

# Polypseudologarithms revisited

Djordje Cvijović

*Atomic Physics Laboratory, Vinča Institute of Nuclear Sciences  
P.O. Box 522, 11001 Belgrade, Republic of Serbia*

**E-Mail:** djurdje@vinca.rs

**Abstract.** Lee, in a series of papers, described a unified formulation of the statistical thermodynamics of ideal quantum gases in terms of the polylogarithm functions,  $\text{Li}_s(z)$ . It is aimed here to investigate the functions  $\text{Li}_s(z)$ , for  $s = 0, -1, -2, \dots$ , which are, following Lee, referred to as the polypseudologarithms (or polypseudologs) of order  $n$ . Various known results regarding polypseudologs, mainly obtained in widely differing contexts and currently scattered throughout the literature, have been brought together along with many new results and insights and they all have been proved in a simple and unified manner. In addition, a new general explicit closed-form formula for these functions involving the Carlitz–Scoville higher tangent numbers has been established.

**PACS numbers:** 05.90.+m, 02.90.+p, 02.30.Gp, 02.10.De.

**Mathematics Subject Classification:** Primary 11M41; Secondary 33E20, 33E99.

**Key Words and Phrases:** Polylogarithms; Polypseudologarithms; Stirling numbers of second kind; Eulerian numbers; Higher tangent numbers

## 1. INTRODUCTION

Lee, in a series of papers [1–6], not only has shown for the first time that statistical thermodynamics of the Bose–Einstein and Fermi–Dirac ideal gases can be unified, but also has obtained a unified description of the main thermodynamical functions in terms of the polylogarithmic functions  $\text{Li}_s(z)$  (for their definitions see Sec. 2) and also examined details of certain physical phenomena revealed by this approach.

The unification is based on the following expression for the reduced density of non-relativistic ideal gas of  $N$  particles confined to a  $d$ -dimensional box of volume  $V$

$$\frac{\rho\lambda^d}{g} = \text{sgn}(\zeta) \text{Li}_{d/2}(\zeta), \quad \zeta = \begin{cases} z & \text{if Bose–Einstein} \\ -z & \text{if Fermi–Dirac,} \end{cases} \quad (1.1)$$

where  $\rho \equiv N/V = N/L^d$  is the particle number density,  $\lambda = (2\pi\beta/m)^{1/2}$  (with  $\hbar = 1$ ) is the thermal wave-length,  $\beta \equiv 1/k_B T$  and  $g = 2s + 1$ , while  $k_B$ ,  $T$ ,  $m$  and  $s$  denote the Boltzman constant, the temperature, the mass and the spin of the particles, respectively. Finally, the fugacity  $z$  is related to the chemical potential  $\mu$  of the system as  $z \equiv e^{\beta\mu}$ . The grand partition function then easily follows from (1.1)  $Q = \exp \{ \text{sgn}(\zeta)(\lambda/L)^d \text{Li}_{d/2+1}(\zeta) \}$ , while the basic thermodynamic functions, such as the energy  $U$ , the entropy  $S$ , the pressure  $P$  and also fluctuation quantities such as the number fluctuations  $Y$ , can be expressed in polylogarithms because of their relationship with the density  $\rho$ .

The important property of the unified formulation is that  $U$ ,  $S$ ,  $P$ ,  $Y$  and the reduced density depend only on two variables, the fugacity and dimensionality, and, more importantly, their values are given by the very same function – polylogarithms (of order  $s$  and argument  $z$ ),  $\text{Li}_s(z)$ ; the fugacity  $z$  determines the argument, while the dimensionality  $d$  enters the polylogarithms through its order—an integral order if  $d$  is even, and a half-integral order if  $d$  is odd. Moreover, by the principle of analytic continuation, the obtained results may be assumed valid for any  $d$ .

It is noteworthy that the polylogarithm formulation of statistical thermodynamics evidently reveals that the thermodynamic properties of ideal quantum gases are described by the structural properties of  $\text{Li}_s(z)$  and, therefore, there is a need to understand better these important but little known functions. In addition, it is not surprising that this formulation is a quite natural framework for dealing with dimensionality effects in ideal quantum gases, such as anomalous physics in  $d = 0$  [4] and a remarkable thermodynamic equivalence (discovered by Lee) between the Bose–Einstein and Fermi–Dirac gases in  $d = 2$  [5, 7]. There are suggestions that negative dimensionality, *i.e.*  $d < 0$ , might be of some theoretical interest [6, 8].

Motivated by the work of Lee [1–6] on the thermodynamics of quantum gases in terms of polylogarithms and, in particular, by his paper on *polypseudologarithms*, *i.e.*  $\text{Li}_s(z)$  for  $s = 0, -1, -2, \dots$ , etc. [6, Sec. 3], we have begun a systematic study of  $\text{Li}_s(z)$  of non-positive integral order  $s$ . Various known results regarding these functions, mainly obtained in different contexts and currently scattered throughout the literature, have been brought together here along with many new results and insights and they all have been proved in a simple and unified manner. However, it is neither intended nor attempted to provide a detailed historical account of this topic, nor to attribute the results to the original authors. Much attention has been given to considering all possible ways of computing  $\text{Li}_s(z)$  for  $s = 0, -1, -2, \dots$ , and, in particular, to the problem raised by Lee [6, Sec. 3] concerning the existence of a general closed-form expression for these functions. It has turned out that there are, in the mathematical literature, several such expressions involving either the Stirling numbers of the second kind or the Eulerian numbers. In Sec. 4 we shall deduce a new explicit closed-form expression involving the Carlitz-Scoville tangent numbers.

## 2. POLYPSEUDOLOGS, THEIR PROPERTIES AND GENERATION

The polylogarithm functions (polylogarithms, or polylogs for short) of order  $s$  and argument  $z$ ,  $\text{Li}_s(z)$ , are defined by

$$\text{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^1 [\log(1/t)]^{s-1} \frac{dt}{1-zt}, \quad (\Im(s) > 0)$$

whenever this integral converges, *i.e.*,  $\Re(s) > 0$ ,  $\Re(z) < 1$ , and elsewhere by analytic continuation. It is assumed that  $\log(1/t)$  has its principal value and, evidently, there is branch cut from  $z = 1$  to  $\infty$ . In the important case where the parameter  $s$  is an integer,  $\text{Li}_s(z)$  will be denoted by  $\text{Li}_n(z)$  (or  $\text{Li}_{-n}(z)$  when the parameter is negative).  $\text{Li}_n(z)$ ,  $n \in \mathbb{N} := \{1, 2, \dots\}$ , are the (*classical*) polylogarithms of order  $n$  (*i.e.* the  $n$ th order polylogarithms). The special case  $n = 1$  is the ordinary logarithm (or monologarithm)  $\text{Li}_1(z) = -\log(1-z)$ , while the cases  $n = 2, 3, 4, \dots$ , are known, respectively, as dilogarithm, trilogarithm, quadrilogarithm, etc.

For more details and an extensive list of references in which polylogarithms appear in physical and mathematical problems we refer the reader to Maximon [9]. The  $n$ th order polylogarithms are thoroughly covered in Lewin's standard text [10], while many formulae involving  $\text{Li}_s(z)$  can be found in Erdélyi *et al.* [11, pp. 30–31] and Prudnikov *et al.* [12, pp. 762–763]. For some new results and applications see [13–15].

Following Lee [6], the functions  $\text{Li}_{-n}(z)$ ,  $n \in \mathbb{N}_0$ , are here referred to as the polypseudologarithms (or polypseudologs) of order  $n$  and we first summarize their main properties and later thoroughly consider different ways of their computation. Polypseudologs exhibit several very important properties which are common to all polylogarithms. (i) The recurrence relation for polylogs can be rewritten as follows

$$z \frac{d}{dz} \text{Li}_{-n}(z) = \text{Li}_{-(n+1)}(z) \quad (n \in \mathbb{N}_0). \quad (2.1)$$

(ii) Polypseudologs satisfy the following particularly simple inversion relation

$$\text{Li}_{-n}\left(\frac{1}{z}\right) = (-1)^{n+1} \text{Li}_{-n}(z) \quad (n \in \mathbb{N}). \quad (2.2)$$

(iii) Duplication relation (or quadratic transformation)

$$\text{Li}_{-n}(z) + \text{Li}_{-n}(-z) = 2^{1+n} \text{Li}_{-n}(z^2) \quad (2.3)$$

is satisfied. (iv) If  $|z| < 1$ , polypseudologs have the series expansions given by  $\sum_{k=1}^{\infty} k^n z^n$ . In addition,  $\text{Li}_{-n}(z)$  have some additional, to these functions, specific properties. (v)  $\text{Li}_0(z)$ ,  $\text{Li}_{-1}(z)$ ,  $\text{Li}_{-2}(z)$ ,  $\dots$ , are rational functions. (vi) They have a pole of order  $n+1$  at  $z=1$ . (vii) The values of polypseudologs at  $z=-1$  are related to the values of the Riemann zeta function  $\zeta(s)$  at negative integers and are expressible in terms of the Bernoulli numbers  $B_n$

$$\text{Li}_{-n}(-1) = (2^{1+n} - 1) \zeta(-n) = (1 - 2^{1+n}) \frac{B_{n+1}}{n+1} \quad (n \in \mathbb{N}). \quad (2.4)$$

Hence,  $\text{Li}_{-2n}(-1) = 0$  and  $\text{Li}_{1-2n}(-1) = (1 - 2^{2n}) B_{2n} / (2n)$  since  $B_{2n+1} = 0$ . (viii)  $\text{Li}_{-n}(z)$  are, for all orders, factorable by  $z$  and also by  $(z+1)$  if  $n$  is an even number.

Lee [6, Sec. III], by applying the recurrence relation to  $\text{Li}_1(z) = -\log(1-z)$  and repeating it over and over again, obtained the  $\text{Li}_{-n}(z)$  to order  $n=8$ :

$$\text{Li}_0(z) = z/(1-z),$$

$$\text{Li}_{-1}(z) = z/(1-z)^2$$

$$\text{Li}_{-2}(z) = z(1+z)/(1-z)^3,$$

$$\text{Li}_{-3}(z) = z(1+4z+z^2)/(1-z)^4,$$

$$\text{Li}_{-4}(z) = z(1+z)(1+10z+z^2)/(1-z)^5,$$

$$\text{Li}_{-5}(z) = z(1+26z+66z^2+26z^3+z^4)/(1-z)^6,$$

$$\text{Li}_{-6}(z) = z(1+z)(1+56z+246z^2+56z^3+z^4)/(1-z)^7,$$

$$\text{Li}_{-7}(z) = z(1+120z+1191z^2+2416z^3+1191z^4+120z^5+z^6)/(1-z)^8,$$

$$\text{Li}_{-8}(z) = z(1+z)(1+246z+4047z^2+11572z^3+4047z^4+246z^5+z^6)/(1-z)^9.$$

Clearly, in this way one can find the polylog of any desired lower order. In this section we consider a half-dozen additional ways for obtaining polypseudologs and majority of them are the closed-form formulae.

The second method to generate  $\text{Li}_{-n}(z)$  will next be described. Note that  $\text{Li}_{-1}(z) = (z \frac{d}{dz}) \sum_{k=0}^{\infty} z^k$ , where we first differentiate and then multiply by  $z$ , and, if we apply the operator  $z \frac{d}{dz}$   $n$  times, which we denote by the symbol  $(z \frac{d}{dz})^n$ , to the summation formula  $\sum_{k=0}^{\infty} z^k = (1-z)^{-1}$ , we have [17, p. 364, Eq. (2)]

$$\text{Li}_{-n}(z) = \left( z \frac{d}{dz} \right)^n \frac{1}{1-z} \quad (n \in \mathbb{N}). \quad (2.5)$$

Moreover, it will be shown below that this relation leads to a closed-form formula involving the Stirling numbers of the second kind.

An alternative way of generating  $\text{Li}_{-n}(z)$  for any  $n$  would be to make use of the generating function method, *i.e.* to generate  $\{\text{Li}_{-n}(z)\}_{n=1}^{\infty}$  from a single function of two variables  $G(z, t)$  by repeated differentiation of that function. It is fortunate that there are several such functions (of which (2.6a) and (2.6b) could be found in the literature; see [20, p. 987] and [21, p. 152, Eq. (19)])

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \text{Li}_{-n}(z) = \frac{1}{1 - z e^t}, \quad (2.6a)$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} (-1)^n \text{Li}_{-n}(z) = \frac{z}{e^t - z}, \quad (2.6b)$$

$$\sum_{n=2}^{\infty} \frac{t^n}{n!} (-1)^{n-1} \text{Li}_{-(n-1)}(z) = \log(e^t - z), \quad (2.6c)$$

$$\sum_{n=1}^{\infty} \frac{t^n}{n!} \text{Li}_{-n}(z) = \frac{1}{2} \frac{1 + z e^t}{1 - z e^t}, \quad (2.6d)$$

so that, for  $n \in \mathbb{N}$ , we have

$$\begin{aligned} \text{Li}_{-n}(z) &\stackrel{\text{a}}{=} \frac{d^n}{dt^n} \left( \frac{1}{1 - z e^t} \right) \Big|_{t=0} \stackrel{\text{b}}{=} (-1)^n z \frac{d^n}{dt^n} \left( \frac{1}{e^t - z} \right) \Big|_{t=0} \\ &\stackrel{\text{c}}{=} (-1)^n \frac{d^{n+1}}{dt^{n+1}} \log(e^t - z) \Big|_{t=0} \stackrel{\text{d}}{=} \frac{1}{2} \frac{d^n}{dt^n} \left( \frac{1 + z e^t}{1 - z e^t} \right) \Big|_{t=0}. \end{aligned} \quad (2.7)$$

Observe that it is in fact enough only to demonstrate that (2.6a) is the generating function for  $\text{Li}_{-n}(z)$ ; the function in (2.6b) follows from (2.6a) (as well as *vice versa*) by appealing to the inversion property in (2.2), while, the functions given by (2.6b) and (2.6c) could be transformed into each other by the recurrence relation (2.1). It is interesting that both functions  $1/(1 - ze^t)$  and  $(1/2)(1 + ze^t)/(1 - ze^t)$  have the same first derivative  $e^t/(1 - ze^t)^2$ , thus we have that (2.6a) and (2.6d) are equivalent. Regarding (2.6a), let  $ze^t < 1$ , where  $|z| < 1$ , then, since  $\sum_{k=0}^{\infty} (ze^t)^k = 1/(1 - ze^t)$ , it is obvious that  $n$ th derivative of  $1/(1 - ze^t)$  with respect to  $t$  evaluated at  $t = 0$  gives  $\text{Li}_{-n}(z)$ .

Another way to compute  $\text{Li}_{-n}(z)$  stems from the fact that all the derivatives indicated in (2.7) are readily expressible analytically in a closed form. Indeed, if we

use the known result ( [21, p. 152] and [22, p. 32])

$$\left. \frac{d^n \Phi(e^t)}{dt^n} \right|_{t=0} = \sum_{k=1}^n S(n, k) \left. \frac{d^k \Phi(t)}{dt^k} \right|_{t=1} \quad (n \in \mathbb{N}), \quad (2.8)$$

where  $S(n, k)$  stands for the Stirling numbers of the second kind defined by means [16, p. 204, Eq. (1b)]

$$S(n, k) = \frac{1}{k!} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \ell^n = \frac{1}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} (k-\ell)^n, \quad (2.9)$$

(also see (2.12) below) and  $\Phi(t)$  is any infinitely differentiable function, then, starting from the corresponding expressions in (2.7), we arrive at the following explicit formulae valid for  $n \in \mathbb{N}$  (for (2.10b), see [21, p. 152])

$$\text{Li}_{-n}(z) = \sum_{k=1}^n \frac{k! S(n, k) z^k}{(1-z)^{k+1}}, \quad (2.10a)$$

$$\text{Li}_{-n}(z) = \sum_{k=1}^n \frac{(-1)^{n+k} k! S(n, k) z}{(1-z)^{k+1}}, \quad (2.10b)$$

$$\text{Li}_{-n}(z) = \sum_{k=0}^n \frac{(-1)^{n+k} k! S(n+1, k+1)}{(1-z)^{k+1}}. \quad (2.10c)$$

We remark that (2.10a) is obtained by (2.7a) as well as by (2.7d), and, furthermore, this formula also follows from (2.5) upon using [17, p. 364, Eq. (3)]

$$\left( z \frac{d}{dz} \right)^n \Psi(z) = \sum_{k=1}^n z^k S(n, k) \frac{d^k \Psi(t)}{dz^k} \quad (n \in \mathbb{N}), \quad (2.11)$$

$\Psi(z)$  being any infinitely differentiable function. It should be noted that it is easy to prove the formulae (2.8) and (2.11) by induction on  $n$ . For  $n = 1$ , they are obviously true and all that is needed for the induction step is the recurrence relation for  $S(n, k)$  [16, p. 208, Eq. (3a)]

$$\begin{aligned} S(n, k) &= k S(n-1, k) + S(n-1, k-1), \\ S(n, 0) &= S(0, k) = 0 \quad \text{and} \quad S(0, 0) = 1 \end{aligned} \quad (k, n \in \mathbb{N}). \quad (2.12)$$

Further, in view of the value of  $S(n, k)$  explicitly given by (2.9), it is straightforward, starting from (2.10), to obtain another group of the closed-form formulae, this time involving double finite sums (for (2.13c), see [20, p. 988, Eq. (7)])

$$\text{Li}_{-n}(z) = \sum_{k=1}^n \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \frac{\ell^n z^k}{(1-z)^{k+1}}, \quad (2.13a)$$

$$\text{Li}_{-n}(z) = \sum_{k=1}^n \sum_{\ell=0}^k (-1)^{n-\ell} \binom{k}{\ell} \frac{\ell^n z}{(1-z)^{k+1}}, \quad (2.13b)$$

$$\text{Li}_{-n}(z) = \sum_{k=0}^n \sum_{\ell=0}^{k+1} \frac{(-1)^{n+1-\ell} \binom{k+1}{\ell}}{k+1} \frac{\ell^{n+1}}{(1-z)^{k+1}}. \quad (2.13c)$$

Next, it is not difficult to show that, in terms of the Eulerian polynomials and numbers,  $A_n(z)$  and  $A(n, k)$ , we have [16, p. 245, Eq. (5n)]

$$\text{Li}_{-n}(z) = \frac{A_n(z)}{(1-z)^{n+1}} = \frac{1}{(1-z)^{n+1}} \sum_{k=0}^n A(n, k) z^k. \quad (2.14)$$

The *Eulerian polynomials*,  $A_n(x)$ , are usually defined by the generating function [16, p. 244, Eq. (5i)]

$$\sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!} = \frac{1-x}{1-xe^{t(1-x)}} \quad (2.15)$$

and they are polynomials in  $x$  of degree  $n$ , explicitly given by

$$A_n(x) = \sum_{k=0}^n A(n, k) x^k, \quad (2.16)$$

where the coefficients  $A(n, k)$  are the *Eulerian numbers* [16, p. 242, Eq. (5f)]

$$A(n, k) = \sum_{\ell=0}^k (-1)^\ell \binom{n+1}{\ell} (k-\ell)^n. \quad (2.17)$$

Observe that the Eulerian polynomials and numbers are not to be confused with the Euler polynomials  $E_n(x)$  and numbers  $E_n$ .

To prove (2.14) we may proceed as follows: upon replacing  $t$  by  $t/(1-x)$  in (2.15) and dividing both sides of the obtained expression by  $(1-x)$ , the right-hand side of (2.15) then becomes  $1/(1-xe^t)$  which is the same as the right-hand side of (2.6a), and hence (2.14).

However, there is one more formula: by (2.14) and (2.17) we get ([18, p. 325], [19] and [20, p. 987, Eq. (3)])

$$\text{Li}_{-n}(z) = \frac{1}{(1-z)^{n+1}} \sum_{k=0}^n \sum_{\ell=0}^k (-1)^\ell \binom{n+1}{\ell} (k-\ell)^n z^k. \quad (2.18)$$

### 3. NEW EXPLICIT CLOSED-FORM FORMULA FOR POLYPSEUDOLOGS

In this section we shall deduce the following new explicit closed-form formula for polypseudologs  $\text{Li}_{-n}(z)$ ,  $n \in \mathbb{N}$ ,

$$\text{Li}_{-n}(z) = \frac{1}{2^{n+1}} \left[ (-1)^{\lfloor n/2-1 \rfloor} T(n, 1) + \sum_{k=1}^{n+1} \frac{(-1)^{\lfloor (n-k)/2-1 \rfloor}}{k} T(n+1, k) \left( \frac{1+z}{1-z} \right)^k \right], \quad (3.1)$$

where  $T(n, k)$  stands for the tangent numbers (of order  $k$ ) or the higher order tangent numbers defined by [23, p. 428]

$$\tan^k(t) = \sum_{n=k}^{\infty} T(n, k) \frac{t^n}{n!} \quad (k \in \mathbb{N}). \quad (3.2)$$

Our derivation of (3.1) goes through two steps. The first step is to show that  $\text{Li}_{-n}(z)$  could be expressed in terms of the *derivative polynomials for tangent*,  $P_n(x)$ ,

which are given by means of the exponential generating function [24]

$$P(x, t) := \frac{x + \tan(t)}{1 - x \tan(t)} = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!} \tag{3.3}$$

or, equivalently, by the recurrence relation [24]

$$P_0(x) = x, \quad P_n(x) = (1 + x^2) P'_{n-1}(x) \quad (n \in \mathbb{N}). \tag{3.3^*}$$

The second step is to make use of a new formula for  $P_n(x)$

$$P_n(x) = T(n, 1) + \sum_{k=1}^{n+1} \frac{1}{k} T(n+1, k) x^k \quad (n \in \mathbb{N}), \tag{3.4}$$

which shall be proved separately at the end of the section.

We are now ready to proceed with the derivation of (3.1). First, consider the generating function for the derivative polynomials  $P_n(x)$  in (3.3). Upon multiplying both sides by  $\iota/2$ ,  $\iota := \sqrt{-1}$ , replacing  $x$  by  $-\iota(1+z)/(1-z)$ , replacing  $t$  by  $it/2$  and noticing that  $\tan(it/2) = -\iota(1-e^t)/(1+e^t)$ , we obtain, after simple algebra, the following

$$\frac{1}{2} \cdot \frac{1+ze^t}{1-ze^t} = \sum_{n=1}^{\infty} \left(\frac{\iota}{2}\right)^{n+1} P_n\left(\frac{1}{\iota} \cdot \frac{1+z}{1-z}\right) \frac{t^n}{n!}. \tag{3.5}$$

Now, if we compare this with the generating function for  $\text{Li}_{-n}(z)$  given by (2.6d), then, clearly, we have

$$\text{Li}_{-n}(z) = \left(\frac{\iota}{2}\right)^{n+1} P_n\left(\frac{1}{\iota} \cdot \frac{1+z}{1-z}\right) \quad (n \in \mathbb{N}). \tag{3.6}$$

Secondly, after combining (3.6) and (3.4), we get a formula similar to the required formula (3.1) and what remains is to evaluate the powers  $\iota^{n+1}$  and  $\iota^{n-k+1}$  which respectively appear as multiplication factors before  $T(n, 1)$  and  $T(n+1, k)$ . At this point it is helpful to examine the tangent numbers in more detail (see Table 1 and Eq. (3.2)) and notice that  $T(n, k) \neq 0$  only when  $1 \leq k \leq n$  and either both  $n$  and  $k$  are even or both  $n$  and  $k$  are odd. In other words,  $T(2m, 2r+1) = 0$  and  $T(2m+1, 2r) = 0$  ( $m, r \in \mathbb{N}_0$ ).

TABLE 1. Tangent numbers  $T(n, k)$

| $n \setminus k$ | 1    | 2    | 3      | 4     | 5      | 6     | 7      | 8     | 9      |
|-----------------|------|------|--------|-------|--------|-------|--------|-------|--------|
| 1               | 1    |      |        |       |        |       |        |       |        |
| 2               | 0    | 2    |        |       |        |       |        |       |        |
| 3               | 2    | 0    | 6      |       |        |       |        |       |        |
| 4               | 0    | 16   | 0      | 24    |        |       |        |       |        |
| 5               | 16   | 0    | 120    | 0     | 120    |       |        |       |        |
| 6               | 0    | 272  | 0      | 960   | 0      | 720   |        |       |        |
| 7               | 272  | 0    | 3696   | 0     | 8400   | 0     | 5040   |       |        |
| 8               | 0    | 7936 | 0      | 48384 | 0      | 80640 | 0      | 40320 |        |
| 9               | 7396 | 0    | 168960 | 0     | 645120 | 0     | 846720 | 0     | 362880 |

This means that the terms involving  $T(n, 1)$  when  $n$  is odd and  $T(n + 1, k)$  when  $n - k$  is odd are the only surviving terms in the obtained formula. Hence, we only need to evaluate the powers  $i^{n+1}$  when  $n$  is odd and  $i^{n-k+1}$  when  $n - k$  is odd. This is how we have found the negative powers, so the proof of (3.1) is now complete.

In order to prove formula (3.4) note that the generating function of  $P_n(x)$  in (3.3) can be rewritten as

$$P(x, t) = [x + \tan(t)] \sum_{k=0}^{\infty} [x \tan(t)]^k = x + (1 + x^2) \sum_{k=1}^{\infty} x^{k-1} \tan^k(t)$$

which, by making use of the definition of  $T(n, k)$  in (3.2) and the following elementary double series identities [25, p. 57, Eq. (2)]

$$\sum_{n=1}^{\infty} \sum_{k=1}^n a(k, n) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a(k, n+k) = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} a(k, n),$$

becomes

$$\begin{aligned} P(x, t) &= x + (1 + x^2) \sum_{k=1}^{\infty} x^{k-1} \sum_{n=k}^{\infty} T(n, k) \frac{t^n}{n!} \\ &= x + \sum_{n=1}^{\infty} (1 + x^2) \left( \sum_{k=1}^n T(n, k) x^{k-1} \right) \frac{t^n}{n!}. \end{aligned} \quad (3.7)$$

On the other hand, by the definition of  $P_n(x)$  in (3.3) in conjunction with the recurrence relation for  $P_n(x)$  (3.3\*), we have

$$P(x, t) = P_0(x) + \sum_{n=1}^{\infty} P_n(x) \frac{t^n}{n!} = x + \sum_{n=1}^{\infty} (1 + x^2) P'_{n-1}(x) \frac{t^n}{n!} \quad (3.8)$$

and thus comparing (3.7) with (3.8) clearly yields

$$P'_{n-1}(x) = \sum_{k=1}^n T(n, k) x^{k-1}, \quad \text{thus} \quad P_n(x) = P_n(0) + \sum_{k=1}^{n+1} \frac{1}{k} T(n+1, k) x^k. \quad (3.9)$$

Finally, the desired result (3.4) follows from (3.9) since, by the definitions (3.2) and (3.3), we have  $P_n(0) = T(n, 1)$ .

#### 4. CONCLUDING REMARKS

It is interesting to note that several special functions, such as the Riemann zeta function  $\zeta(s)$ , the Hurwitz zeta function  $\zeta(s, a)$  and the classical gamma function  $\Gamma(z)$ , when the argument of a function takes on non-positive integer values show an abrupt changes in properties which cannot be easily explained. For instance, at  $z = 0, -1, -2, \dots$ , the function  $\Gamma(z)$  becomes singular, while  $\zeta(s)$  takes on rational values. On the other hand,  $\zeta(s, a)$ , for  $s = 0, -1, -2, \dots$ , reduces to a polynomial in  $a$  which is expressible in terms of the Bernoulli polynomial  $B_n(x)$  as follows

$$\zeta(-n, a) = -\frac{B_{n+1}(a)}{n+1} \quad (n \in \mathbb{N}_0).$$

Similarly, the polypseudologs considered in this paper, *i.e.* the polylogs  $\text{Li}_s(z)$  for  $s = 0, -1, -2, \dots$ , are, in fact, rational functions in  $z$ .



Lee [4] already made use of the nil-log  $\text{Li}_0(z) = z/(1+z)$  in his study of the statistical mechanics of ideal particles in null dimension and discovered several anomalous physical phenomena, among them probably the most remarkable one is that the bosons are not confinable. It remains to be seen whether it would be possible, as it is expected, to employ the polypseudologs  $\text{Li}_{-1}(z)$ ,  $\text{Li}_{-2}(z)$ ,  $\text{Li}_{-3}(z)$ ,  $\dots$ , in some kind of extension of Lee's formalism to the case of negative physical dimensions. It is hoped that this paper will contribute to a better understanding of the analytical properties of these functions and stimulate further work in the field.

#### ACKNOWLEDGEMENTS

The author is very grateful to the three anonymous referees for a careful and thorough reading of the previous version of this paper. Their helpful and valuable comments and suggestions have led to a considerably improved presentation of the results. The author acknowledges financial support from Ministry of Science of the Republic of Serbia under Research Projects 142025 and 144004.

#### REFERENCES

- [1] M.H. Lee and J. Kim, Quantum gases and polylogs, *Physica A* **304** (2002), 421–428.
- [2] M.H. Lee, Polylogarithms and logarithmic diversion in statistical mechanics, *Acta Phys. Polon. B* **40** (2009), 1279–1301.
- [3] M.H. Lee, Polylogarithmic analysis of chemical potential and fluctuations in a  $d$ -dimensional free Fermi gas at low temperatures, *J. Math. Phys.* **36** (1995), 1217–1231.
- [4] M.H. Lee, Statistical mechanics of ideal particles in null dimension and confinement, *Phys. Rev. E* **54** (1995), 946–949.
- [5] M.H. Lee, Equivalence of ideal gases in  $2d$  and Landen's relations, *Phys. Rev. E* **55** (1997), 1518–1520.
- [6] M.H. Lee, Polylogarithms and Riemann's  $\zeta$  function, *Phys. Rev. E* **56** (1997), 3909–3912.
- [7] K.R. Patton, M.R. Geller, and M.P. Blencowe, Thermodynamic equivalence of certain ideal Bose and Fermi gases, *Physica A* **357** (2005), 427–435.
- [8] A. Baram and M. Luban, The Gaussian model of a fluid in dimensions  $D = 0, -2$ : an exact solution *J. Phys. A* **19** (1986), L585–L588.
- [9] L.C. Maximon, The dilogarithm function for complex argument, *Proc. R. Soc. Lond. A* **459** (2003), 2807–2819.
- [10] L. Lewin, *Polylogarithms and associated functions*, North-Holland, New York, 1981.
- [11] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher transcendental functions, volume I*, McGraw-Hill, New York, 1953.
- [12] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, *Integrals and series, volume III: more special functions*, Gordon and Breach, New York, 1990.
- [13] D. Cvijović, Continued-fraction expansions for the Riemann zeta function and polylogarithms, *Proc. Amer. Math. Soc.* **123** (1995), 1527–1535.
- [14] M.H. Lee, Solving certain principal value integrals by reduction to the dilogarithm, *Physica A* **234** (1996), 581–588.
- [15] D. Cvijović, New integral representations of the polylogarithm function, *Proc. R. Soc. A* **463** (2007), 897–905.

- [16] L. Comtet, *Advanced combinatorics. The art of finite and infinite expansions*, D. Reidel, Dordrecht, Holland, 1974.
- [17] P. Knopf, The operator  $(x\frac{d}{dx})^n$  and its application to series, *Math. Mag.* **76** (2003), 364–371.
- [18] R. Stalley, A generalization of the geometric series, *Am. Math. Monthly* **56** (1949), 325–327.
- [19] M.S. Klamkin, On a generalization of the geometric series, *Am. Math. Monthly* **64** (1957), 91–93.
- [20] D. Zeitlin, Two methods for the evaluation of  $\sum_{k=0}^{\infty} k^n x^k$ , *Amer. Math. Monthly* **68** (1961), 986–989.
- [21] C. Truesdell, On a function which occurs in the theory of the structure of polymers, *Ann. of Math.* **46** (1945), 144–157.
- [22] C. Jordan, *Calculus of finite differences*, Chelsea Publ. Co., New York, 1950.
- [23] L. Carlitz and R. Scoville, Tangent numbers and operators, *Duke Math. J.* **39** (1972), 413–429.
- [24] M.E. Hoffman, Derivative polynomials for tangent and secant, *Amer. Math. Monthly* **102** (1995), 23–30.
- [25] E.D. Rainville, *Special functions*, Macmillan, New York, 1960.