Distribution of the Riemann Zeros Represented by the Fermi Gas

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The multiparticle density matrices for degenerate, ideal Fermi gas system in any dimension are calculated. The results are expressed as a determinant form, in which a correlation kernel plays a vital role. Interestingly, the correlation structure of one-dimensional Fermi gas system is essentially equivalent to that observed for the eigenvalue distribution of random unitary matrices, and thus to that conjectured for the distribution of the non-trivial zeros of the Riemann zeta function. Implications of the present findings are discussed briefly.

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In this study we consider the noninteracting N-particle fermion system at zero temperature in any spatial dimension d. In the following analysis the spin-polarized case is considered, and the volume of the system is taken to be unity for notational simplicity. The N-body density matrix[1)] for this Fermi system is then expressed in terms of the product of the Slater determinants with the plane waves[1),2)] as

$$\begin{aligned}
&\rho^{(N)}\left(\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N\right) \\
&= \frac{1}{N!} \sum_{P^{(N)}} \left[\mathrm{sgn} P^{(N)}\right] \exp\left(i\boldsymbol{k}_1 \cdot \boldsymbol{r}_{p_1} + i\boldsymbol{k}_2 \cdot \boldsymbol{r}_{p_2} + \cdots + i\boldsymbol{k}_N \cdot \boldsymbol{r}_{p_N}\right) \\
&\times \sum_{P'^{(N)}} \left[\mathrm{sgn} P'^{(N)}\right] \exp\left(-i\boldsymbol{k}_1 \cdot \boldsymbol{r}_{p'_1} - i\boldsymbol{k}_2 \cdot \boldsymbol{r}_{p'_2} - \cdots - i\boldsymbol{k}_N \cdot \boldsymbol{r}_{p'_N}\right) \\
&= \frac{1}{N!} \sum_{P^{(N)}} \left[\mathrm{sgn} P^{(N)}\right] \exp\left(i\boldsymbol{k}_{p_1} \cdot \boldsymbol{r}_1 + i\boldsymbol{k}_{p_2} \cdot \boldsymbol{r}_2 + \cdots + i\boldsymbol{k}_{p_N} \cdot \boldsymbol{r}_N\right) \\
&\times \sum_{P'^{(N)}} \left[\mathrm{sgn} P'^{(N)}\right] \exp\left(-i\boldsymbol{k}_{p'_1} \cdot \boldsymbol{r}_1 - i\boldsymbol{k}_{p'_2} \cdot \boldsymbol{r}_2 - \cdots - i\boldsymbol{k}_{p'_N} \cdot \boldsymbol{r}_N\right),
\end{aligned} \tag{1}$$

where $\boldsymbol{r}_i$ and $\boldsymbol{k}_i$ refer to the spatial coordinates and the wavenumber vectors, respectively;

$$P^{(N)} = \begin{pmatrix} 1 & 2 & \cdots & N \\ p_1 & p_2 & \cdots & p_N \end{pmatrix} \tag{2}$$

and another $P'^{(N)}$ mean the permutations of order N, and

$$\mathrm{sgn} P^{(N)} = \begin{cases} +1, & \text{for even permutation} \\ -1, & \text{for odd permutation} \end{cases} \tag{3}$$

denotes their signature. The density matrix given by Eq. (1) satisfies the normalization condition as

$$\int d\boldsymbol{r}_1 \cdots d\boldsymbol{r}_N \rho^{(N)} (\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N) = \frac{1}{N!} \sum_{P^{(N)}} \left[\mathrm{sgn} P^{(N)} \right]^2 = 1. \tag{4}$$

Here, noting the identities for $N \times N$ matrices $\boldsymbol{A}$, its transposition $\boldsymbol{A}^T$, and $\boldsymbol{B}$ as

$$\det \boldsymbol{A}^T = \det \boldsymbol{A} \tag{5}$$

and

$$\det \boldsymbol{A} \det \boldsymbol{B} = \det \boldsymbol{A}\boldsymbol{B}, \tag{6}$$

we find

$$\begin{aligned} \rho^{(N)} (\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N) &= \frac{N^N}{N!} \sum_{P^{(N)}} \left[\mathrm{sgn} P^{(N)} \right] K_{1p_1} K_{2p_2} \cdots \times K_{Np_N} \\ &= \frac{N^N}{N!} \det \boldsymbol{K}^{(N)}, \end{aligned} \tag{7}$$

where

$$\boldsymbol{K}^{(N)} = (K_{ij}) \tag{8}$$

is an $N \times N$ matrix whose components are

$$K_{ij} = \frac{1}{N} \sum_{k=1}^{N} \exp \left[i\boldsymbol{k}_k \cdot (\boldsymbol{r}_i - \boldsymbol{r}_j) \right] \tag{9}$$

for $1 \le i, j \le N$. The n-body $(1 \le n \le N)$ density matrix[1)] is then given by

$$\begin{aligned} \rho^{(n)} (\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_n) &= \int d\boldsymbol{r}_{n+1} \cdots d\boldsymbol{r}_N \rho^{(N)} (\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_N) \\ &= \frac{(N-n)!}{N!} N^n \sum_{P^{(n)}} \left[\mathrm{sgn} P^{(n)} \right] K_{1p_1} K_{2p_2} \cdots \times K_{np_n} \\ &= \frac{(N-n)!}{N!} N^n \det \boldsymbol{K}^{(n)} \end{aligned} \tag{10}$$

with an $n \times n$ matrix $\boldsymbol{K}^{(n)}$. It is noted that the prefactor before $\det \boldsymbol{K}^{(n)}$ in Eq. (10) becomes unity for finite n and $N \to \infty$.

A problem in the following is to find an explicit expression for the correlation kernel,

$$K(\boldsymbol{r}) = \frac{1}{N} \sum_{\boldsymbol{k}} \exp (i\boldsymbol{k} \cdot \boldsymbol{r}) \, n_{\boldsymbol{k}}, \tag{11}$$

in any dimension d, which represents Eq. (9) in the limit of infinite N. Here, the $\boldsymbol{k}$ summation is expressed by

$$\sum_{\boldsymbol{k}} = \frac{1}{(2\pi)^d} \int d\boldsymbol{k}, \tag{12}$$

and for the degenerate Fermi gas,

$$n_{\boldsymbol{k}} = \theta\left(k_{\mathrm{F}} - |\boldsymbol{k}|\right) \tag{13}$$

refers to the occupation number density in the wavenumber space with $\theta(x)$ and k_{F} being the step function and the Fermi wavenumber, respectively.

The occupation number density satisfies the normalization condition,

$$\sum_{\boldsymbol{k}} n_{\boldsymbol{k}} = N. \tag{14}$$

Therefore, the Fermi wavenumber k_{F} is given by

$$\frac{1}{(2\pi)^d} V_d\left(k_{\mathrm{F}}\right) = N, \tag{15}$$

where

$$V_d\left(k_{\mathrm{F}}\right) = C_d k_{\mathrm{F}}^d \tag{16}$$

with

$$C_d = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \tag{17}$$

is the volume of the d-dimensional sphere with the radius of k_{F}; $\Gamma(s)$ is the gamma function.[3),4)] The Fermi wavenumber is thus expressed as

$$k_{\mathrm{F}} = 2\sqrt{\pi}\left[\Gamma\left(\frac{d}{2}+1\right)N\right]^{1/d}. \tag{18}$$

By choosing the direction of $\boldsymbol{r}$ in parallel with the d-th component of the wavenumber vector $\boldsymbol{k}$, Eq. (11) is expressed as

$$\begin{aligned} K\left(\boldsymbol{r}\right) &= \frac{1}{N(2\pi)^d}\int_{-1}^{1} dt\ k_{\mathrm{F}} \exp\left(ik_{\mathrm{F}}rt\right) V_{d-1}\left(k_{\mathrm{F}}\sqrt{1-t^2}\right) \\ &= \frac{C_{d-1}k_{\mathrm{F}}^d}{N(2\pi)^d}\int_{-1}^{1} dt \exp\left(ik_{\mathrm{F}}rt\right)\left(1-t^2\right)^{\frac{d-1}{2}} \end{aligned} \tag{19}$$

with $r = |\boldsymbol{r}|$. Then, employing Poisson's formula,[3),4)]

$$\int_{-1}^{1} dt \exp(izt)\left(1-t^2\right)^{\nu-\frac{1}{2}} = \frac{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}{(z/2)^\nu} J_\nu(z), \tag{20}$$

we find an explicit expression for the correlation kernel as

$$K(\boldsymbol{r};\nu) = \Gamma(\nu+1)\left(\frac{2}{k_{\mathrm{F}}r}\right)^{\nu} J_{\nu}(k_{\mathrm{F}}r) \tag{21}$$

with $\nu = d/2$, where $J_\nu(z)$ is the Bessel function of the first kind of order ν.[3),4)] It is remarked that, though Eq. (21) has been derived for integral values of spatial dimension d, $K(\boldsymbol{r};\nu)$ may be regarded as a continuous function of the auxiliary order variable ν, which follows from its series representation.[3),4)]

Let us here consider the case of $d = 3$ and $\nu = 3/2$. The correlation kernel then reads

$$\begin{aligned} K(\boldsymbol{r}) &= \Gamma\left(\frac{5}{2}\right)\left(\frac{2}{k_{\mathrm{F}}r}\right)^{3/2} J_{3/2}(k_{\mathrm{F}}r) \\ &= \frac{3}{k_{\mathrm{F}}r} j_1(k_{\mathrm{F}}r) \\ &= \frac{3}{(k_{\mathrm{F}}r)^3}\left(\sin k_{\mathrm{F}}r - k_{\mathrm{F}}r\cos k_{\mathrm{F}}r\right), \end{aligned} \tag{22}$$

where

$$j_n(z) = \left(\frac{\pi}{2z}\right)^{1/2} J_{n+\frac{1}{2}}(z) \tag{23}$$

is the spherical Bessel function of the first kind of order n.[3),4)] This expression, Eq. (22), combined with Eq. (10) in the case of two-body density matrix, reproduces a well-known result[1),2)] for the pair distribution function of the three-dimensional Fermi gas.

In the case of $d = 1$ and $\nu = 1/2$, we find

$$\begin{aligned} K(r) &= \Gamma\left(\frac{3}{2}\right)\left(\frac{2}{k_{\mathrm{F}}r}\right)^{1/2} J_{1/2}(k_{\mathrm{F}}r) \\ &= j_0(k_{\mathrm{F}}r) \\ &= \frac{\sin k_{\mathrm{F}}r}{k_{\mathrm{F}}r}. \end{aligned} \tag{24}$$

Recalling Eq. (10), this correlation structure is essentially the same as that for the eigenvalues of random matrices in the Gaussian unitary ensemble.[5)–7)] Interestingly, it has been known that this type of correlation structure may hold also for the distribution of zeros in the Riemann zeta function.[8)]

The Riemann zeta function[8)] for complex variable s is defined by

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_p \left(1 - p^{-s}\right)^{-1} \tag{25}$$

for Re $s > 1$, where n and p mean the natural numbers and the prime numbers, respectively. After the analytic continuation over the whole complex plane, the $\zeta(s)$ has non-trivial zeros in the critical strip, $0 < \mathrm{Re}\ s < 1$, and the Riemann hypothesis[8] states that all of them lie on the critical line Re $s = 1/2$; that is,

$$\zeta\left(\frac{1}{2} + it\right) = 0 \tag{26}$$

has non-trivial solutions only when $t = t_j$ $(j = 1, 2, \cdots)$ are real.

The mean density of the non-trivial zeros of $\zeta(s)$ increases logarithmically with height t up to the critical line. We then define unfolded zeros by

$$w_j = \frac{t_j}{2\pi} \log \frac{t_j}{2\pi}. \tag{27}$$

It has then been conjectured[9),10)] that the pair correlation function of the unfolded zeros for $j \to \infty$ has a form,

$$R^{(2)}(w) = 1 - \left(\frac{\sin \pi w}{\pi w}\right)^2, \tag{28}$$

which is analogous to that for the one-dimensional ($d = 1$) Fermi gas, that is, Eqs. (24) and (10) for $n = 2$. This conjecture has also been generalized for all the n-point correlations as a determinant form analogous to Eq. (10).[11)–14)]

Thus, the case of $\nu = d/2 \to 1/2$ in the Fermi gas system gives a special correlation structure observed in the random unitary matrices and the Riemann zeta function. Therefore, the behaviors of the multiparticle correlations of the Fermi gas system with the correlation kernel $K(\boldsymbol{r}; \nu)$ at and around $\nu = 1/2$ may provide useful information about the random matrices and the zeta function, regarding $K(\boldsymbol{r}; \nu)$ as a continuous function of ν. Another challenge, which would be more ambitious, is to look for a family of functions whose zero distributions are described by the correlation functions given by Eqs. (10) and (21) for arbitrary values of ν.

In summary, it has been revealed in this study that the correlation structure similar to those observed in the eigenvalues of the random unitary matrices and in the Riemann zeros is embedded in the Fermi gas system as well.

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1) G.D. Mahan, *Many-Particle Physics* (Plenum Press, New York, 1990).
2) E. Wigner and F. Seitz, Phys. Rev. **43** (1933), 804.
3) M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, New York, 1965).
4) I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, San Diego, 1994).
5) F.J. Dyson, J. Math. Phys. **3** (1962), 140.
6) F.J. Dyson, J. Math. Phys. **3** (1962), 157.
7) F.J. Dyson, J. Math. Phys. **3** (1962), 166.
8) E.C. Titchmarsh and D.R. Heath-Brown, *The Theory of the Riemann Zeta-function* (Clarendon Press, Oxford, 1986).
9) H.L. Montgomery, Proc. Symp. Pure Math. **24** (1973), 181.

10) A.M. Odlyzko, in *Dynamical, Spectral, and Arithmetic Zeta Functions*, edited by M. van Frankenhuysen and M.L. Lapidus (Am. Math. Soc., Contemporary Math. Series, No. 290, 2001), p. 139.
11) Z. Rudnick and P. Sarnak, Duke Math. J. **81** (1996), 269.
12) E.B. Bogomolny and J.P. Keating, Nonlinearity **8** (1995), 1115.
13) E.B. Bogomolny and J.P. Keating, Nonlinearity **9** (1996), 911.
14) J.P. Keating and N.C. Snaith, J. Phys. A **36** (2003), 2859.