

Measurable bounds for Entanglement of Formation

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Abstract

We study the entanglement of formation for arbitrary dimensional bipartite mixed unknown states. Experimentally measurable lower and upper bounds for entanglement of formation are derived.

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Being one of the most striking phenomena in quantum physics, quantum entanglement [1–3] has been extensively investigated in recent years. One of the main tasks in quantum entanglement theory is to quantify the entanglement of quantum systems. Among all the bipartite entanglement measures, entanglement of formation (EOF) is one of the most meaningful and physically motivated measures [4, 5], which quantifies the minimal cost needed to prepare a certain quantum state in terms of EPR pairs, and plays important roles in many physical systems, such as quantum phase transition for various interacting quantum many-body systems [6], macroscopic properties of solids [7], and capacity of quantum channels [8].

Let H_A, H_B be the m, n ($m \leq n$) dimensional vector spaces respectively. A pure quantum state $|\psi\rangle \in H_A \otimes H_B$ is an mn -dimensional vector. Its entanglement of formation is defined by $E(|\psi\rangle) = S(\rho_A)$, where $\rho_A = Tr_B(|\psi\rangle\langle\psi|)$ is the reduced density matrix of $|\psi\rangle\langle\psi|$. $S(\rho_A)$ is the entropy

$$S(\rho_A) = - \sum_{i=1}^m \mu_i \log \mu_i \equiv H(\vec{\mu}), \quad (1)$$

where \log stands for the natural logarithm throughout the paper, μ_i are the eigenvalues of ρ_A and $\vec{\mu}$ is the Schmidt vector $(\mu_1, \mu_2, \dots, \mu_m)$. This definition of entanglement of formation is extended to mixed states ρ by the convex roof,

$$E(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \quad (2)$$

for all possible ensemble realizations $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $p_i \geq 0$ and $\sum_i p_i = 1$.

It is a great challenge to find an analytical formula of the entanglement of formation for general bipartite quantum mixed states ρ . Considerable efforts have been made on deriving entanglement of formation or its lower bound through analytical and numerical approaches. So far the entanglement of formation has been calculated for some particular states like bipartite qubit states [9], isotropic states [10] and Werner states [11] in arbitrary dimensions, and symmetric gaussian states in infinite dimensions [12]. In [13] in order to estimate the entanglement of formation for general states, a lower bound has been presented by using the partial transposition of ρ with respect to the subsystem H_A , ρ^{TA} , and the realignment of ρ , $R(\rho)$. It is shown that (without regard to the normalization coefficient $\log 2$)

$$E(\rho) \geq \begin{cases} 0, & \Omega = 1; \\ H_2[\gamma(\Omega)] + [1 - \gamma(\Omega)] \log_2(m-1), & 1 < \Omega \leq \frac{4(m-1)}{m}; \\ \frac{\log_2(m-1)}{m-2}(\Omega - m) + \log_2 m, & \frac{4(m-1)}{m} < \Omega \leq m; \end{cases} \quad (3)$$

where $\Omega = \max\{\|\rho^{TA}\|, \|R(\rho)\|\}$, H_2 is the standard binary entropy function, $\|A\|$ denotes the trace norm of the matrix A .

Comparing with the entanglement of formation, the entanglement measure concurrence is relatively easier to be dealt with. In [14] a simpler analytical lower bound for concurrence has been presented. And a series of new results related to the bounds of concurrence have been further obtained [15]. In particular in [16–18], the authors derive measurable lower and upper bounds for concurrence for general mixed quantum states,

$$2[\text{Tr}\rho^2 - \text{Tr}\rho_A^2] = \text{Tr}(\rho \otimes \rho V_i) \leq [C(\rho)]^2 \leq \text{Tr}(\rho \otimes \rho K_i) = 2[1 - \text{Tr}\rho_A^2], \quad (4)$$

where $V_1 = 4(P_- - P_+) \otimes P_-$, $V_2 = 4P_- \otimes (P_- - P_+)$ and $K_1 = 4P_- \otimes I$, $K_2 = 4I \otimes P_-$, P_- is the projector on the antisymmetric subspace of the two copies of either subsystem, P_+ the symmetric counterpart of P_- .

Contrary to the concurrence, less has been achieved related to the lower and upper bounds of entanglement of formation. In this paper, we derive analytical lower and upper bounds for entanglement of formation which are measurable experimentally. These bounds supply nice estimation for entanglement of formation for some quantum states.

To derive lower and upper bounds for entanglement of formation, we first consider a pure $|\psi\rangle$ with Schmidt decomposition $|\psi\rangle = \sum_{i=1}^m \sqrt{\mu_i} |ii\rangle$, where $\mu_i \geq 0$, $\sum_i \mu_i = 1$. It is easily verified that

$$1 - \text{Tr}\rho_A^2 = \text{Tr}\rho^2 - \text{Tr}\rho_A^2 = 1 - \sum_i \mu_i^2 \equiv \lambda. \quad (5)$$

Set

$$X(\lambda) = \max\{H(\vec{\mu}) | 1 - \sum_i \mu_i^2 \equiv \lambda\}, \quad Y(\lambda) = \min\{H(\vec{\mu}) | 1 - \sum_i \mu_i^2 \equiv \lambda\}. \quad (6)$$

Let $\varepsilon(x)$ be the largest convex function that is bounded above by $Y(x)$ and $\eta(x)$ the smallest concave function that is bounded below by $X(x)$.

Theorem: For any $m \otimes n$ ($m \leq n$) quantum state ρ , the entanglement of formation $E(\rho)$ satisfies

$$\max\{\varepsilon(\text{Tr}\rho^2 - \text{Tr}\rho_A^2), \varepsilon(\text{Tr}\rho^2 - \text{Tr}\rho_B^2)\} \leq E(\rho) \leq \min\{\eta(1 - \text{Tr}\rho_A^2), \eta(1 - \text{Tr}\rho_B^2)\}. \quad (7)$$

Proof: Without loss of generality, we assume that $\varepsilon(\text{Tr}\rho^2 - \text{Tr}\rho_A^2) \geq \varepsilon(\text{Tr}\rho^2 - \text{Tr}\rho_B^2)$ and $\eta(1 - \text{Tr}\rho_A^2) \leq \eta(1 - \text{Tr}\rho_B^2)$. Note that for any pure state $|\psi\rangle$, the concurrence is given by $C(|\psi\rangle) = \sqrt{2(1 - \text{Tr}(\rho^A)^2)}$. Due to convexity of concurrence, for any pure decomposition $\rho = \sum_{\alpha} p_{\alpha} \rho_{\alpha}$, we have

$$\sum_{\alpha} p_{\alpha} C^2(\rho_{\alpha}) = \sum_{\alpha} 2p_{\alpha} [1 - \text{Tr}(\rho_{\alpha}^A)^2] \geq C^2(\rho).$$

Taking into account of the bounds (4) of $C^2(\rho)$, we obtain

$$\sum_{\alpha} p_{\alpha} [\text{Tr}(\rho_{\alpha})^2 - \text{Tr}(\rho_{\alpha}^A)^2] \geq \text{Tr}\rho^2 - \text{Tr}\rho_A^2 \quad (8)$$

and

$$\sum_{\alpha} p_{\alpha} [1 - \text{Tr}(\rho_{\alpha}^A)^2] \leq 1 - \text{Tr}\rho_A^2. \quad (9)$$

Assume $\rho = \sum_{\alpha} p_{\alpha} \rho_{\alpha}$ be the optimal decomposition of $E(\rho)$. We have

$$E(\rho) = \sum_{\alpha} p_{\alpha} E(|\psi_{\alpha}\rangle) = \sum_{\alpha} p_{\alpha} H(\vec{\mu}_{\alpha}) \geq \sum_{\alpha} p_{\alpha} \varepsilon(\lambda_{\alpha}) \geq \varepsilon(\sum_{\alpha} p_{\alpha} \lambda_{\alpha}) \geq \varepsilon(\Lambda), \quad (10)$$

where $\Lambda = \text{Tr}\rho^2 - \text{Tr}\rho_A^2$. We have used the definition of ε to obtain the first inequality. The second inequality is due to the convex property of $\varepsilon(x)$ and the last one is derived from (8).

On the other hand,

$$E(\rho) = \sum_{\alpha} p_{\alpha} E(|\psi_{\alpha}\rangle) = \sum_{\alpha} p_{\alpha} H(\vec{\mu}_{\alpha}) \leq \sum_{\alpha} p_{\alpha} \eta(\lambda_{\alpha}) \leq \eta(\sum_{\alpha} p_{\alpha} \lambda_{\alpha}) \leq \eta(\Lambda'), \quad (11)$$

where $\Lambda' = 1 - \text{Tr}\rho_A^2$. We have used the definition of η to get the first inequality. The second inequality is derived from the concave property of $\eta(x)$ and the last one is obtained from (9). \square

We calculate now both the maximal admissible $H(\vec{\mu})$ and the minimal admissible $H(\vec{\mu})$ for a given λ , i.e. $X(\lambda)$ and $Y(\lambda)$, by using the Lagrange multipliers approach [10]. The necessary conditions for the maximum and minimum are given by:

$$-\log \mu_k - 1 - 2x\mu_k + y = 0, \quad (12)$$

$$1 - \sum_i \mu_i^2 - \Lambda = 0, \quad 1 - \sum_i \mu_i = 0; \quad (13)$$

where x, y denote the Lagrange multipliers. We get from (12) that

$$\log \mu_k = -2x\mu_k + y - 1. \quad (14)$$

From (14) we know that there are at most two solutions for each μ_k , which will be denoted as α and β in the following.

Let n_1 be the number of entries where $\mu_i = \alpha$ and n_2 the number of entries where $\mu_i = \beta$. If one of the n_1 and n_2 is zero, we have that $\lambda = 1 - \frac{1}{n_1+n_2}$. Otherwise both α and β are nonzero. The problem is now turned to be, for fixed $n_1, n_2, n_1 + n_2 \leq m$, one maximizes or minimizes the function

$$F_{n_1 n_2}(\lambda) = n_1 h(\alpha) + n_2 h(\beta), \quad (15)$$

where $h(x) = -x \log x$, under the constraints (13). By direct computation we obtain

$$\alpha_{n_1 n_2}^\pm = \frac{n_1 \pm \sqrt{n_1^2 - n_1(n_1 + n_2)[1 - n_2(1 - \lambda)]}}{n_1(n_1 + n_2)}, \quad \beta_{n_1 n_2}^\pm = \frac{1 - n_1 \alpha^\pm}{n_2}. \quad (16)$$

To ensure the nonnegativity property of $\alpha_{n_1 n_2}^\pm$ and $\beta_{n_1 n_2}^\pm$, we require that $\max\{1 - \frac{1}{n_1}, 1 - \frac{1}{n_2}\} \leq \Lambda \leq 1 - \frac{1}{n_1+n_2}$. Since $\alpha_{n_2 n_1}^- = \beta_{n_1 n_2}^+, \beta_{n_2 n_1}^- = \alpha_{n_1 n_2}^+$, the function in Eq. (15) takes the same value for $\alpha_{n_1 n_2}^+$ and $\alpha_{n_2 n_1}^-$. Therefore we can restrict ourselves to the solutions $\alpha_{n_1 n_2} = \alpha_{n_1 n_2}^+$. Eq. (15) then turns out to be

$$F_{n_1 n_2}(\lambda) = n_1 h(\alpha_{n_1 n_2}^+) + n_2 h(\beta_{n_1 n_2}^+). \quad (17)$$

When $m = 3$, to find the expressions of upper and lower bounds in (7) is to obtain the max- and minimization over the three functions $F_{12}(\Lambda)$, $F_{21}(\Lambda)$ and $F_{11}(\Lambda)$. From (17) for $m = 3$ we have

$$X(\Lambda) = \begin{cases} F_{11}, & 0 < \Lambda \leq \frac{1}{2}; \\ F_{12}, & \frac{1}{2} < \Lambda \leq \frac{2}{3} \end{cases} \quad Y(\Lambda) = \begin{cases} F_{11}, & 0 < \Lambda \leq \frac{1}{2}; \\ F_{21}, & \frac{1}{2} < \Lambda \leq \frac{2}{3}. \end{cases} \quad (18)$$

From (18) we have that $\eta[\Lambda]$ is the broken line connecting the following points: $[0, 0], [\frac{1}{2}, \log 2], [\frac{2}{3}, \log 3]$.

In order to determine $\varepsilon[\Lambda]$ we solve the following equations: Let $l(\Lambda) = k(\Lambda - 0.5) + 0.868$ be the line crossing through the point $[0.5, F_{12}(0.5)]$. We solve (i) $l(\Lambda) = F_{11}$ and (ii) $\frac{dl(\Lambda)}{d\Lambda} = k = \frac{dF_{11}(\Lambda)}{d\Lambda}$ for k and Λ , and find the values to be 1.65 and 0.091. Thus we derive that $\varepsilon[\Lambda]$ is the curve consisted of F_{11} for $0 < \Lambda \leq 0.091$ and a broken line connecting points $[0.091, F_{11}(0.091)], [0.5, F_{12}(0.5)]$ and $[0.667, \log[3]]$, i.e.

$$\eta[\Lambda] = \begin{cases} 2 \log 2 \times \Lambda, & 0 < \Lambda \leq 0.5; \\ 6 \log \frac{3}{2} \times (\Lambda - \frac{1}{2}) + \log 2, & 0.5 < \Lambda \leq 0.667 \end{cases} \quad (19)$$

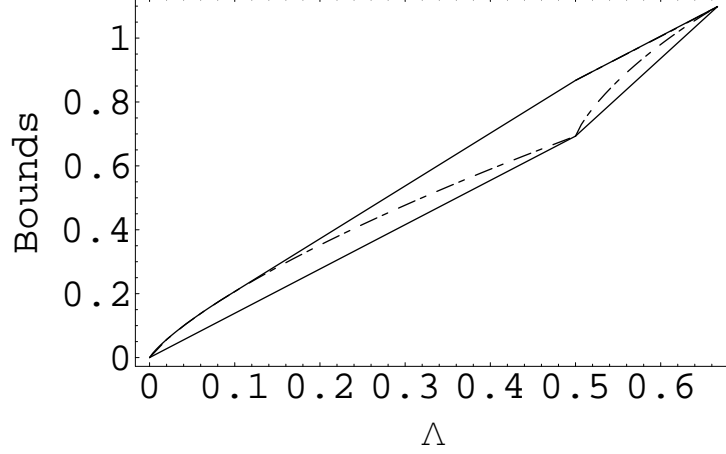


FIG. 1: Upper and lower bounds of $E(\rho)$ (solid lines) for $m = 3$, and F_{11}, F_{12}, F_{21} (dashed line).

and

$$\varepsilon[\Lambda] = \begin{cases} F_{11}, & 0 < \Lambda \leq 0.091; \\ 1.65(\Lambda - 0.5) + 0.868, & 0.091 < \Lambda \leq 0.5; \\ 1.39(\Lambda - 0.667) + 1.099, & 0.5 < \Lambda \leq 0.667, \end{cases} \quad (20)$$

see Fig.1.

When $m = 4$, we need to find the max- and minimization over the six functions $F_{11}, F_{12}, F_{21}, F_{22}, F_{31}$, and F_{13} , which are plotted in Fig.2. We have

$$X(\Lambda) = \begin{cases} F_{11}, & 0 < \Lambda \leq \frac{1}{2}; \\ F_{12}, & \frac{1}{2} < \Lambda \leq \frac{2}{3}; \\ F_{13}, & \frac{2}{3} < \Lambda \leq \frac{3}{4} \end{cases} \quad Y(\Lambda) = \begin{cases} F_{11}, & 0 < \Lambda \leq \frac{1}{2}; \\ F_{21}, & \frac{1}{2} < \Lambda \leq \frac{2}{3}; \\ F_{31}, & \frac{2}{3} < \Lambda \leq \frac{3}{4}. \end{cases} \quad (21)$$

Further more, one obtains that $\eta[\Lambda]$ is the broken line connecting the following points: $[0, 0], [\frac{1}{2}, \log 2], [\frac{2}{3}, \log 3], [\frac{3}{4}, \log 4]$ and $\varepsilon[\Lambda]$ is the curve consisted of F_{11} for $0 < \Lambda \leq 0.062$ and a broken line connecting points $[0.062, 0.142], [0.667, 1.242]$ and $[0.75, 1.386]$ (These points can be derived by using the same processes as that have been done in case $m = 3$), i.e.

$$\eta[\Lambda] = \begin{cases} 2 \log 2 \times \Lambda, & 0 < \Lambda \leq 0.5; \\ 6 \log \frac{3}{2} \times (\Lambda - \frac{1}{2}) + \log 2, & 0.5 < \Lambda \leq 0.667; \\ 12 \log \frac{4}{3} \times (\Lambda - \frac{2}{3}) + \log 3, & 0.667 < \Lambda \leq 0.75 \end{cases} \quad (22)$$

and

$$\varepsilon[\Lambda] = \begin{cases} F_{11}, & 0 < \Lambda \leq 0.062; \\ 1.820(\Lambda - 0.667) + 1.242, & 0.062 < \Lambda \leq 0.667; \\ 1.726(\Lambda - 0.667) + 1.242, & 0.667 < \Lambda \leq 0.75. \end{cases} \quad (23)$$

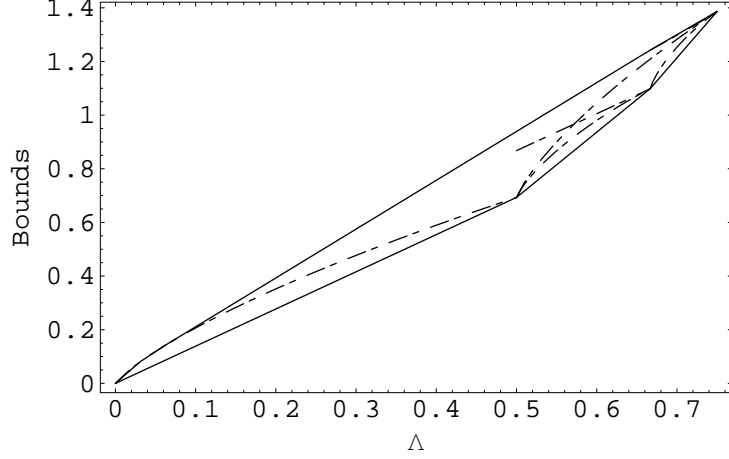


FIG. 2: Upper and lower bounds of $E(\rho)$ (solid lines) for $m = 4$, and $F_{11}, F_{12}, F_{21}, F_{22}, F_{13}, F_{31}$ (dashed line).

Generally, we have the following observation, for any m ,

$$X(\Lambda) = \begin{cases} F_{11}, & 0 < \Lambda \leq \frac{1}{2}; \\ F_{12}, & \frac{1}{2} < \Lambda \leq \frac{2}{3}; \\ \dots & \\ F_{1(m-1)}, & \frac{m-2}{m-1} < \Lambda \leq \frac{m-1}{m}; \end{cases} \quad Y(\Lambda) = \begin{cases} F_{11}, & 0 < \Lambda \leq \frac{1}{2}; \\ F_{21}, & \frac{1}{2} < \Lambda \leq \frac{2}{3}; \\ \dots & \\ F_{(m-1)1}, & \frac{m-2}{m-1} < \Lambda \leq \frac{m-1}{m}; \end{cases} \quad (24)$$

and $\eta[\Lambda]$ is the broken line connecting the following points: $[\frac{i}{i+1}, \log(i+1)], 0 \leq i \leq m-1$, i.e.

$$\eta[\Lambda] = k(k+1) \log \frac{k+1}{k} (\Lambda - \frac{k-1}{k}) + \log k, \quad (25)$$

for $(k-1) < \Lambda \leq k$ and $k = 1, 2, \dots, m-1$. The representation of $\varepsilon[\Lambda]$ can be also figured out numerically.

The measurable upper and lower bounds can be used to estimate the entanglement of formation for an unknown quantum mixed state experimentally. Consider the following mixed quantum state

$$\rho = \frac{x}{9}I + (1-x)|\psi\rangle\langle\psi|. \quad (26)$$

where $|\psi\rangle = (a, 0, 0, 0, \frac{1}{\sqrt{3}}, 0, 0, 0, \frac{1}{\sqrt{3}})^t / \sqrt{\text{Tr}\{|\psi\rangle\langle\psi|\}}$. For $x = 0.1$, one has

$$\Lambda = \text{Tr}\{\rho^2\} - \text{Tr}\{\rho_A^2\} = \text{Tr}\{\rho^2\} - \text{Tr}\{\rho_B^2\} = \frac{1.45 + 9.21a^2 - 0.38a^4}{(2 + 3a^2)^2} \quad (27)$$

$$\Lambda' = 1 - \text{Tr}\{\rho_A^2\} = 1 - \text{Tr}\{\rho_B^2\} = \frac{1.14(0.19 + a^2)(9.67 + a^2)}{(2 + 3a^2)^2}. \quad (28)$$

Substituting Λ and Λ' above into (19) and (20) respectively, we have the upper and lower bounds (7) for $E(\rho)$, see Fig. 3.

For $x = 0.001$, one has

$$\Lambda = \text{Tr}\{\rho^2\} - \text{Tr}\{\rho_A^2\} = \text{Tr}\{\rho^2\} - \text{Tr}\{\rho_B^2\} = \frac{1.99 + 11.97a^2 - 0.004a^4}{(2 + 3a^2)^2} \quad (29)$$

$$\Lambda' = 1 - \text{Tr}\{\rho_A^2\} = 1 - \text{Tr}\{\rho_B^2\} = \frac{0.01(0.17 + a^2)(999.67 + a^2)}{(2 + 3a^2)^2}. \quad (30)$$

The corresponding bounds of $E(\rho)$ is shown in Fig. 4. We see that the lower and upper bounds are closer. And the value for $E(\rho)$ can be estimated more precisely.

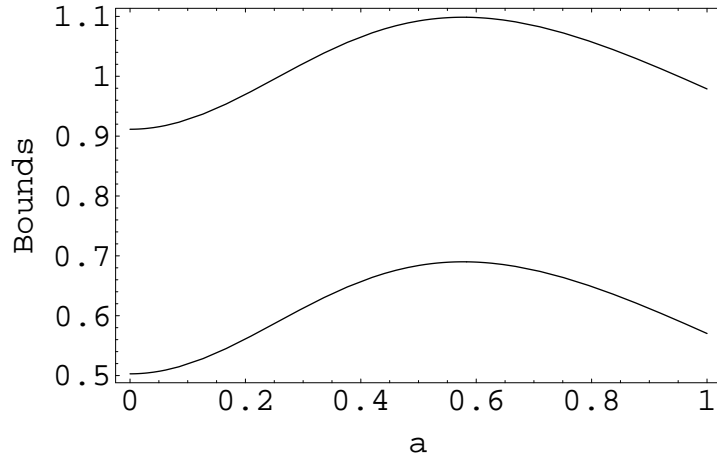


FIG. 3: Upper and lower bounds of $E(\rho)$ with $x = 0.1$.

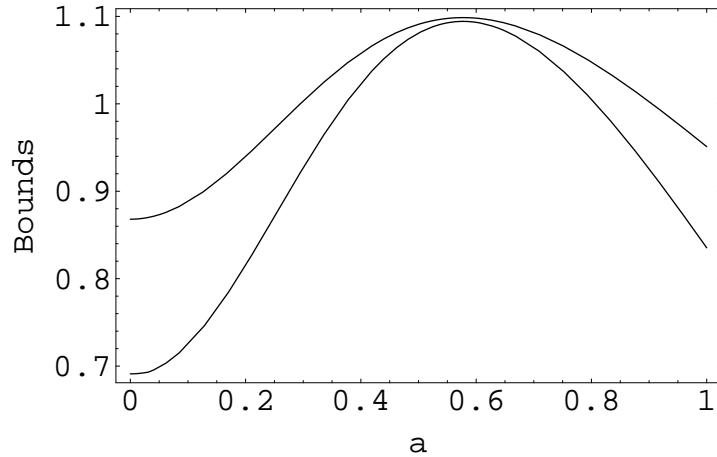


FIG. 4: Upper and lower bounds of $E(\rho)$ with $x = 0.001$.

We have studied the entanglement of formation for mixed quantum states. We have derived upper and lower bounds for entanglement of formation that are experimentally measurable. These bounds together can be used to estimate the entanglement of formation for arbitrary finite dimensional unknown states according to a few measurements on a twofold copy $\rho \otimes \rho$ of the mixed states. These results supplement further the estimation for entanglement of formation, like the case of concurrence for which many lower and upper bounds have been already obtained.

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