SPIN MODELS CONSTRUCTED FROM HADAMARD MATRICES

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Abstract. A spin model (for link invariants) is a square matrix W which satisfies certain axioms. For a spin model W, it is known that W^TW^{-1} is a permutation matrix, and its order is called the index of W. F. Jaeger and K. Nomura found spin models of index 2, by modifying the construction of symmetric spin models from Hadamard matrices.

The aim of this paper is to give a construction of spin models of an arbitrary even index from any Hadamard matrix. In particular, we show that our spin models of indices a power of 2 are new.

1. Introduction

The notion of spin model was introduced by V.F.R. Jones [13] to construct invariants of knots and links. The original definition due to Jones requires that a spin model be a symmetric matrix, but later by K. Kawagoe, A. Munemasa, and Y. Watatani [14], a proper definition allowing non-symmetric matrices is given. In this paper, we consider spin models which are not necessarily symmetric.

Let X be a non-empty finite set. We denote by $Mat_X(\mathbb{C}^*)$ the set of square matrices with non-zero complex entries whose rows and columns are indexed by X. For $W \in Mat_X(\mathbb{C}^*)$ and $x, y \in X$, the (x, y)-entry of W is denoted by $W(x, y)$. A spin model $W \in Mat_X(\mathbb{C}^*)$ is defined to be a matrix which satisfies two conditions (type II and type III; see Section 2).

One of the examples of spin models is a Potts model, defined as follows. Let X be a finite set with r elements, and let $I, J \in Mat_X(\mathbb{C}^*)$ be the identity matrix and the all 1's matrix, respectively. Let u be a complex number satisfying

$$\begin{aligned} (u^2 + u^{-2})^2 &= r \text{ if } r \geq 2, \\ u^4 &= 1 \text{ if } r = 1. \end{aligned} \tag{1}$$

Then a Potts model A_u is defined as

$$A_u = u^3 I - u^{-1}(J - I).$$

As examples of spin models, we know only Potts models [13, 10], spin models on finite abelian groups [3, 5], Jaeger's Higman-Sims model [10], Hadamard models [17, 12], non-symmetric Hadamard models [12], and tensor products of these. Apart from spin models on finite abelian groups, non-symmetric Hadamard models are essentially the only known family of non-symmetric spin models.

If W is a spin model, then by [12, Proposition 2], $R = W^TW^{-1}$ is a permutation matrix. The order of R as a permutation is called the *index* of the spin model

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W. Let H be a *Hadamard matrix,* that is, H is a square matrix of order r with entries ± 1 satisfying $HH^T = I$. In [12], F. Jaeger and K. Nomura constructed *non-symmetric Hadamard models,* which are spin models of index 2:

$$W = \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes A_u & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \xi H \\ \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \xi H^T & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes A_u \end{pmatrix}, \tag{2}$$

where ξ is a primitive 8-th root of unity, and A_u is a Potts model of size r. Note that non-symmetric Hadamard models are a modification of the earlier Hadamard models ([12], see also [12, Section 5]), defined by

$$W' = \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes A_u & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \omega H \\ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \omega H^T & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes A_u \end{pmatrix}, \tag{3}$$

where ω is a 4-th root of unity.

To construct spin models of index $m > 2$, it seems natural to consider an $m \times m$ block matrix W such that each block W_{ij} is the tensor product of two matrices like those in (2) and (3):

$$W_{ij} = S_{ij} \otimes T_{ij} \qquad (i, j \in \mathbb{Z}_m). \tag{4}$$

Such matrices appeared in [9, Proposition 6.2], with the matrices $S_{ij} \in Mat_{\mathbb{Z}_m}(\mathbb{C}^*)$ given by

$$S_{ij}(\ell, \ell') = \eta^{(\ell-\ell')(i-j)} \qquad (\ell, \ell' \in \mathbb{Z}_m), \tag{5}$$

where η is a primitive m-th root of unity.

In this paper, we construct an infinite class of spin models of even index containing non-symmetric Hadamard models. Also, we construct an infinite class of symmetric spin models containing Hadamard models. Our main result is as follows:

Theorem 1.1. *Let r be a positive integer, and let m be an even positive integer. Define $Y = \{1, \dots, r\}$, $X_i = \{(i, \ell, x) \mid \ell \in \mathbb{Z}_m,\ x \in Y\}$ for $i \in \mathbb{Z}_m$, and $X = X_0 \cup \dots \cup X_{m-1}$. Let A_u, $H \in \mathrm{Mat}_Y(\mathbb{C}^*)$ be a Potts model, a Hadamard matrix, respectively. Define V_{ij} for $i, j \in \mathbb{Z}_m$ by*

$$V_{ij} = \begin{cases} A_u & \text{if } i-j \text{ is even,} \\ H & \text{if } (i,j) \equiv (0,1) \pmod 2, \\ H^T & \text{if } (i,j) \equiv (1,0) \pmod 2. \end{cases} \tag{6}$$

Then the following statements hold:

(i) *Let a be a primitive $2m^2$-th root of unity. Let $W \in \mathrm{Mat}_X(\mathbb{C}^*)$ be the matrix whose (α, β) entry is given by $a^{2m(\ell-\ell')(i-j)+\epsilon(i,j)} V_{ij}(x, y)$ for $\alpha = (i, \ell, x), \beta = (j, \ell', y) \in X$, where $\epsilon(i, j) = (i-j)^2 + m(i-j)$. Then W is a spin model of index m.*

(ii) *Let η be a primitive m-th root of unity, and let b be an m^2-th root of unity. Let $W' \in \mathrm{Mat}_X(\mathbb{C}^*)$ be the matrix whose (α, β) entry is given by $\eta^{(\ell-\ell')(i-j)} b^{\delta(i,j)} V_{ij}$ for $\alpha = (i, \ell, x), \beta = (j, \ell', y) \in X$, where $\delta(i, j) = (i-j)^2$. Then W' is a symmetric spin model.*

Note that, in order for $a^{\epsilon(i,j)}$ and $b^{\delta(i,j)}$ to be well-defined, we need to identify $\mathbb{Z}_m$ with the subset $\{0, 1, \dots, m-1\}$ of integers.

Remark 1.2. In Theorem 1.1 (i), if we define S_{ij} by (5) with $\eta = a^{2m}$, and T_{ij} by $T_{ij} = a^{\epsilon(i,j)} V_{ij}$, then the (X_i, X_j)-block of the matrix W is given by (4). Similarly, in Theorem 1.1 (ii), (4) holds with $T_{ij} = b^{\delta(i,j)} V_{ij}$.

The spin models W, W' given in Theorem 1.1 are determined by a Hadamard matrix H of order r, a complex number u satisfying (1), and a primitive $2m^2$-th root of unity a or an m^2-th root of unity b, respectively. Throughout this paper, we denote by $W_{H,u,a}$, $W'_{H,u,b}$ the spin models given by Theorem 1.1 (i), (ii), respectively.

Observe that, for any spin models W_i $(i = 1, 2)$ of indices m_i, their tensor product $W_1 \otimes W_2$ is also a spin model of index $\mathrm{LCM}(m_1, m_2)$. In Section 5, we show that the non-symmetric spin model $W_{H,u,a}$ whose index is a power of 2 is new in the following sense:

Theorem 1.3. *Let H be a Hadamard matrix of order r. Let $W_{H,u,a}$ be a spin model given in Theorem* 1.1 (i)*, whose index m is a power of 2. If $r > 4$, then $W_{H,u,a}$ cannot be decomposed into a tensor product of known spin models.*

We note that the list of known spin models is given in Section 5. Jaeger and Nomura [12, p.278] expected that new non-symmetric spin models of index a power of 2 should be found, and our results give an answer to this expectation.

2. Type II and Type III conditions on block matrices of tensor products

First we define a spin model. A *type II matrix* on a finite set X is a matrix $W \in Mat_X(\mathbb{C}^*)$ which satisfies the *type II condition*:

$$\sum_{x \in X} \frac{W(\alpha, x)}{W(\beta, x)} = n\delta_{\alpha,\beta} \qquad \text{(for all } \alpha, \beta \in X). \tag{7}$$

Let $W^- \in Mat_X(\mathbb{C}^*)$ be defined by $W^-(x, y) = W(y, x)^{-1}$. Then the type II condition is written as $WW^- = nI$. Hence, if W is a type II matrix, then W is non-singular with $W^{-1} = n^{-1} W^-$.

A type II matrix $W \in Mat_X(\mathbb{C}^*)$ is called a *spin model* if W satisfies the *type III condition*:

$$\sum_{x \in X} \frac{W(\alpha, x)W(\beta, x)}{W(\gamma, x)} = D \frac{W(\alpha, \beta)}{W(\alpha, \gamma)W(\gamma, \beta)} \qquad \text{(for all } \alpha, \beta, \gamma \in X) \tag{8}$$

for some nonzero real number D with $D^2 = n$, which is independent of the choice of $\alpha, \beta, \gamma \in X$.

Let m be a positive integer. In this section, assuming that W is an $m \times m$ block matrix with blocks of the form (4), we will establish conditions on T_{ij} under which W satisfies the type II and type III conditions. Some parts of these conditions are already given in [9, Proposition 5.1, Proposition 6.2].

Let η be a primitive m-th root of unity, and let S_{ij} be the matrix of size m defined by (5) for $i, j \in \mathbb{Z}_m$. Let r be a positive integer, and define $Y = \{1, \ldots, r\}$, $X_i = \{(i, \ell, x) \mid \ell \in \mathbb{Z}_m,\ x \in Y\}$ for $i \in \mathbb{Z}_m$, and $X = X_0 \cup \ldots \cup X_{m-1}$. Let $T_{ij} \in Mat_Y(\mathbb{C}^*)$ be a matrix for $i, j \in \mathbb{Z}_m$, and let W_{ij} be the matrix defined by (4). Let $W \in Mat_X(\mathbb{C}^*)$ be the matrix whose (X_i, X_j)-block is W_{ij} for $i, j \in \mathbb{Z}_m$. Then

$$W((i, \ell, x), (j, \ell', y)) = S_{ij}(\ell, \ell') T_{ij}(x, y). \tag{9}$$

Lemma 2.1 ([9, Proposition 5.1]). *The matrix W is a type II matrix if and only if T_{ij} is a type II matrix for all $i,j\in\mathbb{Z}_m$.*

Lemma 2.2. *The matrix W satisfies the type III condition* (8) *if and only if the following equality holds for all $i_1,i_2,i_3\in\mathbb{Z}_m$ and $x_1,x_2,x_3\in Y$:*

$$\sum_{x\in Y}\frac{T_{i_1,i_0}(x_1,x)T_{i_2,i_0}(x_2,x)}{T_{i_3,i_0}(x_3,x)}=\frac{D}{m}\cdot\frac{T_{i_1,i_2}(x_1,x_2)}{T_{i_1,i_3}(x_1,x_3)T_{i_3,i_2}(x_3,x_2)}, \tag{10}$$

where $i_0=i_1+i_2-i_3 \bmod m$.

Proof. The type III condition (8) for $\alpha=(i_1,\ell_1,x_1)$, $\beta=(i_2,\ell_2,x_2)$, $\gamma=(i_3,\ell_3,x_3)$ is equivalent to

$$\begin{aligned}&\sum_{i,\ell\in\mathbb{Z}_m}\frac{\eta^{(\ell_1-\ell)(i_1-i)}\eta^{(\ell_2-\ell)(i_2-i)}}{\eta^{(\ell_3-\ell)(i_3-i)}}\sum_{x\in Y}\frac{T_{i_1,i}(x_1,x)T_{i_2,i}(x_2,x)}{T_{i_3,i}(x_3,x)}\\&=D\frac{\eta^{(\ell_1-\ell_2)(i_1-i_2)}}{\eta^{(\ell_1-\ell_3)(i_1-i_3)}\eta^{(\ell_3-\ell_2)(i_3-i_2)}}\cdot\frac{T_{i_1,i_2}(x_1,x_2)}{T_{i_1,i_3}(x_1,x_3)T_{i_3,i_2}(x_3,x_2)}.\end{aligned}$$

By a direct computation, we obtain

$$\begin{aligned}&\frac{\eta^{(\ell_1-\ell)(i_1-i)}\eta^{(\ell_2-\ell)(i_2-i)}}{\eta^{-(\ell_3-\ell)(i_3-i)}}\cdot\frac{\eta^{(\ell_1-\ell_3)(i_1-i_3)}\eta^{(\ell_3-\ell_2)(i_3-i_2)}}{\eta^{(\ell_1-\ell_2)(i_1-i_2)}}\\&=\eta^{(\ell_1+\ell_2-\ell_3-\ell)(i_1+i_2-i_3-i)}.\end{aligned}$$

So (8) is equivalent to

$$\begin{aligned}&\sum_{i\in\mathbb{Z}_m}\Big(\sum_{\ell\in\mathbb{Z}_m}\eta^{(\ell_1+\ell_2-\ell_3-\ell)(i_1+i_2-i_3-i)}\Big)\sum_{x\in Y}\frac{T_{i_1,i}(x_1,x)T_{i_2,i}(x_2,x)}{T_{i_3,i}(x_3,x)}\\&=D\frac{T_{i_1,i_2}(x_1,x_2)}{T_{i_1,i_3}(x_1,x_3)T_{i_3,i_2}(x_3,x_2)}.\end{aligned} \tag{11}$$

Since η is a primitive m-th root of unity and $i_0=i_1+i_2-i_3 \bmod m$, we have

$$\sum_{\ell\in\mathbb{Z}_m}\eta^{(\ell_1+\ell_2-\ell_3-\ell)(i_1+i_2-i_3-i)}=m\delta_{i,i_0}.$$

Thus (11) is equivalent to (10). □

We remark that in [9, Proposition 6.2] only the necessity of (10) for the type III condition is proved.

Let z_m be the permutation matrix of order m:

$$z_m=\begin{pmatrix} & & & 1\\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}.$$

We define the permutation matrix R of size $n=m^2r$ by $R=I_m\otimes z_m\otimes I_r$, where I_m and I_r are the identity matrices of size m and r, respectively. The order of R is m.

Lemma 2.3. *The matrix W satisfies $W^TW^{-1}=R$ if and only if $T_{ij}=\eta^{i-j}T_{ji}^T$ holds for all $i,j\in\mathbb{Z}_m$.*

Proof. For $\alpha = (i, \ell, x)$ and $\beta = (j, \ell', y) \in X$,

$$\begin{aligned} W^T(\alpha,\beta) &= W(\beta,\alpha) \\ &= \eta^{(\ell'-\ell)(j-i)} T_{j,i}(y,x), \\ (RW)(\alpha,\beta) &= ((I_m \otimes z_m \otimes I_r)W)((i,\ell,x),(j,\ell',y)) \\ &= W((i,\ell-1,x),(j,\ell',y)) \\ &= \eta^{(\ell-1-\ell')(i-j)} T_{ij}(x,y) \\ &= \eta^{(\ell'-\ell)(j-i)} \eta^{-(i-j)} T_{ij}(x,y). \end{aligned}$$

Therefore $R = W^T W^{-1}$ if and only if $T_{ji}(y,x) = \eta^{-(i-j)} T_{ij}(x,y)$ holds for all $i, j \in \mathbb{Z}_m$ and $x, y \in Y$. □

3. Proof of Theorem 1.1

From Remark 1.2, the results in Section 2 can be used for the matrices W and W' given in Theorem 1.1, if we define T_{ij} according to Remark 1.2.

For a mapping g from $\mathbb{Z}^2$ to $\mathbb{Z}$, we denote by λ_g the mapping from $\mathbb{Z}^4$ to $\mathbb{Z}$ defined by

$$\lambda_g(i_1,i_2,i_3,i_4) = g(i_1,i_4)+g(i_2,i_4)-g(i_3,i_4)+g(i_1,i_3)+g(i_3,i_2)-g(i_1,i_2). \quad (12)$$

Recall that we regard $\mathbb{Z}_m$ as the subset $\{0, 1, \ldots, m-1\}$ of $\mathbb{Z}$, and $\delta, \epsilon : \mathbb{Z}^2 \to \mathbb{Z}$ are defined by $\delta(i,j) = (i-j)^2$, $\epsilon(i,j) = \delta(i,j) + m(i-j)$, respectively.

Lemma 3.1. *For all $i_1, i_2, i_3, i_4 \in \mathbb{Z}$, we have*

$$\begin{aligned} \lambda_\delta(i_1,i_2,i_3,i_4) &= (i_1+i_2-i_3-i_4)^2, \\ \lambda_\epsilon(i_1,i_2,i_3,i_4) &= (i_1+i_2-i_3-i_4)(i_1+i_2-i_3-i_4+m). \end{aligned}$$

In particular, if $i_0 = i_1 + i_2 - i_3 \pmod m$, then

$$\begin{aligned} \lambda_\delta(i_1,i_2,i_3,i_0) &\equiv 0 \pmod{m^2}, \\ \lambda_\epsilon(i_1,i_2,i_3,i_0) &\equiv 0 \pmod{2m^2}. \end{aligned}$$

Proof. Straightforward. □

In [12, §5.1], the following is used to construct non-symmetric or symmetric Hadamard models:

Lemma 3.2 ([12, §5.1])**.** *Let $A_u, H \in \mathrm{Mat}_Y(\mathbb{C}^*)$ be a Potts model, a Hadamard matrix, respectively. Then the following holds for all $x_1, x_2, x_3 \in Y$:*

$$\sum_{y\in Y} \frac{A_u(x_1,y)A_u(x_2,y)}{A_u(x_3,y)} = D_u \frac{A_u(x_1,x_2)}{A_u(x_1,x_3)A_u(x_3,x_2)}, \tag{13}$$

$$\sum_{y\in Y} A_u(x_1,y)H(y,x_2)H(y,x_3) = D_u \frac{H(x_1,x_2)H(x_1,x_3)}{A_u(x_2,x_3)}, \tag{14}$$

$$\sum_{y\in Y} A_u(x_1,y)H(x_2,y)H(x_3,y) = D_u \frac{H(x_2,x_1)H(x_3,x_1)}{A_u(x_2,x_3)}, \tag{15}$$

$$\sum_{y\in Y} \frac{H(y,x_1)H(y,x_2)}{A_u(x_3,y)} = D_u A_u(x_1,x_2)H(x_3,x_1)H(x_3,x_2), \tag{16}$$

$$\sum_{y\in Y} \frac{H(x_1,y)H(x_2,y)}{A_u(x_3,y)} = D_u A_u(x_1,x_2)H(x_1,x_3)H(x_2,x_3), \tag{17}$$

where

$$D_u = \begin{cases} -u^2 - u^{-2} & \text{if } |Y| \geq 2, \\ u^2 & \text{if } |Y| = 1. \end{cases}$$

We now prove Theorem 1.1. Since A_u and H are type II matrices, so are the matrices $T_{ij} = a^{\epsilon(i,j)}V_{ij}$ or $b^{\delta(i,j)}V_{ij}$. Thus, Lemma 2.1 implies that $W_{H,u,a}$ and $W'_{H,u,b}$ are type II matrices.

We claim

$$\sum_{y\in Y} \frac{V_{i_1,i_0}(x_1,y)V_{i_2,i_0}(x_2,y)}{V_{i_3,i_0}(x_3,y)} = D_u \frac{V_{i_1,i_2}(x_1,x_2)}{V_{i_1,i_3}(x_1,x_3)V_{i_3,i_2}(x_3,x_2)} \tag{18}$$

for all $i_1, i_2, i_3 \in \mathbb{Z}_m$ and $x_1, x_2, x_3 \in Y$, where $i_0 = i_1 + i_2 - i_3 \bmod m$. Indeed, let $i_1, i_2, i_3 \in \mathbb{Z}_m$. Then

$$(18) \iff \begin{cases} (13) & \text{if} \quad (i_1,i_2,i_3) \equiv (0,0,0),(1,1,1) \pmod 2, \\ (14) & \text{if} \quad (i_1,i_2,i_3) \equiv (0,1,1) \pmod 2, \\ (15) & \text{if} \quad (i_1,i_2,i_3) \equiv (1,0,0) \pmod 2, \\ (16) & \text{if} \quad (i_1,i_2,i_3) \equiv (1,1,0) \pmod 2, \\ (17) & \text{if} \quad (i_1,i_2,i_3) \equiv (0,0,1) \pmod 2. \end{cases}$$

Moreover, when $(i_1,i_2,i_3) \equiv (1,0,1),(0,1,0) \pmod 2$, (18) is equivalent to (14), (15), respectively, with x_1 and x_2 switched. Therefore, (18) holds in all cases by Lemma 3.2.

Firstly, we show that $W_{H,u,a}$ and $W'_{H,u,b}$ satisfy the condition (10). From Lemma 3.1 we have

$$a^{\lambda_\epsilon(i_1,i_2,i_3,i_0)} = 1 \quad \text{and} \quad b^{\lambda_\delta(i_1,i_2,i_3,i_0)} = 1.$$

In view of (12), these imply

$$c^{g(i_1,i_0)+g(i_2,i_0)-g(i_3,i_0)} = c^{g(i_1,i_2)-g(i_1,i_3)-g(i_3,i_2)}, \tag{19}$$

where $(c, g) = (a, \epsilon), (b, \delta)$. Combining (18) and (19), we obtain

$$\sum_{y \in Y} \frac{c^{g(i_1,i_0)} V_{i_1,i_0}(x_1, y) c^{g(i_2,i_0)} V_{i_2,i_0}(x_2, y)}{c^{g(i_3,i_0)} V_{i_3,i_0}(x_3, y)}$$
$$= D_u \frac{c^{g(i_1,i_2)} V_{i_1,i_2}(x_1, x_2)}{c^{g(i_1,i_3)} V_{i_1,i_3}(x_1, x_3) c^{g(i_3,i_2)} V_{i_3,i_2}(x_3, x_2)}.$$

for all $i_1, i_2, i_3 \in \mathbb{Z}_m$ and $x_1, x_2, x_3 \in Y$. Thus (10) holds by setting $D = mD_u$. It follows from Lemma 2.2 that $W_{H,u,a}$ and $W'_{H,u,b}$ satisfy the type III condition (8), and hence they are spin models. Since $\delta(i, j) = \delta(j, i)$, $W'_{H,u,b}$ is symmetric.

Finally, we show that $W_{H,u,a}$ has index m. Since $a^{2m} = \eta$, we have $a^{\epsilon(i,j)-\epsilon(j,i)} = a^{2m(i-j)} = \eta^{i-j}$. So, $T_{ij} = \eta^{i-j} T_{ji}^T$ holds for all $i, j \in \mathbb{Z}_m$. From Lemma 2.3, $W_{H,u,a}$ has index m. This completes the proof of Theorem 1.1.

4. Properties of spin models in Theorem 1.1

For a positive integer r, we let u be a complex number satisfying (1).

Lemma 4.1. *If $r \le 4$, then u is a root of unity. Otherwise, $|u| \ne 1$. If $r \ge 4$ or $r = 1$, then $u^4 > 0$.*

Proof. If u is a root of unity and $r > 1$, then $r = (u^2 + u^{-2})^2 \le |u|^4 + 2 + |u|^{-4} = 4$. It is easy to see that u is indeed a root of unity if $r \le 4$. If $r \ge 4$ or $r = 1$, then we have $u^4 > 0$ from (1). □

For a matrix $W \in Mat_X(\mathbb{C}^*)$, we define

$$E(W) = \{\frac{|W(x, y)|}{|W(x, x)|} \mid x, y \in X\} \subset \mathbb{R}_{>0}.$$

Then

$$E(W_1 \otimes W_2) = E(W_1)E(W_2) \tag{20}$$

holds for any matrices W_1, W_2 with nonzero entries.

For the remainder of this section, let $W_{H,u,a}$, $W'_{H,u,b}$ be the spin models given in Theorem 1.1(i), (ii), respectively. This means that m is an even positive integer, a is a primitive $2m^2$-th root of unity, b is an m^2-th root of unity, and H is a Hadamard matrix of order r.

Lemma 4.2. *We have*

$$E(W_{H,u,a}) = E(W'_{H,u,b}) = \begin{cases} \{1, |u|^{-4}, |u|^{-3}\} & \text{if } r > 4, \\ \{1\} & \text{otherwise.} \end{cases}$$

Proof. Immediate from Theorem 1.1 and Lemma 4.1. □

Lemma 4.3. (i) *Suppose $r \ge 4$ or $r = 1$. Then the entries of $W_{H,u,a}$, $W'_{H,u,b}$ which have absolute value 1 are $2m^2$-th roots of unity, m^2-th roots of unity, respectively. Moreover, $W_{H,u,a}$ contains a primitive $2m^2$-th root of unity as one of its entries.*

(ii) *Suppose $r = 2$, and put $\nu = \mathrm{LCM}(2m^2, 16)$, $\nu' = \mathrm{LCM}(m^2, 16)$. Then the entries of $W_{H,u,a}$, $W'_{H,u,b}$ are ν-th roots of unity, ν'-th roots of unity, respectively. Moreover, $W_{H,u,a}$ contains a primitive ν-th root of unity as one of its entries.*

W	index	size	r	$\mu(W)$	$E(W)$
$W_{H,u,a}$	m	m^2r	$r=1$	$2m^2$	$\{1\}$
			$r=2$	$\mu(W)\mid \mathrm{LCM}(2m^2,16)$	$\{1\}$
			$r=4$	$2m^2$	$\{1\}$
			$r>4$	$2m^2$	$\{1,\lvert u\rvert^{-4},\lvert u\rvert^{-3}\}$
$W'_{H,u,b}$	1	m^2r	$r=1$	$\mu(W)\lvert m^2$	$\{1\}$
			$r=2$	$\mu(W)\mid \mathrm{LCM}(m^2,16)$	$\{1\}$
			$r=4$	$\mu(W)\lvert m^2$	$\{1\}$
			$r>4$	$\mu(W)\lvert m^2$	$\{1,\lvert u\rvert^{-4},\lvert u\rvert^{-3}\}$

TABLE 1. Summary of Properties

Proof. Firstly, suppose $r>4$. From Lemma 4.1, the entries of $W_{H,u,a}$, $W'_{H,u,b}$ with absolute value 1 are

$$\begin{aligned} \pm\, a^{2m(\ell-\ell')(i-j)+\epsilon(i,j)} &\qquad (i-j:\text{odd}), \\ \pm\eta^{(\ell-\ell')(i-j)}b^{\delta(i,j)} &\qquad (i-j:\text{odd}), \end{aligned} \tag{21}$$

which are $2m^2$-th roots of unity, m^2-th roots of unity, respectively. Putting $i=1$, $j=\ell=\ell'=0$ in (21), we obtain a^{1+m} which is a primitive $2m^2$-th root of unity.

Next, suppose $r\le 4$. Then the entries of $W_{H,u,a}$, $W'_{H,u,b}$ are given by

$$va^{2m(\ell-\ell')(i-j)+\epsilon(i,j)} \qquad (v\in\{u^3,-u^{-1},\pm1\}), \tag{22}$$

$$v\eta^{(\ell-\ell')(i-j)}b^{\delta(i,j)} \qquad (v\in\{u^3,-u^{-1},\pm1\}), \tag{23}$$

respectively, all of which are roots of unity.

If $r=4$ or 1, then from (1), $u^4=1$. From (22), (23), the entries of $W_{H,u,a}$, $W'_{H,u,b}$ are $2m^2$-th roots of unity, m^2-th roots of unity, respectively. Putting $i=1$, $j=\ell=\ell'=0$ in (22), we obtain a^{1+m} which is a primitive $2m^2$-th root of unity.

Finally, suppose $r=2$. Since u is a primitive 16-root of unity by (1), the expressions in (22), (23) are ν-th roots of unity, an ν'-th roots of unity, respectively. Putting $v=u^3$, $i=1$, $j=\ell=\ell'=0$ in (22), we obtain u^3a^{1+m} which is a primitive ν-th root of unity. □

For $S\in Mat_X(\mathbb{C}^*)$, we denote by $\mu(S)$ the least common multiple of the orders of the entries of S which have a finite order. If none of the entries of S has a finite order, then we define $\mu(S)=\infty$. For a nonzero complex number ζ, we denote by the same symbol $\mu(\zeta)$ the order of ζ if ζ has a finite order.

Lemma 4.4. *Suppose* $m\equiv 0 \pmod 4$. *Then for* $W=W_{H,u,a}$ *or* $W=W'_{H,u,b}$, *we have* $\mu(W)\mid 2m^2$.

Proof. Immediate from Lemma 4.3. □

In Table 1, we summarize the properties of $W=W_{H,u,a}$, $W'_{H,u,b}$ obtained from Lemmas 4.1, 4.2, and 4.4.

For $W\in Mat_X(\mathbb{C}^*)$ and for a permutation σ of X, we define W^σ by $W^\sigma(\alpha,\beta)=W(\sigma(\alpha),\sigma(\beta))$ for $\alpha,\beta\in X$. Observe that if W is a spin model, then W^σ is also a spin model. If W is a spin model, then from (7), (8), $-W$ and $\pm\sqrt{-1}W$ are also spin models. Two spin models W_1, W_2 are said to be *equivalent* if $cW_1^\sigma=W_2$ for some permutation σ of X and a complex number c with $c^4=1$.

Two Hadamard matrices are said to be *equivalent* if one can be obtained from the other by negating rows and columns, or and permuting rows and columns.

Lemma 4.5. *Let $H_1, H_2 \in \mathrm{Mat}_Y(\mathbb{C}^*)$ be equivalent Hadamard matrices. Then $W_{H_1,u,a}$ is equivalent to $W_{H_2,u,a}$, and $W'_{H_1,u,b}$ is equivalent to $W'_{H_2,u,b}$.*

Proof. Let $(W_1, W_2, c, g) = (W_{H_1,u,a}, W_{H_2,u,a}, a, \epsilon)$ or $(W'_{H_1,u,b}, W'_{H_2,u,b}, b, \delta)$.

If H_2 is obtained by a permutation of columns of H_1, then there exists a permutation π of Y such that $H_2(x, \pi(y)) = H_1(x, y)$ for all $x, y \in Y$. We define a permutation σ of X by

$$\sigma((i,\ell,x)) = \begin{cases} (i,\ell,\pi(x)) & \text{if } i \text{ is odd,} \\ (i,\ell,x) & \text{otherwise.} \end{cases}$$

Then for $\alpha = (i, \ell, x), \beta = (j, \ell', y) \in X$,

$$\begin{aligned} W_2^\sigma(\alpha,\beta) &= W_2(\sigma(\alpha),\sigma(\beta)) \\ &= \begin{cases} c^{g(i,j)} S_{ij}(\ell,\ell') A_u(x,y) & \text{if } i \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell,\ell') H_2(x,\pi(y)) & \text{if } i \equiv j+1 \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell,\ell') H_2^T(\pi(x),y) & \text{if } i+1 \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell,\ell') A_u(\pi(x),\pi(y)) & \text{if } i \equiv j \equiv 1 \pmod 2 \end{cases} \\ &= \begin{cases} c^{g(i,j)} S_{ij}(\ell,\ell') A_u(x,y) & \text{if } i \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell,\ell') H_1(x,y) & \text{if } i \equiv j+1 \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell,\ell') H_1^T(x,y) & \text{if } i+1 \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell,\ell') A_u(x,y) & \text{if } i \equiv j \equiv 1 \pmod 2 \end{cases} \\ &= W_1(\alpha,\beta). \end{aligned}$$

If H_2 is obtained by a permutation of rows of H_1, then there exists a permutation π' of Y such that $H_2(\pi'(x), y) = H_1(x, y)$ for all $x, y \in Y$. We define a permutation σ' of X by

$$\sigma'((i,\ell,x)) = \begin{cases} (i,\ell,\pi'(x)) & \text{if } i \text{ is even,} \\ (i,\ell,x) & \text{otherwise.} \end{cases}$$

Similar calculation shows $W_2^{\sigma'}(\alpha, \beta) = W_1(\alpha, \beta)$.

If H_2 is obtained by negating a column y_1 of H_1, then $H_2(x, y_1) = -H_1(x, y_1)$, $H_2(x, y) = H_1(x, y)$ for all $x \in Y$ and $y \in Y - \{y_1\}$. We define a permutation ρ of X by

$$\rho((i,\ell,x)) = \begin{cases} (i,\ell+\delta_{x,y_1}\frac{m}{2},x) & \text{if } i \text{ is odd,} \\ (i,\ell,x) & \text{otherwise.} \end{cases}$$

Note that $S_{ij}(\ell, \ell') = (-1)^{i-j} S_{ij}(\ell + \frac{m}{2}, \ell') = (-1)^{i-j} S_{ij}(\ell, \ell' + \frac{m}{2})$. Thus for $\alpha = (i, \ell, x), \beta = (j, \ell', y) \in X$,

$$\begin{aligned} &W_2^\rho(\alpha,\beta) \\ &= W_2(\rho(\alpha),\rho(\beta)) \end{aligned}$$

$$= \begin{cases} c^{g(i,j)} S_{ij}(\ell,\ell') A_u(x,y) & \text{if } i \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell,\ell' + \delta_{y,y_1}\frac{m}{2}) H_2(x,y) & \text{if } i \equiv j+1 \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell + \delta_{x,y_1}\frac{m}{2},\ell') H_2^T(x,y) & \text{if } i+1 \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell + \delta_{x,y_1}\frac{m}{2},\ell' + \delta_{y,y_1}\frac{m}{2}) A_u(x,y) & \text{if } i \equiv j \equiv 1 \pmod 2 \end{cases}$$

$$= \begin{cases} c^{g(i,j)} S_{ij}(\ell,\ell') A_u(x,y) & \text{if } i \equiv j \equiv 0 \pmod 2, \\ (-1)^{\delta_{y,y_1}} c^{g(i,j)} S_{ij}(\ell,\ell') H_2(x,y) & \text{if } i \equiv j+1 \equiv 0 \pmod 2, \\ (-1)^{\delta_{x,y_1}} c^{g(i,j)} S_{ij}(\ell,\ell') H_2^T(x,y) & \text{if } i+1 \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell,\ell') A_u(x,y) & \text{if } i \equiv j \equiv 1 \pmod 2 \end{cases}$$

$$= \begin{cases} c^{g(i,j)} S_{ij}(\ell,\ell') A_u(x,y) & \text{if } i \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell,\ell') H_1(x,y) & \text{if } i \equiv j+1 \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell,\ell') H_1^T(x,y) & \text{if } i+1 \equiv j \equiv 0 \pmod 2, \\ c^{g(i,j)} S_{ij}(\ell,\ell') A_u(x,y) & \text{if } i \equiv j \equiv 1 \pmod 2 \end{cases}$$

$$= W_1(\alpha,\beta).$$

If H_2 is obtained by negating a row x_1 of H_1, then $H_2(x_1,y) = -H_1(x_1,y)$, $H_2(x,y) = H_1(x,y)$ for all $x \in Y - \{x_1\}$ and $y \in Y$. We define a permutation ρ' of X by

$$\rho'((i,\ell,x)) = \begin{cases} (i,\ell + \delta_{x,x_1}\frac{m}{2},x) & \text{if } i \text{ is even}, \\ (i,\ell,x) & \text{otherwise}. \end{cases}$$

Similar calculation shows $W_2^{\rho'}(\alpha,\beta) = W_1(\alpha,\beta)$. □

5. Decomposability

Lemma 5.1. *Let S_1, S_2 be finite subsets of positive real numbers. Suppose $1 \in S_1 \cap S_2$ and $|S_1S_2| = 3$. Then*

$$(|S_1|, |S_2|) \in \{(2,2),(1,3),(3,1)\}.$$

If $|S_1| = |S_2| = 2$, then $S_1S_2 = \{1,a,a^2\}$ or $\{1,a,a^{-1}\}$ for some positive real number $a \neq 1$.

Proof. By way of contradiction, we prove that if $|S_1| \geq 3$ and $|S_2| \geq 2$ then $|S_1S_2| > 3$. Since $S_1 \cup S_2 \subset S_1S_2$, we obtain $S_2 \subset S_1 = S_1S_2$. Let $S_1 = \{1,\lambda,\mu\}$ $(\lambda,\mu \neq 1, \lambda \neq \mu)$. Then we may put $S_2 = \{1,\lambda\}$ without loss of generality. Then we have $\lambda^2 \in S_1S_2 = S_1$, so $\mu = \lambda^2$ and $S_1S_2 = \{1,\lambda,\lambda^2,\lambda^3\}$. This implies $|S_1S_2| = 4$, a contradiction.

Suppose $|S_1| = |S_2| = 2$. Then $S_1 = \{1,a\}$, $S_2 = \{1,b\}$ for some $a, b \neq 1$. Then $|S_1S_2| = 3$ implies $a = b$ or $a = b^{-1}$. □

Lemma 5.2. *Let $A \in \mathrm{Mat}_{Z_1}(\mathbb{C}^*)$ be a matrix all of whose entries have a finite order. Let $B \in \mathrm{Mat}_{Z_2}(\mathbb{C}^*)$ be a matrix which satisfies $\mu(B) < \infty$. Then $\mu(A \otimes B)$ is a divisor of $\mathrm{LCM}(\mu(A),\mu(B))$.*

Proof. Let $Z_2' = \{(x_2,y_2) \in Z_2 \times Z_2 \mid o(B(x_2,y_2)) < \infty\}$. Then

$$\mu(A \otimes B) = \mathrm{LCM}(\{o(A(x_1,y_1)B(x_2,y_2)) \mid x_1,y_1 \in Z_1, (x_2,y_2) \in Z_2'\}),$$

which is a divisor of $\mathrm{LCM}(\mu(A),\mu(B))$. □

Some examples of spin models are listed in Section 1, i.e., Potts model, non-symmetric Hadamard models, and Hadamard models. We remark that non-symmetric Hadamard models and Hadamard models are special cases of spin models given in Theorem 1.1 (i), (ii), respectively. In addition to these examples, the following spin models are known.

Spin models on finite abelian groups. Bannai-Bannai-Jaeger [3] gives solutions to modular invariance equation for finite abelian groups, and every solution gives a spin model. Let U be a finite abelian group, and $e = \exp(U)$ denote the exponent of U. Let $\{\chi_a \mid a \in U\}$ be the set of characters of U with indices chosen so that $\chi_a(b) = \chi_b(a)$ for all $a, b \in U$. Let $U = U_1 \oplus \ldots \oplus U_h$ be a decomposition of U into a direct sum of cyclic groups $U_1, U_2, \ldots, U_h$. For each $i \in \{1, 2, \ldots, h\}$ let a_i be a generator and n_i be the order of the cyclic group U_i. For each $x \in U$, we define the matrix $A_x \in Mat_U(\mathbb{C})$ by

$$A_x(\alpha, \beta) = \delta_{x,\beta-\alpha} \quad (\alpha, \beta \in U).$$

For any $x = \sum_{i=1}^{h} x_i a_i$ $(0 \leq x_i < n_i)$, let

$$t_x = t_0 \prod_{i=1}^{h} \eta_i^{x_i} \chi_{a_i}(a_i)^{\frac{x_i(x_i-1)}{2}} \prod_{1 \leq \ell < k \leq h} \chi_{a_\ell}(a_k)^{x_\ell x_k}, \tag{24}$$

where $\eta_i^{n_i} = \chi_{a_i}(a_i)^{-\frac{n_i(n_i-1)}{2}}$ and

$$t_0^2 = D^{-1} \sum_{x \in U} \prod_{j=1}^{h} \eta_j^{-x_j} \chi_{a_j}(a_j)^{-\frac{x_j(x_j-1)}{2}} \prod_{1 \leq \ell < k \leq h} \chi_{a_\ell}(a_k)^{-x_\ell x_k}, \tag{25}$$

where $D^2 = |U|$. Let $\theta_x = t_x / t_0$ for any $x \in U$. Then, for any $x \in U$, θ_x is a root of unity and $\theta_x^{2e} = 1$. Especially, we get

$$\theta_x^{2|U|} = 1. \tag{26}$$

The matrix

$$W = \sum_{x \in U} t_x A_x. \tag{27}$$

is a spin model.

Jaeger's Higman-Sims model. In [10], F. Jaeger constructed a spin model W_J on the Higman-Sims graph of size 100. We denote by A the adjacency matrix of the Higman-Sims graph. We put $W_J = -\tau^5 I - \tau A + \tau^{-1}(J - A - I)$, where τ satisfies $\tau^2 + \tau^{-2} = 3$. Then W_J is a symmetric spin model.

Now every known spin model belongs to one of the following five families:

(a) A_u: Potts model of size $r \geq 2$. If $r = 2$, then $\mu(A_u) = 16$. If $r = 4$, then $\mu(A_u) = 2$ or 4. If $r = 2, 4$, then $E(A_u) = \{1\}$. If $r > 4$, then $E(A_u) = \{1, |u|^{-4}\}$, and hence $|E(A_u)| = 2$.
(b) W_U: spin model on a finite abelian group U. We have various kinds of indices and $E(W_U) = \{1\}$.
(c) W_J: Jaeger's Higman-Sims model of size 100. We have $E(W_J) = \{1, \tau^{-4}, \tau^{-6}\}$ with $\tau^2 + \tau^{-2} = 3$. and hence $|E(W_J)| = 3$.
(d) $W_{H,u,a}$: spin models given in Theorem 1.1(i).
(e) $W'_{H,u,b}$: spin models given in Theorem 1.1(ii).

By way of contradiction, we now give a proof of Theorem 1.3. Let H be a Hadamard matrix of order $r > 4$. Let s be a positive integer and a a primitive 2^{2s+1}-th root of unity. For the remainder of this section, we denote by W the spin model $W_{H,u,a}$ given in Theorem 1.1 (i) of index 2^s. By Lemma 4.2 we obtain

$$E(W) = \{1, |u|^{-4}, |u|^{-3}\}. \tag{28}$$

We assume that

$$W = W_1 \otimes W_2 \otimes \ldots \otimes W_v, \tag{29}$$

where each of W_1, W_2, …, W_v is a known spin model listed in (a)–(e) and their sizes are not equal to 1. Since $|E(W)| = 3$ from (28), using Lemma 5.1 we may assume without loss of generality

$$(|E(W_1)|, |E(W_2)|, \ldots, |E(W_v)|) = (1, \ldots, 1, 2, 2) \text{ or } (1, \ldots, 1, 3).$$

A known spin model W' with $|E(W')| = 1$ belongs to the family (b) or to the families (a), (d) and (e) with $r \le 4$. Therefore, (29) can be reduced to the following cases:

$$W = W_1 \otimes W_2 \otimes W_3 \text{ with } E(W_1) = \{1\}, |E(W_2)| = |E(W_3)| = 2, \tag{30}$$
$$W = W_1 \otimes W_2 \text{ with } E(W_1) = \{1\}, |E(W_2)| = 3, \tag{31}$$

where in (30), (31), W_1 is a tensor product of spin models on finite abelian groups and spin models in the families (a), (d) and (e) with $r \le 4$. Note that W_1 could possibly be of size 1 in (30).

First, we treat the case (30). Then Lemma 5.1 implies $E(W_2 \otimes W_3) = \{1, \beta, \beta^2\}$, or $\{1, \beta, \beta^{-1}\}$ for some β. On the other hand, $E(W_2 \otimes W_3) = E(W_1)E(W_2 \otimes W_3) = E(W) = \{1, |u|^{-4}, |u|^{-3}\}$ by (28). This is a contradiction.

Next, we treat the case (31). We have $E(W_2) = E(W_1)E(W_2) = E(W) = \{1, |u|^{-4}, |u|^{-3}\}$ from (28). Since $\{1, |u|^{-4}, |u|^{-3}\} \neq \{1, \tau^{-4}, \tau^{-6}\}$, W_2 cannot be the spin model (c). Therefore, W_2 belongs to the family (d) or (e). This means $W_2 = W_{H',u',a'}$ or $W_2 = W'_{H',u',b'}$, where H' is a Hadamard matrix of order $r' = (u'^2 + u'^{-2})^2$. Since $|E(W_2)| = 3$, Lemma 4.2 implies $r' > 4$ and $E(W_2) = \{1, |u'|^{-4}, |u'|^{-3}\}$. Then we have $|u'| = |u|$, as $E(W) = E(W_2)$. Now the second part of Lemma 4.1 implies $u^4 > 0$ and $u'^4 > 0$, hence

$$u^4 = u'^4, \tag{32}$$

and further $r = r'$ by (1). Therefore the size of W_2 is $2^{2s'}r$ for some integer s' with $0 < s' < s$, and the size of W_1 is $2^{2(s-s')}$. In particular, we obtain $s > 1$.

Since the tensor product of spin models on finite abelian groups is also a spin model on a finite abelian group, we may suppose that

$$W_1 = W_{11} \otimes W_{12} \otimes W_{13}, \tag{33}$$

where W_{11} is a spin model on a finite abelian group U, W_{12} is a tensor product of spin models in the family (a) with $r \le 4$, and W_{13} is a tensor product of spin models in the families (d) and (e) with $r \le 4$.

We put $|U| = 2^{n_1}$. Since the size 2^{n_1} of W_{11} cannot exceed that of W_1, we have $n_1 \le 2(s - s')$. Then the size of $W_{12} \otimes W_{13}$ is $2^{2(s-s')-n_1}$. The diagonal entry of W_{11} is a complex number t_0 given by (25). The diagonal entries of W_{12}, W_{13} are 16-th roots of unity. We denote by κ_2, κ_3 the diagonal entries of W_{12}, W_{13}, respectively. Comparing the diagonal entries of (33), we have $u^3 = t_0\kappa_2\kappa_3 u'^3$, thus

$$W = (t_0^{-1}W_{11}) \otimes (\kappa_2^{-1}W_{12}) \otimes (\kappa_3^{-1}W_{13}) \otimes (u^3u'^{-3}W_2). \tag{34}$$

From (26), we have

$$\mu(t_0^{-1}W_{11}) \mid 2^{n_1+1}. \tag{35}$$

From (a), we have

$$\mu(\kappa_2^{-1}W_{12}) \mid 2^4. \tag{36}$$

From (a) and Lemma 4.4, we have

$$\mu(\kappa_3^{-1}W_{13}) \mid 2^{2(s-s')-n_1+1}. \tag{37}$$

Since W_2 is a spin model belonging to the family (d) or (e), Lemma 4.3 and (32) imply

$$\mu(u^3u'^{-3}W_2) \mid 2^{2s'+1}. \tag{38}$$

From (34)–(38) and Lemma 5.2, we have

$$\mu(W) \mid \mathrm{LCM}(2^{n_1+1}, 2^4, 2^{2(s-s')-n_1+1}, 2^{2s'+1}).$$

Since $n_1 < 2s$, we have $\max(n_1+1, 4, 2(s-s')-n_1+1, 2s'+1) \le 2s$. This implies $\mu(W) \mid 2^{2s}$, which contradicts Lemma 4.3 (i).

6. Concluding remarks

In this section, we treat the case of $r \le 4$ in Theorem 1.3. We show that if $r = 1, 4$ in Theorem 1.3, then $W_{H,u,a}$ is not new.

If $r = 4$ in Theorem 1.1 (i), then $W_{H,u,a}$ is a tensor product of a Hadamard matrix of order 4 and $W_{(1),u,a}$. Indeed, up to equivalence, there is a unique Hadamard matrix of order $r = 4$. By Lemma 4.5, we may assume without loss of generality

$$H = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$

Then $A_u = u^3H$ with $(u^2+u^{-2})^2 = 4$. Therefore we have $W_{H,u,a} = H \otimes W_{(1),u,a}$. Similarly, a spin model $W'_{H,u,b}$ in Theorem 1.1 (ii) can be decomposed as $H \otimes W'_{(1),u,b}$.

Lemma 6.1. *Let $m \equiv 0 \pmod 4$. Let $W_{(1),u,a}$ be a spin model given in Theorem 1.1 of index m, where $u^4 = 1$ and a is a primitive $2m^2$-th root of unity. Then $W_{(1),u,a}$ is equivalent to $W_{(1),1,au^3}$.*

Proof. First we assume that $u = -1$. Then $a^{\epsilon(i,j)}(-1)^{i-j-1} = -(-a)^{\epsilon(i,j)}$ holds for all $i, j \in \mathbb{Z}_{m^2}$. From this, we have $W_{(1),-1,a} = -W_{(1),1,-a}$. Therefore $W_{(1),-1,a}$ is equivalent to $W_{(1),1,-a}$.

Next we assume that $u^2 = -1$. Since $m \equiv 0 \pmod 4$, we have

$$u(au^3)^{\epsilon(i,j)} = \begin{cases} a^{\epsilon(i,j)}u & \text{if } i-j \text{ is even,} \\ a^{\epsilon(i,j)} & \text{if } i-j \text{ is odd.} \end{cases}$$

From this, we have $uW_{(1),1,au^3} = W_{(1),u,a}$. Therefore $W_{(1),u,a}$ is equivalent to $W_{(1),1,au^3}$. □

Lemma 6.2. *Let m be even, and ξ be a primitive $2m^2$-th root of unity. Then we have*

$$\sum_{x=0}^{m^2-1} \xi^{-x(x-m)} = m. \tag{39}$$

Proof. If (39) holds for $\xi=\exp(2\pi\sqrt{-1}/(2m^2))$, then by considering the action of the Galois group, we see that (39) holds for any primitive $2m^2$-th root of unity ξ. Therefore we may assune $\xi=\exp(2\pi\sqrt{-1}/(2m^2))$ without loss of generality. Since m is even, we may write $m=2k$. Then

$$\begin{aligned}\sum_{x=0}^{m^2-1}\xi^{-x(x-m)} &= \sum_{x=0}^{m^2-1}\xi^{-((x-k)^2-k^2)}\\ &= \xi^{k^2}\sum_{x=0}^{m^2-1}\xi^{-(x-k)^2}\\ &= \frac{\xi^{k^2}}{2}\sum_{x=0}^{m^2-1}(\xi^{-(x-k)^2}+\xi^{-(x-k+m^2)^2})\\ &= \frac{\exp(\pi\sqrt{-1}/4)}{2}\sum_{x=0}^{2m^2-1}\xi^{-(x-k)^2}\\ &= \frac{1+\sqrt{-1}}{2\sqrt{2}}\sum_{x=0}^{2m^2-1}\xi^{-x^2}.\end{aligned}$$

Now the result follows from [16, Theorem 99]. □

Of particular interest among spin models on finite abelian groups are spin models on finite cyclic groups. The spin model defined below is a special case of spin models on finite cyclic groups constructed by [1]. Let m be even, and a be a primitive $2m^2$-th root of unity. We restrict (24) and (25) to $\mathbb{Z}_{m^2}$, that is, $h=1$. In (24) and (25), we put $\eta_1=a^{-m+1}$, $\chi_{a_1}(a_1)=a^2$. Then (24) and (25) become

$$t_x = t_0 a^{x(x-m)} \quad (x\in\mathbb{Z}_{m^2}), \tag{40}$$

$$t_0^2 = m^{-1}\sum_{x=0}^{m^2-1}a^{-x(x-m)}=1, \tag{41}$$

respectively, where we used Lemma 6.2 in (41). Thus we may take $t_0=1$. Then the matrix W given in (27) has entries

$$W(\alpha,\beta)=a^{(\beta-\alpha)(\beta-\alpha-m)} \quad (\alpha,\beta\in\mathbb{Z}_{m^2}). \tag{42}$$

We note that this spin model W on $\mathbb{Z}_{m^2}$ was constructed originally in [2, Theorem 2].

Proposition 6.3. *Let* $m\equiv 0 \pmod 4$. *Let* $W_{(1),u,a}$ *be a spin model given in Theorem* 1.1 (i) *of index* m, *where* $u^4=1$ *and* a *is a primitive* $2m^2$*-th root of unity. Then* $W_{(1),u,a}$ *is equivalent to* W *defined in* (42).

Proof. From Lemma 6.1 it is sufficient to prove that $W_{(1),1,au^3}$ is equivalent to W. By assumption, $m=4k$ for some positive integer k. Since a^{8k^2} is a primitive 4-th root of unity, there exists $t\in\mathbb{Z}_4$ such that $u^3=a^{8k^2t}$. We define a bijection $\psi:\mathbb{Z}_m^2\to\mathbb{Z}_{m^2}$ by

$$\psi(i,\ell)=(4k^2t+1)i+4k\ell$$

for $(i,\ell) \in \mathbb{Z}_m^2$. Then for all $i,j,\ell,\ell' \in \mathbb{Z}_m$,

$$\begin{aligned}
&(\psi(j,\ell') - \psi(i,\ell))(\psi(j,\ell') - \psi(i,\ell) - m) \\
&= ((4k^2t+1)(j-i) + 4k(\ell'-\ell))((4k^2t+1)(j-i) + 4k(\ell'-\ell) - 4k) \\
&= (8k^2t+1)(8k(\ell-\ell')(i-j) + (i-j)^2 + 4k(i-j)) \\
&\quad + 32k^2\left(-kt(j-i)(l'-l) + \frac{kt(j-i)(kt(j-i)+1)}{2}\right. \\
&\qquad\qquad \left. + \frac{(l'-l)(l'-l-1)}{2}\right) \\
&\equiv (8k^2t+1)(8k(\ell-\ell')(i-j) + (i-j)^2 + 4k(i-j)) \pmod{32k^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
W(\psi(i,\ell),\psi(j,\ell')) &= a^{(\psi(j,\ell')-\psi(i,\ell))(\psi(j,\ell')-\psi(i,\ell)-m)} \\
&= a^{(8k^2t+1)(8k(\ell-\ell')(i-j)+(i-j)^2+4k(i-j))} \\
&= (au^3)^{2m(\ell-\ell')(i-j)+(i-j)^2+m(i-j)} \\
&= W_{(1),1,au^3}((i,\ell,1),(j,\ell',1)),
\end{aligned}$$

and we conclude that W is equivalent to $W_{(1),1,au^3}$. □

To conclude the paper, we note that the decomposability and identification with known spin models are yet to be determined for the following cases.

(1) $W_{H,u,a}$: $r=1$, $m \equiv 2 \pmod 4$,
(2) $W'_{H,u,b}$: $r=1$,
(3) $W_{H,u,a}$ and $W'_{H,u,b}$: $r=2$,
(4) $W_{H,u,a}$ and $W'_{H,u,b}$: $r>4$ and m is not a power of 2.

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