Correlation and Entanglement of Multipartite States

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We derive a classification and a measure of classical- and quantum-correlation of multipartite qubit, qutrit, and in general, n-level systems, in terms of SU(n) representations of density matrices. We compare the measure for the case of bipartite correlation with concurrence and the entropy of entanglement. The characterization of correlation is in terms of the number of nonzero singular values of the correlation matrix, but that of mixed state entanglement requires additional invariant parameters in the density matrix. For the bipartite qubit case, the condition for mixed state entanglement is written explicitly in terms of the invariant paramters in the density matrix. For identical particle systems we analyze the effects of exchange symmetry on classical and quantum correlation.

Quantum entanglement is an information resource; it plays an important role in many protocols for quantum-information processing, including quantum computation [1], quantum cryptography [2], teleportation [3], super-dense coding [4], and quantum error correction protocols [5]. Techniques for characterizing the bipartite entanglement and correlation of pure and mixed quantum states have enabled many advances in quantum information and the study of decoherence [6]. Many quantum information protocols use bipartite entanglement, but multipartite entanglement [7], also has quantum-information applications, e.g., controlled secure direct communication [8], quantum error correction [9], controlled teleportation [10] and secret sharing [11]. It has been shown that any inseparable two-qubit states can be distilled to a singlet-state form with enough copies of the qubit-pairs [12] and algorithms for multi-copy entanglement distillation for pairs of qubits have been developed [13]. Moreover, multipartite entanglement offers a means of enhancing interferometric precision beyond the standard quantum limit and is therefore relevant to increasing the precision of atomic clocks by decreasing projection noise in spectroscopy [14]. Here we use a representation of the density matrix for qubit, qutrit, and more generally, n-level systems containing 2, 3, ..., and N-parts, in terms of the correlations between the subsystems to quantify the classical and quantum correlation of multipartite systems. Our classification of correlation is in terms of the correlation matrix and its singular values, and our classification of entanglement of mixed states [15] is associated with the Peres-Horodecki criterion [19], which we express in terms of additional invariant parameters in the density matrix. Separate measures of bipartite, tripartite, etc., correlation are required, since general mixed states can have bipartite correlation as well as higher subsystem-number-correlation.

Werner [15] defined a mixed state of an N-partite system as *separable*, i.e., *classically-correlated*, if it can be written as a convex sum,

$$\rho = \sum_k p_k \, \rho_k^A \rho_k^B \ldots \rho_k^N \ , \quad p_k > 0 \ , \quad \sum_k p_k = 1 \ , \quad (1)$$

where ρ_k^A is a valid density matrix of subsystem A, etc. Otherwise, Werner defined it to be *entangled*, i.e., *quantum-correlated*. Unfortunately, this definition of entanglement for mixed states is not constructive, since, in general, it cannot be used to decide whether a given density matrix is separable or entangled. Moreover, a quantitative measure of entanglement of multi-partite systems has proven to be difficult to devise. Note that studies of the best separable approximation to an arbitrary density have been carried out and have led to a proposal of a measure for entanglement [16]. Furthermore, aspects of the geometry of separability and entanglement based on Schmidt decomposition have been studied and led to an analysis of the question of separability for the two-qubit case [17].

In what follows, we categorize classically-correlated and quantum-correlated states and characterize their correlation in terms the number of nonzero singular values [18], $\{d_i\}$, of the correlation matrix $\mathbf{C}$, and characterize entanglement of bitpartite qubit systems using the Peres-Horodecki criterion [19] which is reformulated totally in terms of the parameters used in forming the density matrix.

First, let us consider a bipartite qubit system. For two uncorrelated qubits, call them A and B, we can write the density matrix as a product, $\rho_{AB} = \rho_A \rho_B$, where the individual qubit density matrices can be written as $\rho_J = \frac{1}{2} \left(1 + \mathbf{n}_J \cdot \boldsymbol{\sigma}_J\right)$, where $J = A, B$, the $\boldsymbol{\sigma}_J$ are Pauli matrices for particle J and the Bloch vectors are $\mathbf{n}_J = \langle \boldsymbol{\sigma}_J \rangle = \mathrm{Tr} \, \boldsymbol{\sigma}_J \rho_J$ [20]. For two correlated qubits,

$$\rho_{AB} = \frac{1}{4} \left[\left(1 + \mathbf{n}_A \cdot \boldsymbol{\sigma}_A\right)\left(1 + \mathbf{n}_B \cdot \boldsymbol{\sigma}_B\right) + \boldsymbol{\sigma}_A \cdot \mathbf{C}^{AB} \cdot \boldsymbol{\sigma}_B\right], \quad (2)$$

where the tensor $\mathbf{C}^{AB}$ specifies the qubit correlations,

$$C_{ij}^{AB} \equiv \langle \sigma_{i,A} \sigma_{j,B} \rangle - \langle \sigma_{i,A} \rangle \langle \sigma_{j,B} \rangle = \langle \sigma_{i,A} \sigma_{j,B} \rangle - n_{i,A} \, n_{j,B}. \quad (3)$$

The density matrix ρ_{AB} is a 4×4 Hermitian matrix with trace unity, so 15 parameters are required to parameterize it. The 3 components of $\mathbf{n}_A$, the 3 components of $\mathbf{n}_B$, and the 9 components C_{ij} of the 3×3 matrix $\mathbf{C}$, where we have no longer explicitly shown the subsystem superscripts, are sufficient for this purpose.

Similarly for the bipartite qutrit case. The 3×3 density matrix of a single qutrit can be written as $\rho = \frac{1}{3}\left(1+\frac{3}{2}\langle\lambda_i\rangle\lambda_i\right)$ where the λ_i are the eight traceless Hermitian Gellman matrices familiar from SU(3) [21], and $\langle\lambda_i\rangle = \mathrm{Tr}\,\lambda_i\rho$. A bipartite qutrit density matrix can be parameterized in the form

$$\rho_{AB} = \frac{1}{9}\left[(1+\frac{3}{2}\langle\lambda_{i,A}\rangle\lambda_{i,A})\,(1+\frac{3}{2}\langle\lambda_{j,B}\rangle\lambda_{j,B})+\frac{9}{4}\lambda_{i,A}C_{ij}\lambda_{j,B}\right],$$

(4)

$$C_{ij} \equiv \langle\lambda_{i,A}\lambda_{j,B}\rangle - \langle\lambda_{i,A}\rangle\langle\lambda_{j,B}\rangle\;,$$

(5)

where C_{ij} specifies the correlation between $\lambda_{i,A}$ and $\lambda_{j,B}$. Here, ρ_{AB} is a 9×9 Hermitian matrix with trace unity, so 80 parameters are required to parameterize it. The eight components of $\langle\lambda_{i,A}\rangle$, eight components of $\langle\lambda_{i,B}\rangle$, and 64 components C_{ij} of the 8×8 matrix $\mathbf{C}$ are sufficient for this purpose. The same procedure can be used for bipartite 4-level systems using the 15 traceless 4×4 Hermitian generator matrices for SU(4), and bipartite n-level systems with the n^2-1 traceless $n\times n$ Hermitian matrices. Likewise, a general qubit-qutrit 6×6 density matrix takes the form $\rho_{AB} = \frac{1}{6}\left[(1+n_{i,A}\sigma_{i,A})\,(1+\frac{3}{2}\langle\lambda_{j,B}\rangle\lambda_{j,B})+\frac{6}{4}\sigma_{i,A}C_{ij}\lambda_{j,B}\right]$ with $C_{ij} \equiv \langle\sigma_{i,A}\lambda_{j,B}\rangle - \langle\sigma_{i,A}\rangle\langle\lambda_{j,B}\rangle$, $i = 1,2,3$ and $j = 1,\ldots,8$.

Our bipartite correlation measure for an n-level and m-level system is based on the $(n^2-1)\times(m^2-1)$ correlation matrix $\mathbf{C}$:

$$\mathcal{E}_C \equiv \frac{n_<^2}{4(n_<^2-1)}\mathrm{Tr}\,\mathbf{C}\mathbf{C}^T = \frac{n_<^2}{4(n_<^2-1)}\sum_{i,j}C_{ij}C_{ji}^T\;,$$

(6)

where $n_< = \min(n,m)$. $\mathcal{E}_C = \frac{n_<^2}{4(n_<^2-1)}\mathrm{Tr}\,(\rho_{AB}-\rho_A\rho_B)^2$ is a nonnegative real number. If $\mathbf{C}$ is a normal matrix [18], $\mathrm{Tr}\,\mathbf{C}\mathbf{C}^T$ equals to the sum of the squares of its eigenvalues, but $\mathbf{C}$ need not be normal. $\mathcal{E}_C$ is basis-independent; any rotation in Hilbert space leaves it unchanged. The normalization factor $n_<^2/[4(n_<^2-1)]$ in (6) is such that the maximum possible value of $\mathcal{E}_C$ is unity. $\mathcal{E}_C$ measures both classical- and quantum-correlation. This measure of bipartite correlation was suggested in Ref. [22] for pure states and $n = m$.

The correlation matrix $\mathbf{C}$ quantifies the correlation and the entanglement of bipartite states. For pure two-qubit states, the number of nonzero singular values (NSVs) of $\mathbf{C}$ is zero for non-entangled states ($\mathbf{C}$ vanishes), and three for entangled states. For classically-correlated states with two terms in the sum [see Eq. (16)], only one NSV occurs, two NSVs occur for three terms, three NSVs occur for four or more terms, and for entangled (i.e., quantum-correlated) mixed states there are three NSVs. These cases are summarized in Fig. 1. Entangled mixed

states can be differentiated from classically-correlated states with 3 NSVs by applying the Peres-Horodecki (PH) partial transposition condition [19] [which corresponds to changing the sign of $n_{y,B}$ and the matrix elements C_{iy}^{AB} that multiply $\sigma_{y,B}$ in (2), and determining whether the resulting ρ is still a genuine density matrix — if it is, the state is classically correlated, i.e., unentangled but correlated] to the density matrices with 3 NSVs. The *only* categories that cannot be distinguished without use of the PH condition are the mixed-entangled and the classically correlated states with ≥ 4 NSVs.
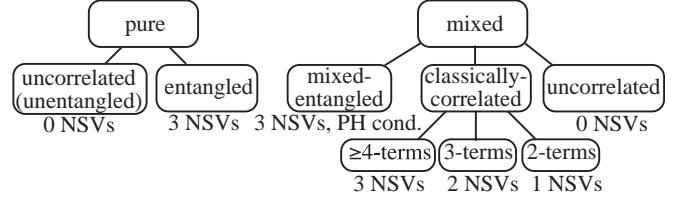

FIG. 1: Classification of two-qubit states. Categories can be experimentally distinguished by measuring $\mathbf{n}_A$, $\mathbf{n}_B$, and using Bell measurements [7] to determine the $\mathbf{C}$ matrix.

Similarly, for a two qutrit pure state, the number of NSVs of $\mathbf{C}$ is zero for non-entangled states (the $\mathbf{C}$ matrix vanishes), three, if only two basis states are present in the entangled state, five, if one of the qutrits contains only two basis states but the other contains three, and eight if all three basis states are present. For classically correlated qutrit states, there are 1, 2, $\ldots$, 8 NSVs for 2, 3, $\ldots$, and 9 or more terms in the sum, etc. A similar classification in terms of the number of NSVs exists for qubit-qutrit and n-level systems.

A general three-qubit density matrix can be written as

$$\rho_{ABC} = \frac{1}{8}\left[(1 + \mathbf{n}_A\cdot\boldsymbol{\sigma}_A)\,(1 + \mathbf{n}_B\cdot\boldsymbol{\sigma}_B)\,(1 + \mathbf{n}_C\cdot\boldsymbol{\sigma}_C)\right.$$

$$+\boldsymbol{\sigma}_A\cdot\mathbf{C}^{AB}\cdot\boldsymbol{\sigma}_B + \boldsymbol{\sigma}_A\cdot\mathbf{C}^{AC}\cdot\boldsymbol{\sigma}_C + \boldsymbol{\sigma}_B\cdot\mathbf{C}^{BC}\cdot\boldsymbol{\sigma}_C$$

$$\left.+ \sum_{ijk}\sigma_{i,A}\sigma_{j,B}\sigma_{k,C}D_{ijk}\right]\;,$$

(7)

where $\mathbf{C}^{AB}$, $\mathbf{C}^{AC}$, and $\mathbf{C}^{BC}$ are the bipartite correlation matrices and the tensor that specifies the tripartite correlations is

$$D_{ijk} \equiv \langle\sigma_{i,A}\sigma_{j,B}\sigma_{k,C}\rangle - \langle\sigma_{i,A}\rangle\langle\sigma_{j,B}\rangle\langle\sigma_{k,C}\rangle\;.$$

(8)

A tripartite qutrit state can be similarly parameterized:

$$\rho_{ABC} = \frac{1}{27}\left[\prod_I(1+\frac{3}{2}\sum_i\langle\lambda_{i,I}\rangle\lambda_{i,I})+\frac{9}{4}\sum_{I,J}\sum_{i,j}\lambda_{i,I}C_{ij,IJ}\lambda_{j,J}\right.$$

$$\left.+ \frac{27}{8}\sum_{I,J,K}\sum_{i,j,k}\lambda_{i,A}\lambda_{j,B}\lambda_{k,C}D_{ijk,IJK}\right]\;,$$

(9)

$$D_{ijk,IJK} \equiv \langle \lambda_{i,I} \lambda_{j,J} \lambda_{k,K} \rangle - \langle \lambda_{i,I} \rangle \langle \lambda_{j,J} \rangle \langle \lambda_{k,K} \rangle . \quad (10)$$

Our tripartite correlation measure $\mathcal{E}_D$ is based on the correlation matrix $\mathbf{D}$, $\mathcal{E}_D \equiv K \sum_{i,j,k} D_{ijk}^2$, which can also be written as

$$\mathcal{E}_D = K \operatorname{Tr} \left(\rho_{ABC} - \rho_A \rho_B \rho_C - \sum_{I,J(I \neq J)} \sum_{i,j} C_{ij}^{I,J} \sigma_{i,I} \sigma_{j,J} \right)^2 ,$$
$$(11)$$

where $K = 1/4$ for qubits, and $K = 27/160$ for qutrits with σs replaced by λs. $\mathcal{E}_D$ is also a basis-independent nonnegative real number; any rotation in Hilbert space leaves it unchanged. A tripartite system may have bipartite- as well as tripartite-correlation. The bipartite correlation of a tripartite system is the sum of the correlation for the three bipartite pairs,

$$\mathcal{E}_C \equiv \frac{n^2}{4(n^2-1)} \sum_{I,J(I \neq J)} \operatorname{Tr} \mathbf{C}^{I,J} (\mathbf{C}^{I,J})^T , \quad (12)$$

where $I, J = A, B, C$. The density matrices of four-particle and higher qubit, qutrits, and n-level system states can be constructed similarly, but with increased complexity. For example, it is clear from Eq. (11) how to generalize and obtain the four-particle correlation of four-particle systems: $\mathcal{E}_E \equiv K' \sum_{i,j,k,l} E_{ijkl}^2$, where the four-particle-correlation term of the four-qubit density matrix ρ_{ABCD} is $\sum_{ijkl} \sigma_{i,A} \sigma_{j,B} \sigma_{k,C} \sigma_{l,D} E_{ijkl}$ and $K' = 1/8$.

We now present some examples of qubit and qutrit bipartite and tripartite correlated states. The maximally entangled bipartite qubit states are the Bell states,

$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}} [|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle] , \quad |\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}} [|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle] . \quad (13)$$

For all these states, $\langle \boldsymbol{\sigma}_A \rangle = \langle \boldsymbol{\sigma}_B \rangle = \mathbf{0}$, i.e., $\mathbf{n}_A = \mathbf{n}_B = \mathbf{0}$. For the singlet, $\langle \sigma_{i,A} \sigma_{j,B} \rangle = -\delta_{ij}$ (the spins are oppositely polarized). The density matrices of the Bell states are:

$$\rho_{\Psi^-} = \frac{1}{4} \left(1_A 1_B - \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B \right) ,$$
$$\rho_{\Psi^+} = \frac{1}{4} \left(1_A 1_B + \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B - 2\sigma_{z,A} \sigma_{z,B} \right) ,$$
$$\rho_{\Phi^+} = \frac{1}{4} \left(1_A 1_B + \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B - 2\sigma_{y,A} \sigma_{y,B} \right) ,$$
$$\rho_{\Phi^-} = \frac{1}{4} \left(1_A 1_B + \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B - 2\sigma_{x,A} \sigma_{x,B} \right) . \quad (14)$$

The correlation matrices of the Bell's states are diagonal and the correlation measure is $\mathcal{E}_C = 1$, i.e., they are maximally entangled.

Let us now consider the Rashid pure states [23],

$$|\phi^+\rangle = (2\cosh(2\theta))^{-1/2} \left(e^{-\theta} |\uparrow\uparrow\rangle + e^{\theta} |\downarrow\downarrow\rangle \right) , \quad (15)$$

whose density matrix is $\rho_{\phi^+} = \frac{1}{4} \{ [1_A - \tanh(2\theta)\sigma_{z,A}][1_B - \tanh(2\theta)\sigma_{z,B}] + \operatorname{sech}(2\theta)(\sigma_{x,A}\sigma_{x,B} - \sigma_{y,A}\sigma_{y,B}) + \operatorname{sech}^2(2\theta)\sigma_{z,A}\sigma_{z,B} \}$. When $\theta = 0$,

$|\phi^+\rangle = |\Phi^+\rangle$, and as $\theta \to \pm\infty$, an unentangled state results. The nonvanishing correlation matrix elements are: $C_{xx} = \operatorname{sech}(2\theta), C_{yy} = -\operatorname{sech}(2\theta), C_{zz} = \operatorname{sech}^2(2\theta)$. Using (6) we obtain the correlation measure $\mathcal{E}_C(|\phi^+\rangle) = \frac{1}{3} \operatorname{Tr} \mathbf{CC}^T = \frac{1}{3}(2\operatorname{sech}^2(2\theta) + \operatorname{sech}^4(2\theta))$. The concurrence $\mathcal{C}$ [19, 24] is

$$\mathcal{C}(|\phi^+\rangle) = \sqrt{2(1 - \operatorname{Tr}[\rho_A^2])} = \operatorname{sech}(2\theta) ,$$

since

$$\rho_A = \frac{1}{2} \begin{pmatrix} 1 - \tanh(2\theta) & 0 \\ 0 & 1 + \tanh(2\theta) \end{pmatrix} ,$$

and the entropy is $S \equiv -\operatorname{Tr}[\rho_A \log_2 \rho_A]$. These results are graphically presented in Fig. 2. All the measures equal unity for $\theta = 0$ and decrease rapidly vs. θ.
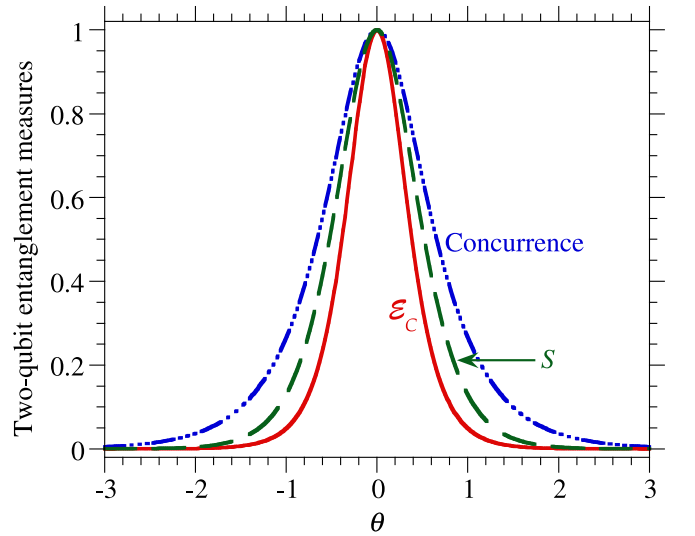

FIG. 2: (color online) Comparison of the $\mathcal{E}_C$ measure of correlation, the concurrence $\mathcal{C}$ and the entanglement entropy S for the Rashid pure states.

Two-qubit classically-correlated states take the form

$$\rho^{CC} = \frac{1}{4} \sum_{k \geq 2} p_k \left(1 + \mathbf{n}_{A,k} \cdot \boldsymbol{\sigma}_A \right) \left(1 + \mathbf{n}_{B,k} \cdot \boldsymbol{\sigma}_B \right) , \quad (16)$$

with $\sum_k p_k = 1$ and $p_k > 0$. The density matrix for the classically-correlated state can be written in the form of Eq. (2) with Bloch vectors

$$\mathbf{n}_A = \sum_k p_k \mathbf{n}_{A,k} , \quad \mathbf{n}_B = \sum_k p_k \mathbf{n}_{B,k} , \quad (17)$$

and correlation matrix

$$C_{ij} = \sum_k p_k n_{i,A,k} \left[n_{j,B,k} - \sum_l p_l n_{j,B,l} \right] . \quad (18)$$

For example, for classically-correlated mixed states of the form $\rho^{CC} = (2\operatorname{sech}^2(2\theta))^{-1/2}(e^{-\theta}|\downarrow\uparrow\rangle \langle\downarrow\uparrow| + e^{\theta}|\uparrow\downarrow\rangle \langle\uparrow\downarrow|)$,

we find that all the correlation coefficients vanish, except for $C_{zz} = -\mathrm{sech}^2(2\theta)$, the density matrix in representation (2) is $\rho^{\mathrm{CC}} = \frac{1}{4}\left(1_A 1_B - \mathrm{sech}^2(2\theta)\,\sigma_{z,A}\sigma_{z,B}\right)$, and the classical-correlation measure is $\mathcal{E}_C^{\mathrm{CC}} = \frac{1}{3}\mathrm{sech}^4(2\theta)$.

It is elucidating to consider the Werner two-qubit density matrix composed of a sum of a singlet state and the maximally mixed state, $\rho^W = p|\Psi^-\rangle\langle\Psi^-| + \frac{1-p}{4}\mathbf{1}$, or, the more general Werner two-qubit density matrix,

$$\rho^{GW} = p\,|\psi^-\rangle\langle\psi^-| + \frac{1-p}{4}\mathbf{1}\,, \qquad (19)$$

where $|\psi^-\rangle = (2\cosh(2\theta))^{-1/2}\left(e^{-\theta}|\uparrow\downarrow\rangle - e^{\theta}|\downarrow\uparrow\rangle\right)$. ρ^{GW} reduces to ρ^W for $\theta = 0$. For ρ^{GW},

$$\mathbf{n}_A = -\mathbf{n}_B = p\tanh(2\theta)\,\hat{\mathbf{z}}\,, \qquad (20)$$

and

$$\mathbf{C}^{GW} = -p\begin{pmatrix} \mathrm{sech}(2\theta) & 0 & 0 \\ 0 & \mathrm{sech}(2\theta) & 0 \\ 0 & 0 & 1-p+p\,\mathrm{sech}^2(2\theta) \end{pmatrix}. \qquad (21)$$

The PH entanglement criterion [19] shows that this state is entangled if $p[(1+2\mathrm{sech}(2\theta)] \geq 1$. Figure 3 plots the PH criterion limit and the correlation measure, $\mathcal{E}_C(p,\theta) = \sum_i d_i^2 = 1 - p + (2p^2 + p)\mathrm{sech}^2(2\theta)$, for the generalized Werner state. Note that the PH criterion is not obtainable from $\mathbf{C}$ alone, but can be obtained using the invariant parameters $\xi \equiv \sum_i d_i - \frac{\mathbf{n}_A\cdot\mathbf{C}\cdot\mathbf{n}_B}{\mathbf{n}_A\cdot\mathbf{n}_B}$ and $\mathbf{n}_A\cdot\mathbf{n}_B$. More explicitly, $p[1 + 2\mathrm{sech}(2\theta)] = -\xi + \sqrt{\xi^2/4 - \mathbf{n}_A\cdot\mathbf{n}_B}$, so the PH condition reads

$$-\frac{\xi}{2} + \frac{-\xi + \sqrt{\xi^2 - 4\,\mathbf{n}_A\cdot\mathbf{n}_B}}{2} \geq 1\,, \qquad (22)$$

which can be written as the condition: the largest root of the quadratic equation, $(x+\xi/2)^2 + \xi(x+\xi/2) + \mathbf{n}_A\cdot\mathbf{n}_B = 0$, is greater than unity. Thus, mixed state entanglement is determined not only by $\mathbf{C}$ but by additional invariant characteristics of the density matrix, i.e., invariant characteristics composed of the parameters $\mathbf{C}$, $\mathbf{n}_A$ and $\mathbf{n}_B$ used to form the density matrix (whereas the correlation is determined only in terms of $\mathbf{C}$). The physical significance of the scalar product $\mathbf{n}_A\cdot\mathbf{n}_B$ as the projection of the expectation value of the spin of one qubit on the other, is clear, as is the physical significance of ξ as a specific projection of the singular values of the correlation matrix that depends on the average spins $\mathbf{n}_A$ and $\mathbf{n}_B$ [25]. However, the physical significance of the PH entanglement criterion is not yet clear; i.e., the physical interpretation of Eq. (22) [or the quadratic equation] remains to be uncovered. But at least the PH condition is now expressed only in terms of the physical parameters appearing in the density matrix, rather than by the partial transposition condition, which is more removed from physical interpretation.

As an example of a tripartite pure qutrit state, consider

$$|\psi^{E3}\rangle = \frac{e^{\theta_1}e^{\theta_2}|v_1 v_1 v_1\rangle + e^{-\theta_1}|v_2 v_2 v_2\rangle + e^{-\theta_2}|v_3 v_3 v_3\rangle}{\sqrt{e^{2\theta_1}e^{2\theta_2} + e^{-2\theta_1} + e^{-2\theta_2}}}, \qquad (23)$$
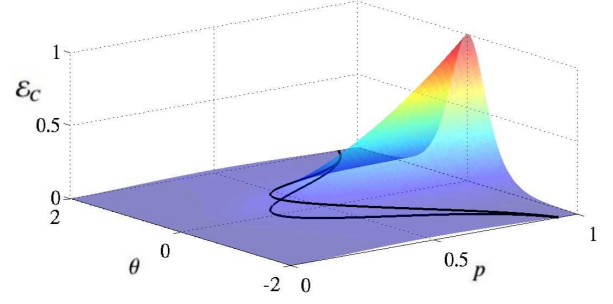

FIG. 3: (color online) $\mathcal{E}_C(p,\theta)$ versus p and θ for the generalized Werner density matrix ρ^{GW}, and the Peres-Horodecki entanglement criterion limit, $p[1 + 2\mathrm{sech}(2\theta)] = 1$, drawn on the p-θ plane and projected onto the $\mathcal{E}_C$ surface.

where $|v_1\rangle = (1,0,0)^T$, $|v_2\rangle = (0,1,0)^T$, $|v_3\rangle = (0,0,1)^T$. The qutrit bipartite correlation $\mathcal{E}_C$ and tripartite correlation $\mathcal{E}_D$ are plotted in Fig. 4. The maximum of $\mathcal{E}_D$ is at $\theta_1 = \theta_2 = 0$, where $\mathcal{E}_D = 1$, but the bipartite correlation dips there. Three ridges of high bipartite and tripartite correlation occur, one at $\theta_1 = \theta_2 =$ negative, and two others at 120 degrees rotation from the first.
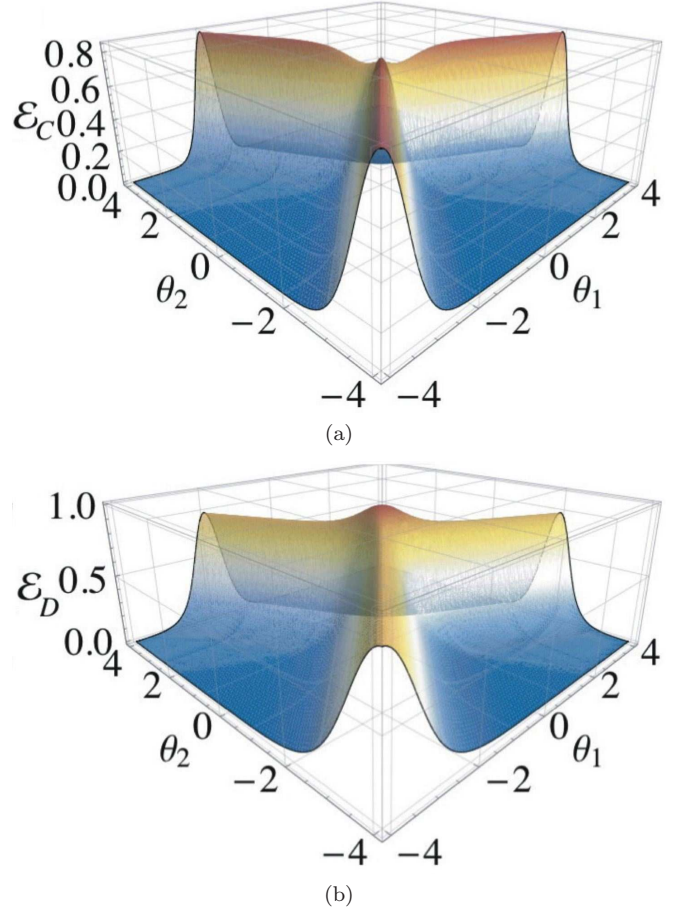

(a)

(b)

FIG. 4: (color online) The bipartite and tripartite correlation measures, $\mathcal{E}_C$ and $\mathcal{E}_D$ for the tripartite-qutrit pure state (23).

In most quantum information systems, qubits are distinguishable, so there is no need to account for exchange symmetry, but for *identical* bosonic or fermionic systems, the density matrix must be properly symmetrized, e.g., $\rho_{AB}^{\text{sym}} = \mathcal{S}\rho_{AB}\mathcal{S}$. For two qubits, the symmetrization operator is $\mathcal{S} = \frac{1}{2}(1 + P_{AB}) = \frac{3}{4} + \frac{1}{4}\boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B$ and the anti-symmetrization operator is $\mathcal{A} = \frac{1}{2}(1 - P_{AB}) = \frac{1}{4} - \frac{1}{4}\boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B$, hence, $\rho_{AB}^{\text{sym}} = \left(\frac{3}{4} + \frac{1}{4}\boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B\right)\rho_{AB}\left(\frac{3}{4} + \frac{1}{4}\boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B\right)$, $\rho_{AB}^{\text{anti}} = \left(\frac{1}{4} - \frac{1}{4}\boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B\right)\rho_{AB}\left(\frac{1}{4} - \frac{1}{4}\boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B\right)$. If the qubits are antisymmetric, $\rho_{AB} = \rho_{AB}^{\text{anti}} = \mathcal{A}\rho_{AB}\mathcal{A}$, the state must be pure singlet, $\rho_{AB}^{\text{anti}} = \frac{1}{4}(1 - \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B)$; it cannot be a mixed state, as opposed to ρ_{AB}^{sym} which can be mixed. If spatial degrees of freedom need to be included in the description, in addition to the internal degrees of freedom, a bipartite density matrix can always be written as a product of an internal (i.e., spin) part and an external (i.e., space) part. Hence, a symmetric density matrix ρ_{AB}^{sym} for the internal degrees of freedom must be multiplied by a symmetric [antisymmetric] spatial density matrix for the spatial degrees of freedom $\{\mathbf{r}_A, \mathbf{r}_B\}$

for bosons [fermions], and ρ_{AB}^{anti} must be multiplied by an antisymmetric [symmetric] spatial density matrix for bosons [fermions], so that the full density matrix has the right exchange symmetry. We show elsewhere that this has relevance to collisional shifts in atomic clocks [26].

In summary, we have developed a classification of correlation for multipartite N-level quantum systems by writing their density matrices in terms of SU(N) generators, and we defined a measure of correlation for such systems, based upon their correlation matrices. The entanglement involves not just the correlation matrix but also other invariant parameters in the density matrix for the system. This formulation can now be used for a variety of applications, e.g., in the optimization of quantum gates and in the calculation of collisional clock shifts.

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