i), so (i), (ii), and (iii) are equivalent. Finally, the equivalence of (iii) and (iv) is seen by comparing the identity $(\Phi \circ \sigma)^* \omega = \sigma^* \omega$ (since $\Phi \in \mathrm{Aut}(\mathcal{G}, \omega)$) and $(\sigma \circ \Phi_0)^* \omega = \Phi_0^* \sigma^* \omega$.

To see the final statement of the Lemma, let σ be an exact Weyl structure and let us denote by $s_\sigma \in \Gamma(\mathcal{L}^\lambda)$ the global scale which induces the trivial connection $\sigma^\lambda \in \Omega^1(\mathcal{L}^\lambda)$. That is, for any point $p = s_\sigma(x).r \in \mathcal{L}^\lambda$, for $x \in M, r \in \mathbb{R}^+$, we have the decomposition

$$T_p \mathcal{L}^\lambda = (R_r)_*((s_\sigma)_*(T_x M)) \oplus \mathbb{R} \zeta_1(p),$$

for ζ_1 the fundamental vector field on $\mathcal{L}^\lambda$ of the vector $1 \in \mathbb{R}$; the value of σ^λ on a tangent vector $v \in T_p \mathcal{L}^\lambda$ is given by the coefficient of $\zeta_1(p)$ determined by this decomposition. Then the holonomy reduction of $(\mathcal{L}^\lambda, \sigma^\lambda)$ to the trivial structure group is given by $s_\sigma(M) \subset \mathcal{L}^\lambda$, and the reduction of $(\mathcal{G}_0, \sigma^* \omega_0)$ to $\mathrm{Ker}(\lambda)$ is given by

$$\overline{\mathcal{G}}_0 = \pi_\lambda^{-1}(s_\sigma(M)) \subset \mathcal{G}_0.$$

Now if $\Phi \in \mathrm{Aut}(\overline{\sigma})$, then by definition $\Phi_0(\overline{\mathcal{G}}_0) \subset \overline{\mathcal{G}}_0$. So for $u \in \overline{\mathcal{G}}_0$ with $\pi_0(u) = x \in M$, we have $\pi_\lambda(u) = s_\sigma(x)$ and (since $\Phi_0(u) \in \overline{\mathcal{G}}_0$ as well) $\pi_\lambda \circ \Phi_0(u) = s_\sigma \circ \pi_0(\Phi_0(u))$. Thus, by the definitions of $\Phi_\lambda : \mathcal{L}^\lambda \to \mathcal{L}^\lambda$ and $\varphi : M \to M$ from Φ_0, we have

$$\Phi_\lambda \circ s_\sigma(x) = \Phi_\lambda \circ \pi_\lambda(u) = \pi_\lambda \circ \Phi_0(u) = s_\sigma \circ \pi_0(\Phi_0(u))) = s_\sigma \circ \varphi(x),$$

which shows one implication claimed. For the other implication, note that from the invariance $\Phi_\lambda \circ s_\sigma = s_\sigma \circ \varphi$, the invariance $\Phi_\lambda^* \sigma^\lambda = \sigma^\lambda$ of the $\mathbb{R}^+$-connection follows directly from the definition of σ^λ in terms of s_σ. (Hence $\Phi \in \mathrm{Aut}(\sigma)$ by the equivalence of (i) and (ii).) And in the same way as we just computed, it also follows that Φ_0 preserves the sub-bundle $\overline{\mathcal{G}}_0$, so $\Phi \in \mathrm{Aut}(\overline{\sigma})$. q.e.d.

Remark 2.1. It follows, from the final statement in Lemma 2.1, that Definition 1.1 recovers the classical definition of essential conformal structures and essential conformal vector fields when the regular infinitesimal flag structure is given by a conformal semi-Riemannian structure (M, c) of signature (p, q). In that case, G_0 is just the conformal group $\mathbb{R}^+ \times O(p, q)$, $\mathcal{G}_0$ is the bundle of frames which are semi-orthonormal with respect to some metric $g \in c$, and the choice of scale representation,

$$\lambda : \mathbb{R}^+ \times O(p, q) \to \mathbb{R}^+$$
$$\lambda : (s, A) \mapsto s^{-1},$$

identifies $\mathcal{L}^\lambda \cong \mathcal{G}_0/\mathrm{Ker}(\lambda)$ with the ray bundle $\mathcal{Q} \to M$ of metrics in the conformal class, with the standard $\mathbb{R}^+$-action given by $s.g_x := s^2 g_x$ for any $g \in c$ and $x \in M$ corresponding to $g_x \in \mathcal{Q}$. Global sections of $\mathcal{L}^\lambda$ thus correspond to choices of a metric in the conformal class.

Remark 2.2. In fact, it follows from the proof of Lemma 2.1 that an automorphism $\Phi \in \mathrm{Aut}(\mathcal{G}, \omega)$ is an exact automorphism of an exact Weyl structure σ whenever Φ_0 preserves the sub-bundle $\overline{\mathcal{G}}_0$, i.e. the requirement in Definition 1.1 that $\Phi \in \mathrm{Aut}(\sigma)$ is superfluous to guarantee that $\Phi \in \mathrm{Aut}(\overline{\sigma})$. In the conformal case this is familiar, as the sub-bundle $\overline{\mathcal{G}}_0 \subset \mathcal{G}_0$ is simply the bundle of orthonormal frames with respect to a choice of metric g in the conformal equivalence class, and a conformal diffeomorphism which preserves this sub-bundle must also preserve g.

On the other hand, the requirement that $\Phi_0(\overline{\mathcal{G}}_0) \subset \overline{\mathcal{G}}_0$ is necessary to guarantee that an automorphism Φ of an exact Weyl structure σ is in fact an *exact* automorphism. An instructive example is the conformal structure induced by the Euclidean metric on $\mathbb{R}^n$: The diffeomorphism given by dilation by a positive constant r is always an automorphism of the exact Weyl structure corresponding to the Euclidean metric. However, for $r \neq 1$, this diffeomorphism is not an isometry, hence not an *exact* automorphism. We are grateful to Felipe Leitner for bringing this to our attention, which led to a modification of Definition 1.1.

2.2 Lichnérowicz Theorem for rank one parabolic geometries

In this section, we prove the following (cf. Section 1.3 for the definition of rank one parabolic geometries, which includes conformal Riemannian structures):

Theorem 2.2. *Let $(\mathcal{G} \to M, \omega)$ be a regular rank one parabolic geometry, with M connected. If this parabolic structure is essential, then M is geometrically isomorphic to either the compact homogeneous model G/P or the noncompact space $G/P \backslash \{eP\}$.*

The key result needed to prove this theorem is the following, proved by C. Frances (Theorem 3 of [7]), which generalizes theorems of Ferrand [6] and Schoen [11] in the cases of conformal Riemannian and strictly pseudo-convex CR structures:

Theorem 2.3. *(Frances, [7]) Let $(\mathcal{G} \to M, \omega)$ be a regular rank one parabolic geometry, with M connected. If $\mathrm{Aut}(\mathcal{G}, \omega)$ acts improperly on M, then M is geometrically isomorphic to either the compact homogeneous model G/P or the noncompact space $G/P \backslash \{eP\}$.*

(In both statements, "geometrically isomorphic" means there is a diffeomorphism of M onto the space in question, which is covered by a morphism of Cartan bundles which pulls back the Maurer-Cartan connection to ω.) Theorem 2.2 now follows as a result of Theorem 2.3 and the following proposition:

Proposition 2.4. *If $(\mathcal{G} \to M, \omega)$ is an essential parabolic structure, then $\mathrm{Aut}(\mathcal{G}, \omega)$ acts improperly on M.*

Proof. Fix a bundle of scales $\mathcal{L}^\lambda \to M$ for $(\mathcal{G}, \omega)$. Assume that $\mathrm{Aut}(\mathcal{G}, \omega)$ acts properly on M and let us show that the parabolic structure is not essential. By Lemma 2.1, it suffices to construct a global scale $s : M \to \mathcal{L}^\lambda$

which is $\mathrm{Aut}(\mathcal{G},\omega)$-invariant, i.e. such that $\Phi_\lambda \circ s = s \circ \varphi$ holds for all $\Phi \in \mathrm{Aut}(\mathcal{G},\omega)$ and $\Phi_\lambda : \mathcal{L}^\lambda \to \mathcal{L}^\lambda$, $\varphi : M \to M$ the induced diffeomorphisms.

We construct this $\mathrm{Aut}(\mathcal{G},\omega)$-invariant scale s using classical properties of proper group actions. The so called "Tube Theorem" (cf. e.g. Theorem 2.4.1 in [5]) guarantees the following, for a C^∞-action of a Lie group H on a manifold M which is proper at $x \in M$: There exists a H-invariant neighborhood U of x on which the H-action is equivalent to the left H-action on the quotient space $H \times_K B$ – for $K \subset H$ a compact subgroup and B a K-invariant neighborhood of 0 in a K-module V – given by $h_1.[h_2, b] = [h_1 h_2, b]$ for $h_i \in H, b \in B$ and $[h, b]$ the equivalence class of the $(h, b) \in H \times B$ under the left K-action $k.(h, b) := (h.k^{-1}, k.b)$. Starting from a choice of global scale $s_0 : M \to \mathcal{L}^\lambda$, and letting $H = \mathrm{Aut}(\mathcal{G},\omega)$, $e \in H$ the identity automorphism and $\Phi \in H$ arbitrary, set:

$$s_U([e,b]) := \int_{\Psi\in K} (\Psi_\lambda)^{-1}(s_0([\Psi,b]))d\Psi; \tag{9}$$

$$s_U([\Phi,b]) := \Phi_\lambda(s_U([e,b])). \tag{10}$$

One verifies that this gives a well-defined local section $s_U : U \to \mathcal{L}^\lambda_{|U}$, which involves checking that for $(e, b) \sim (\Phi, b')$ (i.e. for $\Phi \in K$ and $b = \varphi(b')$) the values $s_U([e, b])$ given by (9) and $s_U([\Phi, b'])$ given by (10), agree. This follows by unwinding the definitions, and using a bi-invariant Haar measure $d\Psi$ on the compact group K. And since $[\Phi, b] = \Phi.[e, b]$ corresponds to the point $\varphi(x')$ for $x' \simeq [e, b]$, the defining equation (10) automatically gives us the invariance property, $s_U \circ \varphi = \Phi_\lambda \circ s_U$. Now we can cover M with a finite number of H-invariant open sets like U above, and construct a global scale s with this invariance property by taking a finite average. q.e.d.

3 Proof of local results

A key reference for the study of infinitesimal automorphisms of parabolic geometries is [2]. In that text, A. Čap generalised to arbitrary parabolic geometries a bijective correspondence between conformal vector fields and adjoint tractors (sections of the associated bundle to the canonical Cartan bundle, $\mathcal{G} \to M$, induced by the adjoint representation on $\mathfrak{g}$) satisfying an identity involving the Cartan curvature, which was first discovered by A. R. Gover in [10]. Moreover, the text of Čap relates this general bijective correspondence to the first splitting operator of a so-called curved BGG-sequence for the parabolic geometry, cf. Theorem 3.4 of [2]. While for a general Cartan geometry we cannot hope to have this kind of bijective correspondence between a class of vector fields on the base manifold and adjoint tractors or other objects defined in terms of the total space of the Cartan bundle, yet we see in Section 3.1 that the curvature identity of [2] extends without difficulty to general infinitesimal automorphisms of Cartan geometries, in the sense defined in Section 1.2. This allows us to apply this fundamental identity to the Cartan geometries $(\mathcal{G}_0, \sigma^*\omega_\leq)$ occurring in the definition of essential infinitesimal automorphisms, which we do in Section 3.2 to establish a general "dictionary" between essentiality of an infinitesimal automorphism near a singularity, and its holonomy.

3.1 Preliminary results on infinitesimal automorphisms

We begin with some general notions, mainly following the development of [2] but in the setting of a general Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) (for now not assumed to be of parabolic type). For any representation $\rho : P \to Gl(V)$, we have the associated vector bundle $V(M) := \mathcal{G} \times_\rho V$. The smooth sections of such a bundle are identified with P-equivariant, V-valued smooth functions on $\mathcal{G}$ in the standard manner, and we will simply treat them as such:

$$\Gamma(V(M)) = \{f \in C^\infty(\mathcal{G}, V) \,|\, f(u.p) = \rho(p^{-1})(f(u))\} =: C^\infty(\mathcal{G}, V)^P.$$

For the most part, the important associated bundles we are dealing with are *tractor bundles*, which for our purposes simply means that the representation (ρ, V) is the restriction to P of a G-representation $\tilde{\rho} : G \to Gl(V)$. And the primary tractor bundle is the adjoint bundle induced by the restriction of the adjoint representation $Ad : G \to Gl(\mathfrak{g})$ to P, which we will denote by $\mathcal{A} = \mathcal{A}(M)$ if there is no danger of confusion about which

Lie algebra $\mathfrak{g}$ is meant, and otherwise by $\mathfrak{g}(M)$. Note that the Lie bracket $[,]_{\mathfrak{g}}$ of $\mathfrak{g}$, by $Ad(P)$-invariance, determines an algebraic bracket on fibers of $\mathcal{A}$ as well as on sections, which we denote with curly brackets $\{,\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

The Cartan connection determines an identification of right-invariant vector fields

$$\mathfrak{X}(\mathcal{G})^P = \{\mathbf{X} \in \mathfrak{X}(\mathcal{G}) \,|\, \mathbf{X}(u.p) = (R_p)_*(\mathbf{X}(u))\},$$

with sections of the adjoint bundle. Namely, consider the maps

$$\begin{aligned} &\mathfrak{X}(\mathcal{G}) \to C^\infty(\mathcal{G}, \mathfrak{g}), && \mathbf{X} \mapsto s_{\mathbf{X}}(u \mapsto \omega(\mathbf{X}(u))); \\ &C^\infty(\mathcal{G}, \mathfrak{g}) \to \mathfrak{X}(\mathcal{G}), && s \mapsto \mathbf{X}_s(u \mapsto \omega_u^{-1}(s(u)); \end{aligned}$$

the property of (3) of a Cartan connection insures that both maps are well-defined, they are inverse, and the property (1) of ω, which can be rewritten as

$$\omega_{u.p}((R_p)_*(\mathbf{X}(u))) = Ad(p^{-1})(\omega_u(\mathbf{X}(u))),$$

implies immediately that these maps restrict to an isomorphism $\mathfrak{X}(\mathcal{G})^P \cong_\omega \Gamma(\mathcal{A})$.

More generally, denote the set of right-invariant (r,s)-tensors on $\mathcal{G}$ by

$$\mathfrak{X}^{r,s}(\mathcal{G})^P := \{\mathbf{t} \in \mathfrak{X}^{r,s}(\mathcal{G}) \,|\, R_p^*\mathbf{t} = (TR_p)^{\otimes r} \circ \mathbf{t}\,\}$$

and identify the space of smooth sections $\Gamma((\mathcal{A}^*)^{\otimes s} \otimes (\mathcal{A})^{\otimes r})$ with the P-equivariant smooth functions

$$C^\infty(\mathcal{G}, (\mathfrak{g}^*)^{\otimes s} \otimes (\mathfrak{g})^{\otimes r})^P := \{s \in C^\infty(\mathcal{G}, (\mathfrak{g}^*)^{\otimes s} \otimes (\mathfrak{g})^{\otimes r}) \,|\, s(u.p) = [(Ad^*)^{\otimes s} \otimes (Ad)^{\otimes r}](p^{-1})(s(u))\,\}.$$

Then we have the following:

Lemma 3.1. *The Cartan connection ω induces isomorphisms*

$$\mathfrak{X}^{r,s}(\mathcal{G})^P \cong_\omega \mathfrak{X}^{0,s}(\mathcal{G}; (\mathfrak{g})^{\otimes r})^P \cong_\omega C^\infty(\mathcal{G}, (\mathfrak{g}^*)^{\otimes s} \otimes (\mathfrak{g})^{\otimes r})^P.$$

In particular, restricting to the horizontal, P-equivariant $(\mathfrak{g})^{\otimes r}$-valued s-forms on $\mathcal{G}$, we get

$$\overline{\Omega^s(\mathcal{G}; (\mathfrak{g})^{\otimes r})}^P \cong_\omega \Omega^s(M; (\mathcal{A})^{\otimes r}).$$

Proof. Given $\mathbf{t} \in \mathfrak{X}^{r,s}(\mathcal{G})$, define $s_{\mathbf{t}} \in C^\infty(\mathcal{G}, (\mathfrak{g}^*)^{\otimes s} \otimes (\mathfrak{g})^{\otimes r})$ by:

$$s_{\mathbf{t}}(u)(X_1, \ldots, X_s) := [\omega_u^{\otimes r}](\mathbf{t}(\omega_u^{-1}(X_1), \ldots, \omega_u^{-1}(X_s))),$$

for all $u \in \mathcal{G}$, $X_1, \ldots, X_s \in \mathfrak{g}$. And given $s \in C^\infty(\mathcal{G}, (\mathfrak{g}^*)^{\otimes s} \otimes (\mathfrak{g})^{\otimes r})$, define $\mathbf{t}_s \in \mathfrak{X}^{r,s}(\mathcal{G})$ by:

$$\mathbf{t}_s(v_1, \ldots, v_s) := [\omega_u^{\otimes r}]^{-1}(s(u)(\omega(v_1), \ldots, \omega(v_s))),$$

for all $u \in \mathcal{G}$ and $v_1, \ldots, v_s \in T_u\mathcal{G}$. In the above definitions, $\omega_u^{\otimes r} : (T_u\mathcal{G})^{\otimes r} \to (\mathfrak{g})^{\otimes r}$ is the unique linear map, induced for any $u \in \mathcal{G}$ via the universal property, from the obvious multilinear map $T_u\mathcal{G} \times \ldots \times T_u\mathcal{G} \to (\mathfrak{g})^{\otimes r}$. Using universality again, the properties (3) and (1) of ω imply, respectively, that $\omega_u^{\otimes r}$ is a linear isomorphism, and that for the map $(T_uR_p)^{\otimes r} : (T_u\mathcal{G})^{\otimes r} \to (T_{u.p}\mathcal{G})^{\otimes r}$ induced in an analogous way for any $p \in P$, we have the equivariance property:

$$[\omega_{u.p}^{\otimes r}] \circ [(T_uR_p)^{\otimes r}] = [\mathrm{Ad}^{\otimes r}](p^{-1}) \circ [\omega_u^{\otimes r}] : (T_u\mathcal{G})^{\otimes r} \to (\mathfrak{g})^{\otimes r}.$$

Using these two facts, it is a straightforward matter to verify that the maps $\mathbf{t} \mapsto s_{\mathbf{t}}$ and $s \mapsto \mathbf{t}_s$ are inverse to one another, and that they restrict to the isomorphism $\mathfrak{X}^{r,s}(\mathcal{G})^P \cong_\omega C^\infty(\mathcal{G}, (\mathfrak{g}^*)^{\otimes s} \otimes (\mathfrak{g})^{\otimes r})^P$ claimed. The other claims stated are established in a completely analogous way. q.e.d.

For a tractor bundle $V(M)$, the identification $\Gamma(\mathcal{A}) \cong_\omega \mathfrak{X}(\mathcal{G})^P$ yields two kinds of differentiation of smooth sections with respect to adjoint tractors:

Definition 3.1. The *invariant differentiation* or *fundamental D-operator* of $V(M)$ is the map $D^V : \Gamma(V(M)) \to \Gamma(\mathcal{A}^* \otimes V(M))$ defined, for any $s \in \Gamma(\mathcal{A})$ and any $v \in \Gamma(V(M))$, by:

$$D^V_s v := \mathbf{X}_s(v). \tag{11}$$

The *tractor connection* of $V(M)$ is the map $\nabla^V : \Gamma(V(M)) \to \Gamma(\mathcal{A}^* \otimes V(M))$ defined, for any $s \in \Gamma(\mathcal{A})$ and any $v \in \Gamma(V(M))$, by:

$$\nabla^V_s v := D^V_s v + (d\tilde{\rho} \circ s) \circ v. \tag{12}$$

Recall that $\tilde{\rho} : G \to Gl(V)$ is the G-representation which ρ is a restriction of, given by the definition of a tractor bundle. In fact, the quantity defined by (12) only depends on the equivalence class $[s]$ of s under the quotient $\mathcal{A}/\mathfrak{p}(M) = \mathcal{G} \times_{\bar{\mathrm{Ad}}(P)} \mathfrak{g}/\mathfrak{p}$, and since the Cartan connection determines a natural isomorphism $\mathcal{A}/\mathfrak{p}(M) \cong_\omega TM$, we identify the tractor connection with a covariant derivative on $V(M)$:

$$\nabla^V : \Gamma(V(M)) \to \Gamma(T^*M \otimes V(M)).$$

The curvature tensor of a Cartan connection is the $\mathfrak{g}$-valued two-form on $\mathcal{G}$ defined, for any $u \in \mathcal{G}$ and $v, w \in T_u\mathcal{G}$, by the structure equation $\Omega^\omega(v,w) := d\omega(v,w) + [\omega(v), \omega(w)]$. The curvature tensor is easily seen to be horizontal and P-equivariant, i.e. $\Omega^\omega \in \overline{\Omega^2(\mathcal{G};\mathfrak{g})}^P$ and we may equivalently consider the curvature function $\kappa^\omega \in C^\infty(\mathcal{G}; \lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g})^P \cong \Omega^2(M;\mathcal{A})$ given by $\kappa^\omega \simeq_\omega \Omega^\omega$. The following identity was proved for parabolic geometries in [2] (cf. also Lemma 1.5.12 in [4] for the general case, which came to our attention while completing the present text):

Lemma 3.2. *Under the identifications of Lemma 3.1, let $\mathbf{X} \simeq_\omega s_\mathbf{X}$ for $\mathbf{X} \in \mathfrak{X}(\mathcal{G})^P$ and $s_\mathbf{X} \in \Gamma(\mathcal{A})$. Then for $\mathcal{L}_\mathbf{X}\omega \in \overline{\Omega^1(\mathcal{G};\mathfrak{g})}^P$ we have, under the identification $\overline{\Omega^1(\mathcal{G};\mathfrak{g})}^P \cong_\omega \Omega^1(M;\mathcal{A})$:*

$$\mathcal{L}_\mathbf{X}\omega \simeq_\omega \nabla^\mathcal{A} s_\mathbf{X} + \Pi(s_\mathbf{X}) \lrcorner \kappa^\omega. \tag{13}$$

Proof. For a given point $u \in \mathcal{G}$ and $Y \in T_u\mathcal{G}$, let $\mathbf{Y}$ be a local extension of Y to a right-invariant vector field. Then we have

$$(\mathcal{L}_\mathbf{X}\omega)(Y) = \mathbf{X}(\omega(\mathbf{Y}))(u) - \omega([\mathbf{X},\mathbf{Y}](u)).$$

Using the definitions, we have:

$$\begin{aligned}\kappa^\omega(u)(s_\mathbf{X}(u), s_\mathbf{Y}(u)) &= d\omega(\mathbf{X}(u),\mathbf{Y}(u)) + [\omega(\mathbf{X}(u)), \omega(\mathbf{Y}(u))] \\ &= \mathbf{X}(\omega(\mathbf{Y}))(u) - \mathbf{Y}(\omega(\mathbf{X}))(u) - \omega([\mathbf{X},\mathbf{Y}](u)) + [s_\mathbf{X}(u), s_\mathbf{Y}(u)] \\ &= (\mathcal{L}_\mathbf{X}\omega)(Y) - \mathbf{Y}(s_\mathbf{X})(u) - [s_\mathbf{Y}(u), s_\mathbf{X}(u)],\end{aligned}$$

where the last line follows from comparing the formula for $(\mathcal{L}_\mathbf{X}\omega)(Y)$ given at the outset, and from the relation $\omega \circ \mathbf{X} = s_\mathbf{X}$. But since $\mathbf{Y} = \mathbf{Y}_{s_\mathbf{Y}}$ and $[s_\mathbf{Y}(u), s_\mathbf{X}(u)] = (\mathrm{ad} \circ s_\mathbf{Y}) \circ s_\mathbf{X}(u)$, the last two terms of the last line add to $-(\nabla^\mathcal{A}_{s_\mathbf{Y}} s_\mathbf{X})(u)$ by (12). q.e.d.

3.2 Holonomy and essential infinitesimal automorphisms

Let us return now to the setting of a (regular, normal) parabolic geometry $(\mathcal{G} \to M, \omega)$ of type (G,P) and the corresponding regular infinitesimal flag structure $\mathcal{M} = (M, \{T^iM\}, \mathcal{G}_0)$ of type $(\mathfrak{g}, P)$. As we did for automorphisms of $(\mathcal{G}, \omega^{nc})$ at the opening of Section 2, we may determine an infinitesimal automorphism $\mathbf{X} \in \inf(\mathcal{G},\omega)$ by conditions on the underlying vector field $X \in \mathfrak{X}(M)$, which just amount to imposing the same conditions for the locally defined diffeomorphisms given by flowing along X, i.e. we must have $[X, \Gamma(T^iM)] \subseteq \Gamma(T^iM)$ for all $-k \leq i \leq -1$, and the condition that the local flows of X determine local bundle maps of $\mathcal{F}(\mathrm{gr}(TM))$ which preserve $\mathcal{G}_0$ as a subbundle. We write $X \in \inf(\mathcal{M})$ and consider the lift $\mathbf{X} \in \inf(\mathcal{G},\omega^{nc})$ to be implicitly included. In the different examples of parabolic geometries, this translates into more geometric language. For example, in the conformal case, the former condition is trivial, while the latter condition amounts to requiring the conformal Killing equation, $\mathcal{L}_X g = 2\lambda g$ for any $g \in c$ and some $\lambda = \lambda(X) \in C^\infty(M)$. In the case of CR structures, the conditions are that, $[X, \Gamma(\mathcal{H})] \subseteq \Gamma(\mathcal{H})$, for $\mathcal{H} \subset TM$ the codimension one contact distribution defining the CR structure, and that $\mathcal{L}_X J = 0$ for J the almost complex structure on $\mathcal{H}$.

From Lemma 3.2 and Definition 1.1, we have a bijection between infinitesimal automorphisms $\mathbf{X} \in \inf(\mathcal{G}, \omega)$ and adjoint tractors $s_{\mathbf{X}} \in \Gamma(\mathcal{A})$ satisfying $\nabla^{\mathcal{A}} s_{\mathbf{X}} + \Pi(s_{\mathbf{X}}) \lrcorner \kappa^\omega = 0$. In the present setting, this gives us a bijection between $X \in \inf(\mathcal{M})$ and such $s_{\mathbf{X}}$, and moreover it is easy to verify that $\Pi(s_{\mathbf{X}}) = X \in \mathfrak{X}(M)$. Now, denote by $\mathbf{X}_0 \in \mathfrak{X}(\mathcal{G}_0)^{G_0}$ the invariant vector field induced, via projection by π_0, by $\mathbf{X} \in \mathfrak{X}(\mathcal{G})^P$. For any Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$, the induced Cartan connection $\sigma^* \omega_{\leq}$ and Lemma 3.1 give us an isomorphism denoted $\mathfrak{X}(\mathcal{G}_0)^{G_0} \cong_\sigma \Gamma(\mathcal{A}^\sigma)$, where we denote with $\mathcal{A}^\sigma = \mathfrak{p}^*(M)$ the adjoint bundle of the Cartan geometry $(\mathcal{G}_0 \to M, \sigma^* \omega_{\leq})$. Let us write $\mathbf{X}_0 \simeq_\sigma s_{\mathbf{X}_0}$ for the adjoint tractor corresponding to $\mathbf{X}_0 \in \mathfrak{X}(\mathcal{G}_0)^{G_0}$. If we further denote by $\nabla^\sigma : \Gamma(\mathcal{A}^\sigma) \to \Gamma(T^*M \otimes \mathcal{A}^\sigma)$ the corresponding adjoint tractor connection, then Lemma 3.2 tells us that $X \in \inf(\mathcal{M})$ is inessential if and only if there exists some exact Weyl structure $\sigma : \mathcal{G}_0 \to \mathcal{G}$, such that

$$\nabla^\sigma s_{\mathbf{X}_0} + X \lrcorner \kappa^{\sigma^* \omega_{\leq}} = 0,$$

and such that $(\mathbf{X}_0)_{|\overline{\mathcal{G}}_0} \in \mathfrak{X}(\overline{\mathcal{G}}_0)$ for the reduction $\overline{\mathcal{G}}_0$ to $\mathrm{Ker}(\lambda)$ determined by σ.

At present, we are only interested in local properties of infinitesimal automorphisms, viz the question if some neighborhood of a given point can be found on which X is inessential. The following proposition shows that the points $x \in M$ for which the answer could be "no", must all be singularities of the vector field, i.e. $X(x) = 0$ (where, incidentally the above identity simplifies to $(\nabla^\sigma s_{\mathbf{X}_0})(x) = 0$).

Proposition 3.3. *Let $\mathcal{M} = (M, \{T^i M\}, \mathcal{G}_0)$ be a regular infinitesimal flag structure of type $(\mathfrak{g}, P)$, let $X \in \inf(\mathcal{M})$, and let $x \in M$. If $X(x) \neq 0$, then there exists a neighborhood U of x such that the restriction of X to U is inessential.*

Proof. Take a neighborhood U of x on which flow-box coordinates for the flow of X can be introduced, i.e.

$$U = \{(x_0, \varphi_{X,t}(x_0)) \,|\, x_0 \in M_0 \cap U, \, -\varepsilon < t < \varepsilon\},$$

where M_0 is some locally defined hypersurface transversal to the integral curves $\varphi_{X,t}(x_0)$ of X, which are defined for the interval given. This can all be done since we may first restrict an open neighborhood of x on which X is non-vanishing. Now the final argument in the proof of Proposition 2.4 can be transferred to this situation, to define a scale $s : U \to \pi_\lambda^{-1}(U)$ which is invariant under the flows $\varphi_{X,t}$. q.e.d.

From now on, let us fix a singularity $x \in M$ of an infinitesimal automorphism $X \in \inf(\mathcal{M})$. We also choose a point $u \in \mathcal{G}$ in the fiber over x, and let $u_0 = \pi_+(u) \in \mathcal{G}_0$, likewise in the fiber over x. The remaining text is aimed at relating the local essentiality of X near x, to invariant properties of the holonomy of X at x, which is a one-parameter subgroup $h^t \subset P$:

Definition 3.2. (Cf. [9], Section 6) Given X, x and u as above, *the holonomy h_u^t of X at x with respect to* u is defined, for t sufficiently small, as follows: Let $\Phi_{\mathbf{X},t}(u)$, the integral curve of $\mathbf{X}$ through u, be defined for $t \in (-\varepsilon, \varepsilon)$. Since $\mathbf{X}$ projects to X, $\mathbf{X}(u')$ is tangent to $\mathcal{G}_x$ for all $u' \in \mathcal{G}_x$, and hence all $\Phi_{\mathbf{X},t}(u)$ lie in $\mathcal{G}_x$. Then $h_u^t \in P$ is defined by:

$$\Phi_{\mathbf{X},t}(u) =: u.h_u^t. \tag{14}$$

Since $h_u^{t+s} = h_u^t h_u^s$ whenever both are defined, $h_u^t = \exp(t X_{h,u})$ for some $X_{h,u} \in \mathfrak{p}$, and we define h_u^t via this identity for all $t \in \mathbb{R}$.

Recall that, by definition, $s_{\mathbf{X}}(u) = \omega(\mathbf{X}(u))$. Also, since $h_u^t = \exp(t X_{h,u})$, we have $X_{h,u} = \frac{d}{dt}_{|t=0} h_u^t$. By definition, the integral curve $\Phi_{\mathbf{X},t}(u)$ satisfies $\Phi'_{\mathbf{X}}(0) = \mathbf{X}(u)$. Hence, $s_{\mathbf{X}}(u) = \omega(\frac{d}{dt}_{|t=0}(u.h_u^t))$, and so by property (2) of the Cartan connection ω, we have

$$X_{h,u} = s_{\mathbf{X}}(u). \tag{15}$$

In particular, this implies the following equivariance properties of $X_{h,u}$ and h^t_u with respect to a change of the base point $u \in \mathcal{G}_x$, so it makes sense to speak of the holonomy h^t of X at x as a conjugacy class of one-parameter groups in P:

$$X_{h,u.p} = \mathrm{Ad}(p^{-1})(X_{h,u}) \ \forall\ p \in P; \tag{16}$$
$$h^t_{u.p} = p^{-1} h^t_u p \ \forall\ p \in P. \tag{17}$$

The first part of relating essentiality of X near x to its holonomy, is the following:

Proposition 3.4. *Let $X \in \inf(\mathcal{M})$ have a singularity $x \in M$, and let $u \in \mathcal{G}_x$ be as above. If X is inessential in some neighborhood of x, then its holonomy h^t_u is conjugate under P to a one-parameter subgroup of* $\mathrm{Ker}(\lambda) \subset G_0$ *for a choice of scale representation $\lambda : G_0 \to \mathbb{R}^+$ (equivalently, $s_{\mathbf{X}}(u.p) = \mathrm{Ad}(p^{-1})(s_{\mathbf{X}}(u)) \in$* $\mathrm{Ker}(\lambda') \subset \mathfrak{g}_0$ *for some $p \in P$).*

Moreover, if X is an infinitesimal automorphism for any locally defined Weyl structure σ (not necessarily exact), then X has holonomy h^t_u conjugate under P to a one-parameter subgroup of G_0 (equivalently, $s_{\mathbf{X}}(u.p) = \mathrm{Ad}(p^{-1})(s_{\mathbf{X}}(u)) \in \mathfrak{g}_0$ for some $p \in P$).

Proof. Assume that σ is any locally defined Weyl structure in a neighborhood of the point x which X is an infinitesimal automorphism of, i.e. such that $\mathbf{X}_0 \in \inf(\mathcal{G}_0, \sigma^*\omega_{\leq})$, so we have $\mathcal{L}_{\mathbf{X}_0}\sigma^*\omega_{\leq} = 0$. Then note that the identity (13) from Lemma 3.2 simplifies, for $u \in \mathcal{G}_x$ and $u_0 := \pi_+(u) \in (\mathcal{G}_0)_x$, to give us the following two identities:

$$(\nabla^{\mathcal{A}} s_{\mathbf{X}})(u) = 0; \tag{18}$$
$$(\nabla^{\sigma} s_{\mathbf{X}_0})(u_0) = 0. \tag{19}$$

We first prove the second claim in the proposition, by computing the identity (18) in terms of σ, to show that (19) implies $s_{\mathbf{X}}(\sigma(u_0)) \in \mathfrak{g}_0$. Since $\sigma(u_0), u \in \mathcal{G}_x$, therefore $s_{\mathbf{X}}(\sigma(u_0))$ is $\mathrm{Ad}(P)$-conjugate to $s_{\mathbf{X}}(u)$ and the claim follows.

For the computation, note that in general, a Weyl structure σ allows us to identifiy a section $s \in \Gamma(\mathcal{A})$ with $s \circ \sigma \in C^\infty(\mathcal{G}_0, \mathfrak{g})^{G_0} \cong \oplus_{i=-k}^{k} C^\infty(\mathcal{G}_0, \mathfrak{g}_i)^{G_0}$. We will write $[s]^\sigma = (s^\sigma_{-k}, \ldots, s^\sigma_k) \in \oplus_{i=-k}^{k} C^\infty(\mathcal{G}_0, \mathfrak{g}_i)^{G_0}$.

Now consider any $Y \in T_xM$, and let $\mathbf{Y}_0$ be a local right-invariant vector field on $\mathcal{G}_0$ around $u_0 \in (\mathcal{G}_0)_x$, projecting onto Y at x, and let $\mathbf{Y}$ be a local right-invariant vector field on $\mathcal{G}$ which extends the vector field $\sigma_*\mathbf{Y}_0$. Then by the chain rule, we have $\mathbf{Y}(s)(\sigma(u'_0)) = \mathbf{Y}_0([s]^\sigma)(u'_0)$ for u'_0 near u_0 in $\mathcal{G}_0$. Now compute from the definition, for $s = s_{\mathbf{X}}$ as above:

$$\begin{aligned}(\nabla^{\mathcal{A}}_Y s)(\sigma(u_0)) &= \mathbf{Y}(s)(\sigma(u_0)) + [\omega(\mathbf{Y}(\sigma(u_0))), s(\sigma(u_0))]_{\mathfrak{g}} \\ &= \mathbf{Y}_0([s]^\sigma)(u_0) + [\sigma^*\omega(\mathbf{Y}_0(u_0)), [s]^\sigma(u_0)]_{\mathfrak{g}}.\end{aligned}$$

Now we translate the last line into vector notation, where the top, middle and bottom components correspond, respectively to the projection onto $\mathfrak{p}_+, \mathfrak{g}_0$ and $\mathfrak{g}_-$, respectively (denoted, as usual, with a subscript). Note that from $\Pi(s) = X$, and since $X(x) = 0$, we have $s^\sigma_-(\sigma(u_0)) = 0$, and so we get the following reformulation of the left-hand side of (18):

$$\begin{pmatrix} \mathbf{Y}_0(s^\sigma_+)(u_0) \\ \mathbf{Y}_0(s^\sigma_0)(u_0) \\ \mathbf{Y}_0(s^\sigma_-)(u_0) \end{pmatrix} + \begin{pmatrix} \{\sigma^*\omega(\mathbf{Y}_0), [s]^\sigma\}_+(u_0) \\ \{\omega_0(\mathbf{Y}_0), s^\sigma_0\}(u_0) + \{\omega_-(\mathbf{Y}_0), s^\sigma_+\}_0(u_0) \\ \{\omega_-(\mathbf{Y}_0), s^\sigma_0\}(u_0) + \{\omega_-(\mathbf{Y}_0), s^\sigma_+\}_-(u_0) \end{pmatrix}. \tag{20}$$

On the other hand, let us compute the identity (19). The section $s_{\mathbf{X}_0} \in C^\infty(\mathcal{G}_0, \mathfrak{p}^*)^{G_0}$ is defined by

$$s_{\mathbf{X}_0}(u'_0) = \sigma^*\omega_{\leq}(\mathbf{X}_0(u'_0)) = \omega_{\leq}(\sigma(u'_0))(\sigma_*(\mathbf{X}_0(u'_0))).$$

Using the facts that σ is a section of $\pi_+ : \mathcal{G} \to \mathcal{G}_0$, and that $\mathbf{X}$ projects onto $\mathbf{X}_0$ via π_+, it follows that $\mathbf{X}(\sigma(u'_0)) - \sigma_*(\mathbf{X}_0(u'_0))$ lies in the kernel of $T_{\sigma(u'_0)}\pi_+$. In particular, this means we have:

$$\omega_{\leq}(\sigma(u'_0))(\sigma_*(\mathbf{X}_0(u'_0))) = \omega_{\leq}(\mathbf{X}(\sigma(u'_0))),$$

or equivalently, $s_{\mathbf{X}_0} = s^\sigma_- + s^\sigma_0$. Using this, a similar calculation to the one above gives:

$$(\nabla^\sigma_Y s_{\mathbf{X}_0})(u_0) = \begin{pmatrix} \mathbf{Y}_0(s^\sigma_0)(u_0) \\ \mathbf{Y}_0(s^\sigma_-)(u_0) \end{pmatrix} + \begin{pmatrix} \{\omega_0(\mathbf{Y}_0), s^\sigma_0\}(u_0) \\ \{\omega_-(\mathbf{Y}_0), s^\sigma_0\}(u_0) \end{pmatrix}. \quad (21)$$

Comparing the $\mathfrak{g}_0$-components of (20) and (21), we see that if both terms vanish, we must have

$$\{\omega_-(\mathbf{Y}_0), s^\sigma_+\}_0(u_0) := \mathrm{pr}_{\mathfrak{g}_0}([\omega_-(\mathbf{Y}_0)(u_0), s^\sigma_+(u_0)]) = 0.$$

But we have $\mathfrak{g}_- = \{\omega_-(\mathbf{Y}_0)(u_0) \,|\, Y \in T_xM\}$, and from this it follows that $s^\sigma_+(u_0) = 0$, by using the properties of $|k|$-graded semi-simple Lie algebras: We have the grading element $E \in \mathfrak{g}_0$, which satisfies $[E, Y_j] = jY_j$ for all $Y_j \in \mathfrak{g}_j$. Now using the *Ad*-invariance of the Killing form B, we have, for any $Y \in \mathfrak{g}_-$, and $0 < j \leq k$:

$$B([Y, s^\sigma_j(u_0)], E) = B(Y, [s^\sigma_j(u_0), E]) = -jB(Y, s^\sigma_j(u_0)),$$

and since B induces an isomorphism $\mathfrak{g}_j \cong (\mathfrak{g}_{-j})^*$, this vanishes for all $Y \in \mathfrak{g}_-$ only if $s^\sigma_j(u_0) = 0$ for all $j > 0$, i.e. only if $s(\sigma(u_0)) \in \mathfrak{g}_0$, which is the second claim of the proposition.

The proof of the first claim of the proposition is completely analogous. If σ is exact, then we have the holonomy reduction $\overline{\mathcal{G}}_0 \subset \mathcal{G}_0$ to structure group $Ker(\lambda) \subset G_0$, and denoting the resulting reduction by $\overline{\sigma} : \overline{\mathcal{G}}_0 \to \mathcal{G}$, the condition that X is an *exact* infinitesimal automorphism of σ is, in addition to the above requirements, that $\mathbf{X}_0(\overline{u}) \in T_{\overline{u}}\overline{\mathcal{G}}_0 \subset T_{\overline{u}}\mathcal{G}_0$ for all $\overline{u} \in \overline{\mathcal{G}}_0$ and that the resulting vector field $\overline{\mathbf{X}}_0$ on $\overline{\mathcal{G}}_0$ is an infinitesimal automorphism of the Cartan connection $\overline{\sigma}^*\omega_{\leq}$. This gives

$$(\nabla^{\overline{\sigma}} s^{\overline{\sigma}}_{\overline{\mathbf{X}}_0})(\overline{u}_0) = 0,$$

where $\nabla^{\overline{\sigma}}$ denotes the tractor connection on the adjoint tractor bundle associated to $\overline{\mathcal{G}}_0$ by the adjoint representation on $\mathfrak{g}_- \oplus \mathrm{Ker}(\lambda')$. Then by the same considerations, if we write $s^\sigma_0(\overline{u}_0) = s^{\overline{\sigma}}_0(\overline{u}_0) + z(\overline{u}_0)E_\lambda$, for $\overline{u}_0 \in (\overline{\mathcal{G}}_0)_x$, then this condition implies that

$$0 = [\omega_-(\mathbf{Y}_0(\overline{u}_0)), z(\overline{u}_0)E_\lambda] = \sum_{j=1}^{k} jz(\overline{u}_0)\omega_{-j}(\mathbf{Y}_0(u_0)),$$

for all $Y \in T_xM$, which can only happen if $z(\overline{u}_0) = 0$. Hence we must have $s(\sigma(\overline{u}_0)) \in \mathrm{Ker}(\lambda') \subset \mathfrak{g}_0$, which is the first claim of the proposition. q.e.d.

We conclude by proving the following converse, which completes the correspondence between the holonomy of an essential vector field at a singularity, and its essentiality:

Proposition 3.5. *Let $X \in \inf(\mathcal{M})$ have a singularity $x \in M$, and let $u \in \mathcal{G}_x$, $u_0 = \pi_+(u) \in (\mathcal{G}_0)_x$. If the holonomy h^t_u of X is conjugate under P to a one-parameter subgroup of* $\mathrm{Ker}(\lambda) \subset G_0$ *for a choice of scale representation $\lambda : G_0 \to \mathbb{R}^+$ (equivalently, $s_{\mathbf{X}}(u.p) = \mathrm{Ad}(p^{-1})(s_{\mathbf{X}}(u)) \in \mathrm{Ker}(\lambda') \subset \mathfrak{g}_0$ for some $p \in P$), then X is inessential on some neighborhood of x.*

Moreover, if the holonomy h^t_u is conjugate under P to a one-parameter subgroup of G_0 (equivalently, $s_{\mathbf{X}}(u.p) = \mathrm{Ad}(p^{-1})(s_{\hat{X}}(u)) \in \mathfrak{g}_0$ for some $p \in P$), then X is an infinitesimal automorphism of some local Weyl structure σ around x.

Proof. We will need the following "exponential coordinates" on $\mathcal{G}$ and M around u and x respectively, which are induced by the Cartan connection ω: For any $Y \in \mathfrak{g}$, denote by $\hat{Y}$ the vector field on $\mathcal{G}$ which is determined by the identity, $\omega(\hat{Y}(u')) = Y$ for all $u' \in \mathcal{G}$. Defining

$$\mathcal{W}_u := \{Y \in \mathfrak{g} \,|\, \varphi_{\hat{Y},t}(u) \,\text{is defined for}\, 0 \leq t \leq 1\},$$

then there exist an open neighborhood $\mathcal{V}_u$ of $0 \in \mathfrak{g}$ and an open neighborhood V_u of $u \in \mathcal{G}$ such that the exponential map $\exp^\omega_u$ defined on $\mathcal{W}_u$ is a diffeomorphism of $\mathcal{V}_u$ onto V_u, where by definition:

$$\exp^\omega_u : Y \mapsto \exp^\omega(u, Y) := \varphi_{\hat{Y},1}(u).$$

Restricting $\mathcal{V}_u$ if necessary to a smaller neighborhood of zero, we get a diffeomorphism

$$\overline{\exp}_u^\omega := \pi \circ \exp_u^\omega : \mathcal{V}_u^- \xrightarrow{\approx} U_x,$$

where U_x is a neighborhood of x in M and $\mathcal{V}_u^- := \mathcal{V}_u \cap \mathfrak{g}_-$. Furthermore, for $V_u^- := \exp_u(\mathcal{V}_u^-)$, the restriction of the projection π gives a diffeomorphism of V_u^- onto U_x.

These exponential coordinates can obviously be used to define a local Weyl structure over U_x, since they give a local section of $\pi : \mathcal{G} \to M$ on U_x. This gives us the local trivialisations $\pi^{-1}(U_x) \cong V_u^- \times P$ and $\pi_0^{-1}(U_x) \cong \pi_+(V_u^-) \times G_0$. Then we simply define $\sigma : \pi_0^{-1}(U_x) \to \pi^{-1}(U_x)$ by

$$\sigma : \pi_+(u').g_0 \mapsto u'.g_0; \tag{22}$$

for any $u' \in V_u^-, g_0 \in G_0$. This is by definition a G_0-equivariant local section of $\pi_+ : \mathcal{G} \to \mathcal{G}_0$, that is a Weyl structure. We will now show, assuming $h_u^t \subset G_0$, that the local flows of $\mathbf{X} \in \mathfrak{X}(\mathcal{G})$ commute with σ, i.e. we have $\Phi_{\mathbf{X},t} \circ \sigma = \sigma \circ \Phi_{\mathbf{X}_0,t}$ on $\pi_0^{-1}(U_x)$, for t sufficiently small so that both sides exist. By the equivalence of (i) and (iv) in Lemma 2.1, this suffices to prove the second claim of the proposition.

To show the commutativity of the local flows of $\mathbf{X}$ with the Weyl structure σ given by (22), we need the following general equivariance relation for an infinitesimal automorphism in exponential coordinates (cf. the proof of Proposition 6.2 of [9]):

$$\Phi_{\mathbf{X},t}(\exp^\omega(u, Y)) = \exp^\omega(u, \mathrm{Ad}(h_u^t)(Y)).h_u^t. \tag{23}$$

The identity (23) is based on the observation that we have $[\mathbf{X}, \hat{Y}] = 0$ for any infinitesimal automorphism, and any $Y \in \mathfrak{g}$ (this follows immediately from the defining equation, $\mathcal{L}_{\mathbf{X}}\omega = 0$, of an infinitesimal automorphism). Hence, the flows commute, $\Phi_{\mathbf{X},t} \circ \Phi_{\hat{Y},s} = \Phi_{\hat{Y},s} \circ \Phi_{\mathbf{X},t}$ whenever both sides are defined, which together with equivariance of ω may be used to show that both sides of (23) are given as the endpoint of the same integral curve through $\Phi_{\mathbf{X},t}(u) = u.h_u^t$.

Now consider an arbitrary point $\pi_+(u').g_0 \in \pi_0^{-1}(U_x)$, where $u' = \exp^\omega(u, Y) \in V_u^-$ for $Y \in \mathcal{V}_u^-$. Then $\sigma(\pi_+(u').g_0) := u'.g_0$, and we have, by P-equivariance of $\Phi_{\mathbf{X},t}$ and (23):

$$(\Phi_{\mathbf{X},t} \circ \sigma)(\pi_+(u').g_0) = \Phi_{\mathbf{X},t}(u'.g_0) = \exp^\omega(u, \mathrm{Ad}(h_u^t)(Y)).h_u^t g_0.$$

But since $h_u^t \in G_0$, we have $\mathrm{Ad}(h_u^t)(Y) \in \mathfrak{g}_-$ and for t sufficiently small we may also assume $\mathrm{Ad}(h_u^t)(Y) \in \mathcal{V}_u$ by continuity, so $\exp^\omega(u, \mathrm{Ad}(h_u^t)(Y)) \in V_u^-$. We also have $h_u^t g_0 \in G_0$, so G_0-equivariance of π_+ gives $\pi_+(\Phi_{\mathbf{X},t}(u'.g_0)) = \pi_+(\exp^\omega(u, \mathrm{Ad}(h_u^t)(Y))).h_u^t g_0$, and hence combining the above gives:

$$(\Phi_{\mathbf{X},t} \circ \sigma)(\pi_+(u').g_0) = \Phi_{\mathbf{X},t}(u'.g_0) = (\sigma \circ \pi_+ \circ \Phi_{\mathbf{X},t})(u', g_0).$$

Finally, since $\Phi_{\mathbf{X}_0,t}$ is defined on $\mathcal{G}_0$ via the relation $(\Phi_{\mathbf{X}_0,t} \circ \pi_+) = (\pi_+ \circ \Phi_{\mathbf{X},t})$, this shows that $(\Phi_{\mathbf{X},t} \circ \sigma) = (\sigma \circ \Phi_{\mathbf{X}_0,t})$ on $\pi_0^{-1}(U_x)$.

The claim when h_u^t is conjugate to a one-parameter subgroup of $\mathrm{Ker}(\lambda)$ is quite easy to establish in the same way. The above considerations also show that the local, equivariant section σ of $\pi_+ : \mathcal{G} \to \mathcal{G}_0$ over U_x which was constructed using the exponential map, can be restricted to a map $\overline{\sigma} : (\overline{\mathcal{G}}_0)_{U_x} \to \mathcal{G}_{U_x}$ where the sub-bundle $(\overline{\mathcal{G}}_0)_{U_x} \subset (\mathcal{G}_0)_{U_x}$ is just given by $\pi_+(V_u^-) \times \mathrm{Ker}(\lambda)$ in the local trivialization, giving a locally exact Weyl structure. And since h_u^t is conjugate to a one-parameter subgroup of $\mathrm{Ker}(\lambda)$, it can be arranged (by moving to a different point in the fiber over x, if necessary) that the restriction of $\mathbf{X}_0$ to $(\overline{\mathcal{G}}_0)_{U_x}$ is always tangent to this sub-bundle, so X is locally inessential. q.e.d.

References

[1] O. Biquard, *Métriques d'Einstein asymptotiquement symétriques*, Astérisque **265** (2000); English translation: *Asymptotically Symmetric Einstein Metrics*, SMF/AMS Texts and Monographs, volume 13 (2006).

[2] A. Čap, *Infinitesimal Automorphisms and Deformations of Parabolic Geometries*, J. Eur. Math. Soc. **10** (2008) No. 2, 415-437.

[3] A. Čap, J. Slovák, *Weyl structures for parabolic geometries*, Math. Scand. **93**, No. 1 (2003) 53-90.

[4] A. Čap, J. Slovák, *Parabolic Geometries I: Background and General Theory*, Mathematical Surveys and Monographs vol. 154, American Mathematical Society, 2009.

[5] J. J. Duistermaat, J. A. C. Kolk, *Lie Groups*, Springer Universitext 2000.

[6] J. Ferrand, *The action of conformal transformations on a Riemannian manifold*, Math. Ann. **304** (1996) no. 2, 277-291.

[7] C. Frances, *Sur le groupe d'automorphismes des géométries paraboliques de rang un*, Ann. sci. cole Norm. Sup. (4) **40**, no. 5 (2007) 741-764. English version: *A Ferrand-Obata theorem for rank one parabolic geometries*, arXiv preprint math.DG/0608537.

[8] C. Frances, *Essential conformal structures in Riemannian and Lorentzian geometry*, in Recent Developments in Pseudo-Riemannian Geometry, ESI Lectures in Mathematics and Physics, European Mathematical Society 2008.

[9] C. Frances, *Local dynamics of conformal pseudo-Riemannian transformations*, arXiv preprint math.DG/0909.0044.

[10] A. R. Gover, *Laplacian operators and Q-curvature of conformally Einstein manifolds*, Math. Ann. **336** (2006) 311-334.

[11] R. Schoen, *On the conformal and CR automorphism groups*, Geom. Funct. Anal. **5** (1995), No. 2, 464-481.

SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, P O WITS 2050, JOHANNESBURG, SOUTH AFRICA.
E-mail: `jesse.alt@wits.ac.za`