Intuitionistic Fuzzy Ideal Extensions of Γ-Semigroups

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Abstract. In this paper the concept of the extensions of intuitionistic fuzzy ideals in a semigroup has been extended to a Γ-Semigroups. Among other results characterization of prime ideals in a Γ-Semigroups in terms of intuitionistic fuzzy ideal extension has been obtained.

1. Introduction

Γ-Semigroups was introduced by Sen and Saha[14] as a generalization of semigroup and ternary semigroup. Many results of semigroups could be extended to Γ-Semigroups directly and via operator semigroups[4] of a Γ-Semigroups.The concept of intuitionistic fuzzy set was introduced by Atanassove[2, 3] ,as a generalizion of the notion of fuzzy set. Many results of semigroups have been studied in terms of fuzzy sets[16]. Kuroki[5,6] is the main contributor to this study. Motivated by Kuroki [5,6], Xie[15], Mustafa et all[9] we have initiated the study of Γ-Semigroups in terms of intuitionistic fuzzy sets. This paper is a continuation of [7],[8]. In this paper, the concept of the extensions of intuitionistic fuzzy ideals in a semigroup, introduced by Xie, has been extended to the general situation of Γ-Semigroups. We have investigated some of its properties in terms of intuitionistic fuzzy prime and intuitionistic fuzzy semiprime ideals of Γ-semigroup. Among other results we have obtained characterization of prime ideals in a Γ-Semigroups in terms of intuitionistic fuzzy ideal extension. The above introduction is mostly a part of [11]

2. Preliminaries

2.1. Definition[4]. Let S and Γ be two non-empty sets. S is called a Γ-semigroup[4] if there exist mappings from $S \times \Gamma \times S$ to S, written as $(a, \alpha, b) \to a\alpha b$, and from $\Gamma \times S \times \Gamma$ to Γ, written as $(\alpha, a, \beta) \to \alpha a\beta$ satisfying the following associative laws

$$(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c) \; and \; \alpha(a\beta b)\gamma = (\alpha a\beta)b\gamma = \alpha a(\beta b\gamma)$$

for all $a, b, c \in S$ and for all $\alpha, \beta, \gamma \in \Gamma$.

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2.2. Definition [**2**]**.** An intuitionistic fuzzy set A of a non-empty set X is an object having of the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$$

where the function $\mu_A \colon X \to [0, 1]$ and $\gamma_A \colon X \to [0, 1]$ denote the degree of membership and the degree of nonmembership of each element $x \in X$ to the set A, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$.

For the sake of simplicity, We shall use the symbol $A = (\mu_A, \gamma_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in S\}$. $\mathrm{Im}\,(\mu_A)$ denote the image set of μ_A.Similirly $\mathrm{Im}\,(\gamma_A)$ denote the image set of γ_A.

2.3. Definition [**15**]**.** The set of all intuitionistic fuzzy subsets $A = (\mu_A, \upsilon_A)$ and $B = (\mu_B, \upsilon_B)$ of a set X with the relation

$$A \subseteq B \ \ iff \ \ \mu_A \leq \mu_B \ and \ \upsilon_A \geq \upsilon_B$$

$\forall\ x \in X$ is a complete lattice.

For a nonempty family $\left\{A_i = (\mu_{A_i}, \upsilon_{A_i}) \ : i \in I\right\}$ of intuitionistic fuzzy subsets of X, the inf $A_i = \left\{(\inf \mu_{A_i}, \sup \upsilon_{A_i}) \ : i \in I\right\}$ and the sup $A_i = \left\{(\sup \mu_{A_i}, \inf \upsilon_{A_i}) \ : i \in I\right\}$ are the intuitionistic fuzzy subsets of X defined by:

$\inf A_i : X \longrightarrow [0, 1]$,$x \longrightarrow \left\{(\inf \mu_{A_i}(x), \sup \upsilon_{A_i}(x)) \ : i \in I\right\}$

$\sup A_i : X \longrightarrow [0, 1]$, x$\longrightarrow \left\{(\sup \mu_{A_i}(x), \inf \upsilon_{A_i}(x)) \ : i \in I\right\}$ where $\inf \mu_{A_i}(x) = \inf \left\{\mu_{A_i}(x) : i \in I\right\}$ and $\sup \upsilon_{A_i}(x) = \sup \{\upsilon_{A_i}(x) : i \in I\}$ and similarly for $\sup \mu_{A_i}(x)$ and $\inf \upsilon_{A_i}(x)$.

2.4. Definition [**8**]**.** A non-empty intitutionistic fuzzy subset $A = (\mu_A, \upsilon_A)$ of a Γ-semigroup S is called a intitutionistic fuzzy left ideal(right ideal) of S if

$$\begin{array}{rcll} \mu_A(x\gamma y) & \geq & \mu_A(y) & (\mu_A(x\gamma y) \geq \mu_A(y)) \\ \upsilon_A(x\gamma y) & \leq & \upsilon_A(y) & (\upsilon_A(x\gamma y) \leq \upsilon_A(y)) \end{array}$$

$\forall x, y \in S, \forall \gamma \in \Gamma$.

2.5. Definition [**8**]**.** A non-empty intitutionistic fuzzy subset $A = (\mu_A, \upsilon_A)$ of a Γ-semigroup S is called an intitutionistic fuzzy ideal of S if it is both intitutionistic fuzzy left ideal and intitutionistic fuzzy right ideal of S.

2.6. Definition [**7**]**.** An intitutionistic fuzzy ideal $A = (\mu_A, \upsilon_A)$ of a Γ-semigroup S is called intitutionistic fuzzy prime ideal

$$\inf_{\gamma \in \Gamma} \mu_A(x\gamma y) = \{\mu_A(x) \vee \mu_A(y)\}$$

and

$$\sup_{\gamma \in \Gamma} \upsilon_A(x\gamma y) = \{\upsilon_A(x) \wedge \upsilon_A(y)\}$$

$\forall x, y \in S$.

2.7. Definition. An intuitionistic fuzzy ideal $A = (\mu_A, \upsilon_A)$ of a Γ-Semigroups S is called intuitionistic fuzzy semiprime ideal if

$$\mu_A(x) \geq \inf_{\gamma \in \Gamma} \mu_A(x\gamma x)$$

and

$$\upsilon_A(x) \underset{\gamma \in \Gamma}{\leq} \sup \upsilon_A(x\gamma x)$$

$\forall x, y \in S$.

2.8. Definition [**4**]**.** Let S be a Γ-Semigroups. Then an ideal I of S is said to be

(i) prime if for ideals A, B of S, $A\Gamma B \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$.

(ii) semiprime if for an ideal A of S, $A\Gamma A \subseteq I$ implies that $A \subseteq I$.

2.9. Proposition [**7, 8**]**.** Let S be a Γ-Semigroups and $\phi = I \subseteq S$. Then I is an ideal(prime ideal, semiprime ideal) of S iff $X = (\Phi_I, \Psi_I)$ is an intuitionistic fuzzy ideal(resp. intuitionistic fuzzy prime ideal, intuitionistic fuzzy semiprime ideal) of S, where $X = (\Phi_I, \Psi_I)$ is the characteristic function of I.

2.10. Theorem [**7,8**]**.** Let I be an ideal of a Γ-Semigroups S. Then the following are equivalent:

(i) I is prime(semiprime).

(ii) for $x, y \in S$, $x\Gamma y \subseteq I \implies x \in I$ or $y \in I$ (resp. $x\Gamma x \subseteq I \implies x \in I$).

(ii) for $x, y \in S$, $x\Gamma S\Gamma y \subseteq I \implies x \in I$ or $y \in I$(resp. $x\Gamma S\Gamma x \subseteq I \implies x \in I$).

3. Intuitionistic Fuzzy Ideal Extensions

3.1. Definition. Let S be a Γ-Semigroups, $A = (\mu_A, \upsilon_A)$ be an intuitionistic fuzzy subset of S and $x \in S$, then

$$< x,\ A > (y) = \{(y, < x, \mu_A > (y), < x, \upsilon_A > (y)) : x \in X\}$$

is the intuitionistic fuzzy subset of S.where the function $< x, \mu_A >: S \longrightarrow [0,1]$ and $< x, \upsilon_A >: S \longrightarrow [0,1]$ defined by $< x, \mu_A > (y) = \inf\limits_{\gamma\in\Gamma} \mu_A(x\gamma y)$ and $< x, \upsilon_A > (y) = \sup\limits_{\gamma\in\Gamma} \upsilon_A(x\gamma y)$is called the extension of A by x.

Example (a): Let S be the set of all non-positive integers and Γ be the set of all non-positive even integers. Then S is a Γ-Semigroups where $a\gamma b$ and $\alpha a\beta$ denote the usual multiplication of integers a, γ, b and α, a, β respectively with $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy subset of S, defined as follows

$$\mu_A(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0.1 & \text{if } x = -1, -2 \\ 0.2 & \text{if } x < -2 \end{cases}$$

and

$$\upsilon_A(x) = \begin{cases} 0 & \text{if } x = 0 \\ 0.7 & \text{if } x < 0 \end{cases}$$

Then the intuitionistic fuzzy subset $A = (\mu_A, \gamma_A)$ of S is an intuitionistic fuzzy ideal of S.

For $x = 0 \in S$, $< x, \mu_A > (y) = 1$ and $< x, \upsilon_A > (y) = 0$ $\forall y \in S$. For all other $x \in S$, $< x, \mu_A > (y) = 0.2$ and $< x, \upsilon_A > (y) = 0.7$ $\forall y \in S$.

Thus $< x, A >$ is an intuitionistic fuzzy ideal extension of A by x.

3.2. Proposition. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal of a commutative Γ-Semigroups S and $x \in S$. Then $< x, A >$ is an intuitionistic fuzzy ideal of S.

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy ideal of a commutative Γ-Semigroups S and $p, q \in S$, $\beta \in \Gamma$. Then

$$< x, \mu_A > (p\beta q) = \inf_{\gamma\in\Gamma} \mu_A(x\gamma p\beta q) \geq \inf_{\gamma\in\Gamma} \mu_A(x\gamma p) =< x, \mu_A > (p)$$

and

$$< x, \upsilon_A > (p\beta q) = \sup_{\gamma\in\Gamma}\upsilon_A(x\gamma p\beta q) \leq \sup_{\gamma\in\Gamma}\upsilon_A(x\gamma p) =< x, \upsilon_A > (p)$$

Thus $< x, A >$ is an intuitionistic fuzzy right ideal of S. Hence S being commutative $< x, A >$ is an intuitionistic fuzzy ideal of S. □

3.3. Remark. Commutativity of Γ-Semigroups S is not required to prove that $< x, A >$ is an intuitionistic fuzzy right ideal of S when $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy right ideal of S.

3.4. Proposition. Let S be a commutative Γ-Semigroups and $A = (\mu_A, \upsilon_A)$ be an intuitionistic fuzzy prime ideal of S. Then $< x, A >$ is intuitionistic fuzzy prime ideal of S for all $x \in S$.

Proof. Let $A = (\mu_A, \upsilon_A)$ be an intuitionistic fuzzy prime ideal of S. Then by Proposition 3.2, $< x, A >$ is an intuitionistic fuzzy ideal of S. Let $y, z \in S$. Then

$$\begin{aligned}
\inf_{\beta\in\Gamma} &< x, \mu_A > (y\beta z) = \inf_{\beta\in\Gamma}\inf_{\gamma\in\Gamma}\mu_A(x\gamma y\beta z) \quad by \quad 3.1 \\
&= \inf_{\beta\in\Gamma}\{\mu_A(x) \vee \mu_A(y\beta z)\} \quad by \quad 2.6 \\
&= \{\mu_A(x) \vee \inf_{\beta\in\Gamma}\mu_A(y\beta z)\} \\
&= \{\mu_A(x) \vee \{\mu_A(y) \vee \mu_A(z)\} \\
&= \{(\mu_A(x) \vee \mu_A(y)) \vee (\mu_A(x) \vee \mu_A(z))\} \\
&= \left\{\inf_{\ell\in\Gamma}\mu_A(x\ell y) \vee \inf_{\varepsilon\in\Gamma}\mu_A(x\varepsilon z)\right\} \\
&= < x, \mu_A > (y) \vee < x, \mu_A > (z)
\end{aligned}$$

and

$$\begin{aligned}
\sup_{\beta\in\Gamma} &< x, \upsilon_A > (y\beta z) = \sup_{\beta\in\Gamma}\sup_{\gamma\in\Gamma}\upsilon_A(x\gamma y\beta z) \quad by \quad 3.1 \\
&= \sup_{\beta\in\Gamma}\{\upsilon_A(x) \wedge \upsilon_A(y\beta z)\} \quad by \quad 2.6 \\
&= \{\upsilon_A(x) \wedge \sup_{\beta\in\Gamma}\upsilon_A(y\beta z)\} \\
&= \{\upsilon_A(x) \wedge \{\upsilon_A(y) \wedge \upsilon_A(z)\} \\
&= \{(\upsilon_A(x) \wedge \upsilon_A(y)) \wedge (\upsilon_A(x) \wedge \upsilon_A(z))\} \\
&= \left\{\sup_{\ell\in\Gamma}\upsilon_A(x\ell y) \wedge \sup_{\varepsilon\in\Gamma}\upsilon_A(x\varepsilon z)\right\} \\
&= < x, \upsilon_A > (y) \wedge < x, \upsilon_A > (z)
\end{aligned}$$

Hence by Definition 2.6, $< x, A >$ is an intuitionistic fuzzy prime ideal of S. □

3.5. Definition. Suppose S is a Γ-Semigroups and $A = (\mu_A, \upsilon_A)$ is an intuitionistic fuzzy subset of S. Then we define $supp\ \mu_A = \{x \in S : \mu_A(x) > 0\}$ and $inff\ \upsilon_A = \{x \in S : \upsilon_A(x) < 1\}$

3.6. Proposition. Let S be a Γ-Semigroups, $A = (\mu_A, \upsilon_A)$ be an intuitionistic fuzzy ideal of S and $x \in S$. Then we have the following:

(1) $A \subseteq < x, A >$.

(2) $< (x\alpha)^n x, A > \subseteq < (x\alpha)^{n+!} x, A > \forall \alpha \in \Gamma, \forall n \in N$.

(3)$If\ \ \mu_A(x) > 0$ and $\upsilon_A(x) < 1$ then $supp < x, \mu_A >= S$ and $inff < x, \upsilon_A >= S$.

Proof. (1) .Let $y \in S$. Then

$$< x, \mu_A > (y) = \inf_{\gamma \in \Gamma} \mu_A(x\gamma y) \geq \mu_A(y)$$

and

$$< x, \upsilon_A > (y) = \sup_{\gamma \in \Gamma} \upsilon_A(x\gamma y) \leq \upsilon_A(y)$$

(since A is an intuitionistic fuzzy ideal of S).Hence $A \subseteq < x, A >$.

(2) .In (2) we have to prove that

$$\begin{aligned} (\quad < \quad (x\alpha)^n x, \mu_A >) &\leq (< (x\alpha)^{n+!} x, \mu_A)\ and \\ (\quad < \quad (x\alpha)^n x, \upsilon_A >) &\geq (< (x\alpha)^{n+!} x, \upsilon_A >) \end{aligned}$$

Now

$$\begin{aligned} (< (x\alpha)^{n+1} x, \mu_A >)(y) &= \inf_{\gamma \in \Gamma} \mu_A \left((x\alpha)^{n+1} x\gamma y\right) \\ &= \inf_{\gamma \in \Gamma} \mu_A \left(x\alpha \left(x\alpha\right)^n x\gamma y\right) \\ &\geq \inf_{\gamma \in \Gamma} \mu_A \left(\left(x\alpha\right)^n x\gamma y\right) \\ &= < (x\alpha)^n x, \mu_A > (y) \end{aligned}$$

and

$$\begin{aligned} (< (x\alpha)^{n+1} x, \upsilon_A >)(y) &= \sup_{\gamma \in \Gamma} \upsilon_A \left((x\alpha)^{n+1} x\gamma y\right) \\ &= \sup_{\gamma \in \Gamma} \upsilon_A \left(x\alpha \left(x\alpha\right)^n x\gamma y\right) \\ &\leq \sup_{\gamma \in \Gamma} \upsilon_A \left(\left(x\alpha\right)^n x\gamma y\right) \\ &= < (x\alpha)^n x, \upsilon_A > (y) \end{aligned}$$

Hence $< (x\alpha)^n x, A > \subseteq < (x\alpha)^{n+!} x, A > \forall \alpha \in \Gamma, \forall n \in N$.

(3) .Since $< x, A >$ is an intuitionistic fuzzy subset of S, by definition, supp $< x, A > \subseteq S$.Let $y \in S$. Since A is an intuitionistic fuzzy ideal of S, we have,

$$< x, \mu_A > (y) = \inf_{\gamma \in \Gamma} \mu_A(x\gamma y) \geq \mu_A(x) > 0$$

and

$$< x, \upsilon_A > (y) = \sup_{\gamma \in \Gamma} \upsilon_A(x\gamma y) \leq \upsilon_A(x) < 1$$

Then $< x, \mu_A > (y) > 0$ and $< x, \upsilon_A > (y) < 1$.So $y \in supp < x, \mu_A >$ and $y \in inff < x, \upsilon_A >$. □

3.7. Remark. If we consider $(x\alpha)^0 x = x$ then (2) is also true for $n = 0$.

3.8. Definition. Suppose S is a Γ-Semigroups, $M \subseteq S$ and $x \in S$. We define $< x, M >= \{y \in S | x\Gamma y \subseteq M\}$, where $x\Gamma y := \{x\alpha y : \alpha \in \Gamma\}$.

3.9. Proposition. Let be a Γ-Semigroups and $\phi = M \subseteq S$. Then $< x, \Phi_M >= \Phi_{<x,M>}$ and $< x, \Psi_M >= \Psi_{<x,M>}$ for every $x \in S$, where (Φ_M, Ψ_M) denotes the characteristic function of M, where

$$\Phi_M(x) = \begin{cases} 1 \; if \; x \in M \\ 0 \; if \; x \notin M \end{cases} \quad , \quad \Psi_M(x) = \begin{cases} 0 \; if \; x \in M \\ 1 \; if \; x \notin M \end{cases}$$

Proof. Let $x, y \in S$.Then two cases may arise viz. Case (i) $y \in < x, M >$ .Case (ii) $y \notin < x, M >$.

Case (i) $y \in < x, M >$ .Then $x\Gamma y \subseteq M$. Hence $x\gamma y \in M$ $\forall \gamma \in \Gamma$. This means $\Phi_M(x\gamma y) = 1$ and $\Psi_M(x\gamma y) = 0$ $\forall \gamma \in \Gamma$. Hence $\inf\limits_{\gamma\in\Gamma} \Phi_M(x\gamma y) = 1$ and $\sup\limits_{\gamma\in\Gamma} \Psi_M(x\gamma y) = 0$ whence $< x, \Phi_M >= 1$ and $< x, \Psi_M >= 0$. Also $\Phi_{<x,M>} = 1$ and $\Psi_{<x,M>} = 0$.

Case (ii) $y \notin < x, M >$.Then there exists $\gamma \in \Gamma$ such that $x\gamma y \notin M$.So $\Phi_M(x\gamma y) = 0$ and $\Psi_M(x\gamma y) = 1$.Hence $\inf\limits_{\gamma\in\Gamma} \Phi_M(x\gamma y) = 0$ and $\sup\limits_{\gamma\in\Gamma} \Psi_M(x\gamma y) = 1$.Thus $< x, \Phi_M >= 0$ and $< x, \Psi_M >= 1$.Again $\Phi_{<x,M>} = 0$ and $\Psi_{<x,M>} = 1$.Thus we conclude $< x, \Phi_M >= \Phi_{<x,M>}$ and $< x, \Psi_M >= \Psi_{<x,M>}$. □

3.10. Proposition. Let S be a Γ-Semigroups and $A = (\mu_A, \upsilon_A)$ be a nonempty intuitionistic fuzzy subset of S. Then for any $t \in [0, 1]$, $< x, A_t >=< x, A >_t$for all $x \in S$.where A_t denotes $U(\mu_A : t)$ and $L(\upsilon_A : t)$.

Proof. Let $y \in < x, A >_t$ .This means $y \in U(< x, \mu_A >: t)$ and $y \in L(< x, \upsilon_A >: t)$.Then $< x, \mu_A > (y) \geq t$ and $< x, \upsilon_A > (y) \leq t$.Hence $\inf\limits_{\gamma\in\Gamma} \mu_A(x\gamma y) \geq t$ and $\sup\limits_{\gamma\in\Gamma} \upsilon_A(x\gamma y) \leq t$.This gives $\mu_A(x\gamma y) \geq t$ and $\upsilon_A(x\gamma y) \leq t$ for all $\gamma \in \Gamma$ and hence $x\gamma y \in U(\mu_A : t)$ and $x\gamma y \in L(\upsilon_A : t)$.for all $\gamma \in \Gamma$. Consequently,$y \in < x, U(\mu_A : t) >$ and $y \in < x, L(\mu_A : t) >$ .i.e $y \in < x, A_t >$ .It follows that $< x, A >_t \subseteq < x, A_t >$ .Reversing the above argument we can deduce that $< x, A_t > \subseteq < x, A >_t$ .Hence $< x, A_t >=< x, A >_t$. □

3.11. Proposition. Let S be a commutative Γ-Semigroups i.e., $a\alpha b = b\alpha a$ $\forall a, b \in S$, $\forall \alpha \in \Gamma$ and $A = (\mu_A, \upsilon_A)$ be an intuitionistic fuzzy subset of S such that $< x, A >= A$ for every $x \in S$.Then $A = (\mu_A, \upsilon_A)$ is a constant function.

Proof. Let $x, y \in S$. Then by hypothesis we have

$$\begin{aligned} \mu_A(x) &= < y, \mu_A > (x) \\ &= \inf_{\gamma\in\Gamma} \mu_A(y\gamma x) \\ &= \inf_{\gamma\in\Gamma} \mu_A(x\gamma y) \\ &= < x, \mu_A > (y) = \mu_A(y) \end{aligned}$$

and

$$\begin{aligned} \upsilon_A(x) &= < y, \upsilon_A > (x) \\ &= \sup_{\gamma\in\Gamma} \upsilon_A(y\gamma x) \\ &= \sup_{\gamma\in\Gamma} \upsilon_A(x\gamma y) \\ &= < x, \upsilon_A > (y) = \upsilon_A(y) \end{aligned}$$

Hence $A = (\mu_A, \upsilon_A)$ is a constant function. □

3.11.1. *Corollary.* Let S be a commutative Γ-Semigroups, $A = (\mu_A, \upsilon_A)$ be an intuitionistic fuzzy prime ideal of S. If $A = (\mu_A, \upsilon_A)$ is not constant, then $A = (\mu_A, \upsilon_A)$ is not a maximal intuitionistic fuzzy prime ideal of S.

Proof. Let $A = (\mu_A, \upsilon_A)$ be an intuitionistic fuzzy prime ideal of S. Then, by Proposition 3.4 for each $x \in S$, $< x, A >$ is an intuitionistic fuzzy prime ideal of S.Now by Proposition 3.6 (1) $A \subseteq < x, A >$ for all $x \in S$.If $< x, A > = A$ for all $x \in S$ then by Proposition 3.11 A is constant which is not the case by hypothesis.Hence there exists $x \in S$ such that $A \subset < x, A >$ .This completes the proof. □

3.12. Proposition. Let S be a commutative Γ-Semigroups. If $A = (\mu_A, \upsilon_A)$ is an intuitionistic fuzzy semiprime ideal of S, then $< x, A >$ is an intuitionistic fuzzy semiprime ideal of S for every $x \in S$.

Proof. Let $A = (\mu_A, \upsilon_A)$ be an intuitionistic fuzzy semiprime ideal of S and $x, y \in S$. Then $\inf\limits_{\gamma\in\Gamma} < x, \mu_A > (y\gamma y) = \inf\limits_{\gamma\in\Gamma}\inf\limits_{\delta\in\Gamma}\mu_A(x\delta y\gamma y) \leq \inf\limits_{\gamma\in\Gamma}\inf\limits_{\delta\in\Gamma}\mu_A(x\delta y\gamma y\delta x)$ (since A is an intuitionistic fuzzy ideal of S) $= \inf\limits_{\gamma\in\Gamma}\inf\limits_{\delta\in\Gamma}\mu_A(x\delta y\gamma x\delta y)$ (using commutativity of S and Definition 2.7) $=< x, \mu_A > (y)$. And $\sup\limits_{\gamma\in\Gamma} < x, \upsilon_A > (y\gamma y) = \sup\limits_{\gamma\in\Gamma}\sup\limits_{\delta\in\Gamma}\upsilon_A(x\delta y\gamma y) \geq \sup\limits_{\gamma\in\Gamma}\sup\limits_{\delta\in\Gamma}\upsilon_A(x\delta y\gamma y\delta x)$ (since A is an intuitionistic fuzzy ideal of S) $= \sup\limits_{\gamma\in\Gamma}\sup\limits_{\delta\in\Gamma}\upsilon_A(x\delta y\gamma x\delta y)$ (using commutativity of S and Definition 2.7) $=< x, \upsilon_A > (y)$.

Again by Proposition 3.2, $< x, A >$ is an intuitionistic fuzzy ideal of S. Consequently,$< x, A >$ is an intuitionistic fuzzy semiprime ideal of S for all $x \in S$. □

3.13. Corollary. Let S be a commutative Γ-Semigroups, $\{A_i\}_{i\in I}$ be a non-empty family of intuitionistic fuzzy semiprime ideals of S and let $A = (\mu_A, \upsilon_A) = \left(\inf \mu_{A_i}, \sup \upsilon_{A_i}\right)_{i\in I}$ .Then for any $x \in S$, $< x, A >$ is an intuitionistic fuzzy semiprime ideal of S.

Proof. Since each $A_i = (\mu_{A_i}, \upsilon_{A_i})$ $(i \in I)$ is an intuitionistic fuzzy ideal, $\mu_{A_i}(0) \neq 0$ and $\upsilon_{A_i}(0) \neq 1$ $\forall i \in I$ (Each $\mu_{A_i}$and υ_{A_i} are non-empty, so there exists $xi \in S$ such that $\mu_{A_i}(x_i) \neq 0$ and $\upsilon_{A_i}(x_i) \neq 1$ $\forall i \in I$. Also $\mu_{A_i}(0) = \mu_{A_i}(0\gamma x_i) \geq \mu_{A_i}(x_i)$ and $\upsilon_{A_i}(0) = \upsilon_{A_i}(0\gamma x_i) \leq \upsilon_{A_i}(x_i)$ $\forall i \in I$.Hence $\forall i \in I, \mu_{A_i}(0) \neq 0$ and $\upsilon_{A_i}(0) \neq 1$).Consequently, $\mu_A \neq 0$ and $\upsilon_A \neq 1$.Thus A is non-empty.Now let $x, y \in S$. Then

$$\begin{aligned}
\mu_A(x\gamma y) &= \inf\left\{\mu_{A_i} : i \in I\right\}(x\gamma y) \\
&= \inf\left\{\mu_{A_i}(x\gamma y) : i \in I\right\} \\
&\geq \inf\left\{\mu_{A_i}(x) : i \in I\right\} \\
&= \mu_A(x)
\end{aligned}$$

and

$$\begin{aligned} \upsilon_A(x\gamma y) &= \sup\{\upsilon_{A_i} : i \in I\}(x\gamma y) \\ &= \sup\{\upsilon_{A_i}(x\gamma y) : i \in I\} \\ &\leq \sup\{\upsilon_{A_i}(x) : i \in I\} \\ &= \upsilon_A(x) \end{aligned}$$

Hence S being a commutative Γ-Semigroups A is an intuitionistic fuzzy ideal of S.

Now if $a \in S$ then

$$\begin{aligned} \mu_A(a) &= \inf\left\{\mu_{A_i} : i \in I\right\}(a) \\ &= \inf\left\{\mu_{A_i}(a) : i \in I\right\} \\ &\geq \inf\left\{\inf_{\gamma\in\Gamma}\mu_{A_i}(a\gamma a) : i \in I\right\} \; cf.Definition 2.7 \\ &= \inf_{\gamma\in\Gamma}\{\inf\left\{\mu_{A_i}(a\gamma a) : i \in I\right\} \\ &= \inf_{\gamma\in\Gamma}\{\inf\left\{\mu_{A_i} : i \in I\right\}(a\gamma a) \\ &= \inf_{\gamma\in\Gamma}\mu_A(a\gamma a) \end{aligned}$$

and

$$\begin{aligned} \upsilon_A(a) &= \sup\{\upsilon_{A_i} : i \in I\}(a) \\ &= \sup\{\upsilon_{A_i}(a) : i \in I\} \\ &\leq \sup\left\{\sup_{\gamma\in\Gamma}\upsilon_{A_i}(a\gamma a) : i \in I\right\} \; cf.Definition 2.7 \\ &= \sup_{\gamma\in\Gamma}\{\sup\{\upsilon_{A_i}(a\gamma a) : i \in I\} \\ &= \sup_{\gamma\in\Gamma}\{\sup\{\upsilon_{A_i} : i \in I\}(a\gamma a) \\ &= \sup_{\gamma\in\Gamma}\upsilon_A(a\gamma a) \end{aligned}$$

This means, $A = (\mu_A, \upsilon_A)$ is an intuitionistic fuzzy semiprime ideal of S. Hence by Proposition 3.13, for any $x \in S < x, A >$ is an intuitionistic fuzzy semiprime ideal of S. □

3.14. Remark. The proof of the above Corollary shows that in a Γ-Semigroups intersection of arbitrary family of intuitionistic fuzzy semiprime ideals is an intuitionistic fuzzy semiprime ideal.

3.15. Corollary. Let S be a commutative Γ-Semigroups, $\{S_i\}_{i\in I}$ a non-empty family of semiprime ideals of S and $A := \cap_{i\in I} S_i \neq \phi$. Then $< x, X_A >$ is an intuitionistic fuzzy semiprime ideal of S for all $x \in S$ where $X_A = (\Phi_A, \Psi_A)$ is the characteristic function of A.

Proof. By supposition $A = \phi$. Then for any ideal P of S, $P\Gamma P \subseteq A$ implies that $P\Gamma P \subseteq S_i \; \forall i \in I$. Since each S_i is a semiprime ideal of S, $P \subseteq S_i \; \forall i \in I$ (cf. Definition 2.8). So $P \subseteq \cap_{i\in I} S_i = A$. Hence A is a semiprime ideal of S(cf.Definition 2.8). So the characteristic function $X_A = (\Phi_A, \Psi_A)$ of A is an intuitionistic fuzzy semiprime ideal of S(cf. Proposition 2.9). Hence by Proposition 3.13, $\forall x \in S$ $< x, X_A >$ is an intuitionistic fuzzy semiprime ideal of S.

Alternative Proof: $A := \cap_{i\in I}S_i \neq \phi$ (by the given condition). Hence $X_A \neq \phi$ i.e $\Phi_A \neq \phi$ *and* $\Psi_A \neq \phi$ .Let $x \in S$. Then $x \in A$ or $x \notin A$. If $x \in A$ then $\Phi_A(x) = 1$ and $\Psi_A(x) = 0$ and $x \in S_i$ $\forall i \in I$. Hence

$$\inf\{\Phi_{S_i} : i \in I\}(x) = \inf\{\Phi_{S_i}(x) : i \in I\} = 1 = \Phi_A(x)$$

and

$$\sup\{\Psi_{S_i} : i \in I\}(x) = \sup\{\Psi_{S_i}(x) : i \in I\} = 0 = \Psi_A(x)$$

If $x \notin A$ then $\Phi_A(x) = 0$ and $\Psi_A(x) = 1$and for some $i \in I$, $x \notin S_i$. It follows that $\Phi_{S_i}(x) = 0$ and $\Psi_{S_i}(x) = 1$.Hence

$$\inf\{\Phi_{S_i} : i \in I\}(x) = \inf\{\Phi_{S_i}(x) : i \in I\} = 0 = \Phi_A(x)$$

and

$$\sup\{\Psi_{S_i} : i \in I\}(x) = \sup\{\Psi_{S_i}(x) : i \in I\} = 1 = \Psi_A(x)$$

Thus we see that $\Phi_A = \inf\{\Phi_{S_i} : i \in I\}$ and $\Psi_A = \sup\{\Psi_{S_i} : i \in I\}$ .Again $X_{S_i} = (\Phi_{S_i}, \Psi_{S_i})$ is an intuitionistic fuzzy semiprime ideal of S for all $i \in I$ (cf.Definition 2.9) .Consequently by Corollary 3.14, for all $x \in S$,$< x, X_A >$ is an intuitionistic fuzzy semiprime ideal of S. □

3.16. Theorem. Let S be a Γ-Semigroups. If $A = (\mu_A, \upsilon_A)$ is an intuitionistic fuzzy prime ideal of S and $x \in S$ such that $A(x) = \left(\inf\limits_{y\in S}\mu_A(y), \sup\limits_{y\in S}\upsilon_A(y)\right)$,then $< x, A >= A$.Conversely, if $A = (\mu_A, \upsilon_A)$ is an intuitionistic fuzzy ideal of S such that $< y, A >= A$ $\forall y \in S$ with $A(y)$ not maximal in $A(S)$ then $A = (\mu_A, \upsilon_A)$ is prime.

Proof. Let $A = (\mu_A, \upsilon_A)$ be an intuitionistic fuzzy prime ideal of S and $x \in S$ be such that $\mu_A(x) = \inf\limits_{y\in S}\mu_A(y)$ and $\upsilon_A(x) = \sup\limits_{y\in S}\upsilon_A(y)$(it can be noted here that since each $\mu_A(y)$, $\upsilon_A(y) \in [0, 1]$, a closed and bounded subset of R, $\inf\limits_{y\in S}\mu_A(y)$ and $\sup\limits_{y\in S}\upsilon_A(y)$ exists).Let $z \in S$. Then $\mu_A(x) \leq \mu_A(z)$ and $\upsilon_A(x) \geq \upsilon_A(z)$. Hence

$$\{\mu_A(x) \vee \mu_A(z)\} = \mu_A(z)..........*$$

and

$$\{\upsilon_A(x) \wedge \upsilon_A(z)\} = \upsilon_A(z)..........*'$$

Now $< x, \mu_A > (z) = \inf\limits_{\gamma\in\Gamma}\mu_A(x\gamma z)$ and $< x, \upsilon_A > (z) = \sup\limits_{\gamma\in\Gamma}\upsilon_A(x\gamma z)$.Since $A = (\mu_A, \upsilon_A)$ is an intuitionistic fuzzy prime ideal of S,So $\inf\limits_{\gamma\in\Gamma}\mu_A(x\gamma z) = \{\mu_A(x) \vee \mu_A(z)\}$ and $\sup\limits_{\gamma\in\Gamma}\upsilon_A(x\gamma z) = \{\upsilon_A(x) \wedge \upsilon_A(z)\}$ .This implies $\inf\limits_{\gamma\in\Gamma}\mu_A(x\gamma z) = \mu_A(z)$ and $\sup\limits_{\gamma\in\Gamma}\upsilon_A(x\gamma z) = \upsilon_A(z)$ (using $*, *'$).Hence $< x, \mu_A > (z) = \mu_A(z)$ and $< x, \upsilon_A > (z) = \upsilon_A(z)$.Consequently,$< x, A >= A$.

Conversely, let $A = (\mu_A, \upsilon_A)$ be an intuitionistic fuzzy ideal of S such that $< y, A >= A \; \forall \; y \in S$ with $A(y)$ not maximal in $A(S)$ and let $x_1, x_2 \in S$. Then $A = (\mu_A, \upsilon_A)$ being an intuitionistic fuzzy ideal of S, $\mu_A(x_1\gamma x_2) \geq \mu_A(x_1)$,$\upsilon_A(x_1\gamma x_2) \leq \upsilon_A(x_1)$ and $\mu_A(x_1\gamma x_2) \geq \mu_A(x_2), \upsilon_A(x_1\gamma x_2) \leq \upsilon_A(x_2)$ $\forall\gamma \in \Gamma$. This leads to $\inf\limits_{\gamma\in\Gamma}\mu_A(x_1\gamma x_2) \geq \mu_A(x_1)$, $\sup\limits_{\gamma\in\Gamma}\upsilon_A(x_1\gamma x_2) \leq \upsilon_A(x_1)$..... $(**)$ and $\inf\limits_{\gamma\in\Gamma}\mu_A(x_1\gamma x_2) \geq \mu_A(x_2)$, $\sup\limits_{\gamma\in\Gamma}\upsilon_A(x_1\gamma x_2) \leq \upsilon_A(x_2)$...... $\left(**'\right)$ .Now two cases may arise viz. Case (i) Either $(\mu_A(x_1), \upsilon_A(x_1))$ or $(\mu_A(x_2), \upsilon_A(x_2))$ is maximal in $A(S)$. Case (ii) Neither

$(\mu_A(x_1), \upsilon_A(x_1))$ nor $(\mu_A(x_2), \upsilon_A(x_2))$ is maximal in $A(S)$.Case (i) Without loss of generality, let $(\mu_A(x_1), \upsilon_A(x_1))$ be maximal in $A(S)$.Then $\inf\limits_{\gamma\in\Gamma}\mu_A(x_1\gamma x_2) \leq \mu_A(x_1)$, $\sup\limits_{\gamma\in\Gamma}\upsilon_A(x_1\gamma x_2) \geq \upsilon_A(x_1)$.Consequently $\inf\limits_{\gamma\in\Gamma}\mu_A(x_1\gamma x_2) = \mu_A(x_1) = \{\mu_A(x_1) \vee \mu_A(x_2)\}$, $\sup\limits_{\gamma\in\Gamma}\upsilon_A(x_1\gamma x_2) = \upsilon_A(x_1) = \{\upsilon_A(x_1) \wedge \upsilon_A(x_2)\}$ .Case (ii) By the hypothesis $< x_1, A >= A$ i.e $< x_1, \mu_A >= \mu_A$ and $< x_1, \upsilon_A >= \upsilon_A$ also $< x_2, \mu_A >= \mu_A$ and $< x_2, \upsilon_A >= \upsilon_A$. Hence $< x_1, \mu_A > (x_2) = \mu_A (x_2)$ and $< x_1, \upsilon_A > (x_2) = \upsilon_A (x_2) \implies \inf\limits_{\gamma\in\Gamma}\mu_A(x_1\gamma x_2) = \mu_A (x_2)$ and $\sup\limits_{\gamma\in\Gamma}\upsilon_A(x_1\gamma x_2) = \upsilon_A (x_2)$ .So $\inf\limits_{\gamma\in\Gamma}\mu_A(x_1\gamma x_2) = \{\mu_A(x_1) \vee \mu_A(x_2)\}$ and $\sup\limits_{\gamma\in\Gamma}\upsilon_A(x_1\gamma x_2) = \{\upsilon_A(x_1) \wedge \upsilon_A(x_2)\}$ (using $(**)$, $\left(**'\right)$).Thus we conclude that $A = (\mu_A, \upsilon_A)$ is an intuitionistic fuzzy prime ideal of S. □

To end this paper we get the following characterization theorem of a prime ideal of a Γ-Semigroups which follows as a corollary to the above theorem.

3.17. Corollary. Let S be a Γ-Semigroups and I be an ideal of S. Then I is prime iff for $x \in S$ with $x \notin I$, $< x, X_I >= X_I$, where $X_I = (\Phi_I, \Psi_I)$ is the characteristic function of I.

Proof. Let I be a prime ideal of S. Then, by Proposition 2.9, $X_I = (\Phi_I, \Psi_I)$ is an intuitionistic fuzzy prime ideal of S. For $x \in S$ such that $x \notin I$, we have

$$\Phi_I (x) = 0 = \inf_{y\in S} \Phi_I (y)$$

and

$$\Psi_I (x) = 1 = \inf_{y\in S} \Psi_I (y)$$

Then by Theorem 3.17 $< x, X_I >= X_I$.

Conversely, let $< x, X_I >= X_I$ for all x in S with $x \notin I$,Let $y \in S$ be such that $X_I(y)$ is not maximal in $X_I(S)$. Then $\Phi_I(y) = 0$ and $\Psi_I(y) = 1$ so $y \notin I$. So $< y, X_I >= X_I$.So by the Theorem 3.17, X_I is an intuitionistic fuzzy prime ideal of S. So I is rime ideal of S(cf. Proposition 2.9). □

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