The classicality and quantumness of a quantum ensemble

Xuanmin Zhu[1,2], Shengshi Pang[2], Shengjun Wu[2], and Quanhui Liu[1]
[1]*School for Theoretical Physics, and Department of Applied Physics Hunan University, Changsha 410082, China*
[2]*Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*

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In this paper, we investigate the classicality and quantumness of a quantum ensemble. We define a quantity called classicality to characterize how classical a quantum ensemble is. An ensemble of commuting states that can be manipulated classically has a unit classicality, while a general ensemble has a classicality less than 1. We also study how quantum an ensemble is by defining a related quantity called quantumness. We find that the classicality of an ensemble is closely related to how perfectly the ensemble can be cloned, and that the quantumness of an ensemble is essentially responsible for the security of quantum key distribution(QKD) protocols using that ensemble. Furthermore, we show that the quantumness of an ensemble used in a QKD protocol is exactly the attainable lower bound of the error rate in the sifted key.

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Quantum theory has revealed many counterintuitive features of quantum systems in comparison with classical systems. Classical systems can be copied, deleted and distinguished with unit probability, while it is impossible to perfectly copy or delete an unknown quantum state [1–3] and non-orthogonal quantum states cannot be reliably distinguished [4]. The no-cloning theorem assures the security of quantum key distribution protocols [5] and prohibits superluminal communication[6]. Non-commuting observables described by quantum mechanics cannot be determined simultaneously, and a quantum measurement usually disturbs the quantum systems measured, in striking contrast to the fact that measurements can leave classical systems unperturbed in principle.

Some quantum ensembles can be manipulated like classical ones, whereas others can not. For example, an unknown state from an ensemble $\mathcal{E}_{ort}$ which is composed of orthogonal pure states could be cloned perfectly and determined without being disturbed; on the other hand a state from an ensemble $\mathcal{E}_{non}$ composed of non-orthogonal states cannot be cloned perfectly and determined exactly [7]. Perfect clonability and distinguishability are essential characteristics of the classical ensembles. Intuitively, the ensemble $\mathcal{E}_{ort}$ is more classical than $\mathcal{E}_{non}$, so the following questions naturally arise: what ensembles could be handled like classical ones and what could not? Is there a quantity to quantitatively measure how classical an ensemble is? There have already been some studies on the quantumness of quantum ensembles [8, 9]. In this paper, we study the classicality and quantumness of an ensemble from a different perspective. We start from considering how precisely an unknown state from the ensemble can be cloned and how stable it is under an appropriate measurement,i.e, how close the state after the measurement is to the original state. By classicality of an ensemble, we mean that how well it can be manipulated as a classical one.

Considering the classical cloning process is able to transform a classical state and a blank state into two copies of the original state, we can apply the process to clone orthogonal states of a quantum system. For an arbitrary unknown input state ρ, a quantum cloning process can be defined as the unitary transformation that assures $|j\rangle|0\rangle \to |j\rangle|j\rangle$, where $\{|j\rangle\}$ is a basis we choose for the Hilbert space of the input system and $|0\rangle$ is a blank state of an ancillary system. This cloning strategy was first introduced in [1], and we call it a classical cloning strategy as it is the quantum counterpart of the classical cloning process.

Obviously, this classical cloning strategy is neither perfect nor optimum for cloning an unknown quantum state. The copies produced are generally different from the original state, so it is useful to quantify the distance between a copy and the original state. How to measure the distance is investigated intensively and many proposals have been put forward [4, 10]. One distance measure is the relative entropy [9, 10], which has been used to quantify entanglement and correlations [11, 12]. However, the relative entropy is not a genuine metric as it is not symmetric. Two other widely used distance measures, the trace distance and the fidelity [4] are well defined because both of them are symmetric and have many other properties of good distance measures. In this paper, we use fidelity as the distance measure. The fidelity of ρ and σ is defined as [13]

$$F(\rho, \sigma) = (\mathrm{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}})^2. \qquad (1)$$

(The square root of the above quantity is also frequently defined as the fidelity [4], but we adopt Eq. (1) as the fidelity definition throughout this paper.) It is obvious that $0 \le F(\rho, \sigma) \le 1$ and $F(\rho, \sigma) = 1$ if and only if $\rho = \sigma$.

For the quantum cloning process with the classical strategy, an arbitrary input state can be written in the basis $\{|j\rangle\}$ as $\rho = \sum_{ij} \rho_{ij}|i\rangle\langle j|$, and the output bipartite state is given by $\sum_{ij} \rho_{ij}|ii\rangle\langle jj|$. Therefore, the reduced density matrix of either subsystem is obtained as $\rho' = \sum_j \rho_{jj}|j\rangle\langle j| = \sum_j \langle j|\rho|j\rangle|j\rangle\langle j|$, and the fidelity between the input state and one output state is given by

$F(\rho, \rho')$.

For an ensemble denoted by $\mathcal{E} = \{q_i, \rho_i\}$, we investigate its classicality by studying how well an unknown state from the ensemble can be cloned by the classical cloning strategy with respect to a basis $\{|j\rangle\}$. First, we define the average cloning fidelity for the ensemble $\mathcal{E}$ as

$$F_{ave}(\mathcal{E}, \{|j\rangle\}) = \sum_i q_i F(\rho_i, \rho_i'), \tag{2}$$

where $\rho_i' = \sum_j \langle j|\rho_i|j\rangle |j\rangle\langle j|$. For an ensemble composed of orthogonal pure states, an average cloning fidelity of 1 could be reached if the orthogonal states are chosen as the basis $\{|j\rangle\}$. For a general quantum ensemble, it can be seen that $F_{ave}(\mathcal{E}) \leq 1$ as $F(\rho_i, \rho_i') \leq 1$ and $\sum_i q_i = 1$. The average cloning fidelity F_{ave} represents the performance of a classical copying strategy on a quantum ensemble; meanwhile, F_{ave} can also represent stability of the states in an ensemble under a projective measurement, since $\rho_i' = \sum_j \langle j|\rho_i|j\rangle|j\rangle\langle j|$ is the density matrix after a projective measurement on ρ_i along a basis $\{|j\rangle\}$. In some sense, the average cloning fidelity represents how classical the ensemble is. Therefore, we can define the classicality J of the quantum ensemble $\mathcal{E} = \{q_i, \rho_i\}$ as

$$J(\mathcal{E}) = \max_{\{|j\rangle\}}\{F_{ave}(\mathcal{E}, |j\rangle)\} = \max_{\{|j\rangle\}}\{\sum_i q_i F(\rho_i, \rho_i')\}, \tag{3}$$

where $\{|j\rangle\}$ is an orthonormal basis of the subspace spanned by the states in the ensemble. For an infinite quantum ensemble $\mathcal{E} = \{f(\alpha), \rho(\alpha)\}$, the classicality is similarly defined as

$$J(\mathcal{E}) = \max_{\{|j\rangle\}}\{\int f(\alpha) F(\rho(\alpha)_i, \rho(\alpha)_i') \mathrm{d}\alpha\}, \tag{4}$$

where $\rho(\alpha)' = \sum_j \langle j|\rho(\alpha)|j\rangle|j\rangle\langle j|$ and $f(\alpha)$ is the probability distribution function satisfying $\int f(\alpha)\mathrm{d}\alpha = 1$. It can be seen that the classicality J defined above is an intrinsic property of the ensemble, independent of the cloning basis. It is evident that $\mathcal{E}$ can be manipulated as a pure classical ensemble only when $J(\mathcal{E}) = 1$.

A single state ρ can be considered as an ensemble consisting of just one state, and it can be shown that the classicality J of a single-state ensemble ρ is equal to one. Since the cloning basis states could be chosen as the eigenstates of ρ, then $\rho = \rho'$, and thus $J = F(\rho, \rho') = 1$. Therefore, the classicality of a single-state ensemble is equal to one. In the following theorem, we give the range of J.

Theorem 1. $1/d \leq J(\mathcal{E}) \leq 1$, *where d is the dimension of the Hilbert space. $J(\mathcal{E}) = 1$ if and only if all quantum states in the ensemble commute with each other.*

Proof. We only give the proof for finite ensembles as that for infinite ensembles is similar. For a finite ensemble $\mathcal{E} = \{q_i, \rho_i\}$, from the definition we have $J(\mathcal{E}) = \max_{\{|j\rangle\}}\{\sum_i q_i F(\rho_i, \rho_i')\} \leq \sum_i q_i = 1$ due to the property $F(\rho_i, \rho_i') \leq 1$. Suppose $\{|j^*\rangle\}$ is the basis that maximizes F_{ave} for the ensemble $\mathcal{E}$, i.e., $J(\mathcal{E}) = \sum_i q_i F(\rho_i, \rho_i')$, where $\rho_i' = \sum_{j^*} \langle j^*|\rho_i|j^*\rangle|j^*\rangle\langle j^*|$. The fidelity satisfies the inequality $F(\rho, \rho') \geq \mathrm{tr}\rho\rho'$ [13], then $J(\mathcal{E}) \geq \sum_i q_i \mathrm{tr}\rho_i\rho_i' = \sum_i q_i \mathrm{tr}\rho_i'^2$. If the dimension of the Hilbert space spanned by the states in the ensemble is d, we have $\mathrm{tr}\rho_i'^2 = \sum_{j=1}^d \rho_{jj}^2 \geq (\sum_{j=1}^d \rho_{jj})^2/d = 1/d$. Then $J(\mathcal{E}) \geq \sum_i q_i/d = 1/d$, and we get $1/d \leq J(\mathcal{E}) \leq 1$. Now we prove that $J(\mathcal{E}) = 1$ if and only if all quantum states in the ensemble are mutually commutative. If $J(\mathcal{E}) = 1$, then for each i, $F(\rho_i, \rho_i') = 1$ and thus $\rho_i = \rho_i' = \sum_j \langle j^*|\rho_i|j^*\rangle|j^*\rangle\langle j^*|$. So all the states are diagonal in the same basis $\{|j^*\rangle\}$, and they commute with each other. On the other hand, if all the states in the ensemble commute with each other, all of them can be diagonalized simultaneously. So there exists a basis in which all the states are diagonal and we can use this basis in the classical cloning strategy, then $\rho_i' = \rho_i$ and each $F(\rho_i, \rho_i') = 1$, so we get $J(\mathcal{E}) = 1$. □

The classicality J of an ensemble $\{q_i, \rho_i\}$ quantifies how well states from the ensemble can be cloned by a classical strategy, thus gives a measure of how classical the ensemble is. From another perspective, the classicality J of an ensemble $\{q_i, \rho_i\}$ also tells us to what extent the states from the ensemble commute. The classicality of an ensemble of mutually commuting states is equal to 1, this is in accordance with the fact that commuting states could be broadcasted [14].

Theorem 2. *For the ensembles $\mathcal{E}_A = \{q_i, \rho_{iA}\}$, $\mathcal{E}_B = \{q_j, \rho_{jB}\}$, and $\mathcal{E}_{AB} = \{q_i q_j, \rho_{iA} \otimes \rho_{jB}\}$, there is an inequality*

$$J(\mathcal{E}_{AB}) \geq J(\mathcal{E}_A) J(\mathcal{E}_B); \tag{5}$$

for the infinite ensembles $\mathcal{E}_A = \{f(\alpha), \rho_A(\alpha)\}$, $\mathcal{E}_B = \{f(\beta), \rho_B(\beta)\}$, and $\mathcal{E}_{AB} = \{f(\alpha)f(\beta), \rho_A(\alpha) \otimes \rho_B(\beta)\}$, the inequality (5) *is also valid.*

Proof. Assume that $\{|k\rangle\}$ and $\{|m\rangle\}$ are the bases of the systems A and B which maximize $\sum_i q_i F(\rho_{iA}, \rho_{iA}')$ and $\sum_j q_j F(\rho_{jB}, \rho_{jB}')$ respectively. Then $J(\mathcal{E}_A) = \sum_i q_i F(\rho_{iA}, \rho_{iA}')$, $J(\mathcal{E}_B) = \sum_j q_j F(\rho_{jB}, \rho_{jB}')$, where $\rho_{iA}' = \sum_k \langle k|\rho_{iA}|k\rangle|k\rangle\langle k|$ and $\rho_{jB}' = \sum_m \langle m|\rho_{jB}|m\rangle|m\rangle\langle m|$. The basis $\{|k\rangle \otimes |m\rangle\}$ may not be optimal for $\mathcal{E}_{AB}$, and from the definition we can get

$$\begin{aligned} J(\mathcal{E}_{AB}) &= \max_{\{|l\rangle^{AB}\}}\{F_{ave}(\mathcal{E}_{AB}, \{|l\rangle^{AB}\})\} \\ &\geq F_{ave}(\mathcal{E}_{AB}, |k\rangle \otimes |m\rangle) \\ &= \sum_{ij} q_i q_j F(\rho_{iA} \otimes \rho_{jB}, \rho_{iA}' \otimes \rho_{jB}') \\ &= \sum_{ij} q_i q_j F(\rho_{iA}, \rho_{iA}') F(\rho_{jB}, \rho_{jB}') \\ &= J(\mathcal{E}_A) J(\mathcal{E}_B). \end{aligned} \tag{6}$$

The proof for the infinite ensembles is similar. □

Above we have proved $J(\mathcal{E}_{AB}) \geq J(\mathcal{E}_A)J(\mathcal{E}_B)$ in theorem 2, but so far we have not found any example for which $J(\mathcal{E}_{AB}) > J(\mathcal{E}_A)J(\mathcal{E}_B)$, so we conjecture that $J(\mathcal{E}_{AB}) = J(\mathcal{E}_A)J(\mathcal{E}_B)$ holds true for all ensembles $\mathcal{E}_A$, $\mathcal{E}_B$, $\mathcal{E}_{AB}$ defined in Theorem 2.

Now, we show that J is invariant under unitary operations. For a finite ensemble $\mathcal{E} = \{q_i, \rho_i\}$, after a unitary operation U, the classicality of the new ensemble is given as

$$\begin{aligned} J(U\mathcal{E}U^\dagger) &= \max_{\{|j\rangle\}}\{F_{ave}(U\mathcal{E}U^\dagger, |j\rangle)\} \\ &= \max_{\{U|j\rangle\}}\{F_{ave}(U\mathcal{E}U^\dagger, U|j\rangle)\} \\ &= \max_{\{U|j\rangle\}}\{\sum_i q_i F(U\rho_i U^\dagger, \sum_j \langle j|\rho_i|j\rangle U|j\rangle\langle j|U^\dagger)\} \\ &= \max_{\{|j\rangle\}}\{\sum_i q_i F(\rho_i, \rho_i')\} \\ &= J(\mathcal{E}). \end{aligned} \tag{7}$$

It is obvious that the above equation is also valid for infinite ensembles. Therefore, an ensemble $\mathcal{E}_0$ can be transformed to another ensemble $\mathcal{E}_1$ by a unitary operation only if they have the same classicality, i.e., $J(\mathcal{E}_0) = J(\mathcal{E}_1)$.

From our intuition, we have another interesting conjecture: for an arbitrary ensemble $\{p_i, \rho_i\}$ and a standard state $|0\rangle\langle 0|$, there is an inequality $J(\{p_i, \rho_i \otimes |0\rangle\langle 0|\}) \geq J(\{p_i, \rho_i \otimes \rho_i\})$, with equality if and only if all ρ_i are commuting. Using this conjecture and the property that classicality J is unitarily invariant, we can obtain the non-cloning and no-deleting theorems straightforwardly.

Next, we turn to study an opposite property of an ensemble. Let us define the quantumness Q of an ensemble as

$$Q(\mathcal{E}) = 1 - J(\mathcal{E}) = \min_{\{|j\rangle\}}\{\sum_i q_i(1 - F(\rho_i, \rho_i'))\}. \tag{8}$$

The function Q has similar properties to those of J, and $0 \leq Q \leq (d-1)/d$. It can be seen that the quantumness of a single-state ensemble is equal to zero. The quantumness of an ensemble tells us how much the ensemble is distinct from a pure classical ensemble, and we shall see that the quantumness of an ensemble used for quantum key distribution (QKD) is precisely the attainable lower bound of the error rate.

In the quantum key distribution theory, the error rate is the rate of errors caused by eavesdroppers [15, 16]. Legitimate users can use it to judge whether there exist eavesdroppers. Now we study the relation between the quantumness of the ensemble used in a QKD protocol and the error rate under the intercept-resend eavesdropping strategies [16].

Theorem 3. *The quantumness of the ensemble used in a general QKD protocol is the attainable lower bound of the error rate under the intercept-resend eavesdropping strategy.*

Proof. In a general QKD protocol, Alice sends a state $|\psi_i\rangle$ (*raw* key) to Bob with a probability q_i, and the ensemble used is $\{q_i, |\psi_i\rangle\}$. When Bob's measurement basis is different from Alice's sending basis, the state Bob received is discarded, and when their bases are the same, the received state is reserved and it is called a *sifted* key. The error rate is the average probability that Bob's measurement gives a result different from the state Alice sends after the raw keys are sifted. In the intercept-resend strategy, the eavesdropper Eve intercepts a state from Alice, say $|\psi_i\rangle$, then performs a projective measurement along the basis $\{|j\rangle\}$ and gets an output $|j\rangle$ with the probability $|\langle j|\psi_i\rangle|^2$, and finally resends the output state to Bob. When Bob's measurement basis is in accordance with Alice's sending basis, the probability that Bob gets the original state $|\psi_i\rangle$ is $P = \sum_j |\langle j|\psi_i\rangle|^4 = F(|\psi_i\rangle\langle\psi_i|, \rho_i')$, where $\rho_i' = \sum_j |\langle j|\psi_i\rangle|^2 |j\rangle\langle j|$. Thus the error rate in this strategy is $R = \sum_i q_i(1 - F(|\psi_i\rangle\langle\psi_i|, \rho_i'))$. The quantumness of the ensemble $\{q_i, |\psi_i\rangle\}$ is $Q = \min_{\{|j\rangle\}}\{\sum_i q_i(1 - F(|\psi_i\rangle\langle\psi_i|, \rho_i'))\} \leq R$, therefore, the quantumness Q is the attainable lower bound of the error rate of a general QKD protocol. □

It is obvious that the ensembles whose quantumness is zero or very small are not suitable for QKD protocols, since the eavesdroppers can get the information of the keys without being detected, so we can say that the quantumness of an ensemble is responsible for the security of QKD protocol. Besides the famous BB84 QKD protocol [5], there is an important six-state protocol [17, 18]. The error rates for BB84 and six-state protocols are 1/4 and 1/3 respectively [15]. By simple calculation, we can get that the quantumness of the two ensembles used in these two QKD protocols are 1/4 and 1/3 which are equal to their error rates.

The ensemble $\mathcal{E}_{bloch} = \{1/4\pi, \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\varphi}|1\rangle\}$ consisting of pure states uniformly distributed on the Bloch sphere is an infinite ensemble, where $\theta \in [0, \pi)$ and $\varphi \in [0, 2\pi)$. A general basis for the two dimensional Hilbert space can be given as: $|e_1\rangle = \cos(\theta_1/2)|0\rangle + \sin(\theta_1/2)e^{i\varphi_1}|1\rangle$ and $|e_2\rangle = \sin(\theta_1/2)|0\rangle - \cos(\theta_1/2)e^{i\varphi_1}|1\rangle$. The average cloning fidelity of this ensemble is $F_{ave} = 2/3$ which is independent of the basis used in the classical cloning process, so its classicality is $J(\mathcal{E}_{bloch}) = 2/3$. For a fixed θ, we define a symmetric double-circle ensemble: $\mathcal{E}(\theta) = \{1/4\pi, \cos(\theta/2)|0\rangle \pm \sin(\theta/2)e^{i\varphi}|1\rangle\}$, where $\varphi \in [0, 2\pi)$. The states in the ensemble $\mathcal{E}(\theta)$ lie on two symmetric latitudinal circles of the Bloch sphere with polar angles $\pm\theta$. The average cloning fidelity of this ensemble is $F_{ave}(\theta, \theta_1, \varphi_1) = 1 - \sin^2\theta/2 + \sin^2\theta_1(3\sin^2\theta - 2)/4$. According to the definition of classicality, we have

$$\begin{aligned} J(\theta) &= \max_{\{\theta_1, \varphi_1\}}\{F_{ave}(\theta, \theta_1, \varphi_1)\} \\ &= \begin{cases} 1 - \frac{1}{2}\sin^2\theta & \text{if } 0 \leq \sin\theta \leq \sqrt{2/3} \\ \frac{1}{2} + \frac{1}{4}\sin^2\theta & \text{if } \sqrt{2/3} < \sin\theta \leq 1 \end{cases} \end{aligned} \tag{9}$$

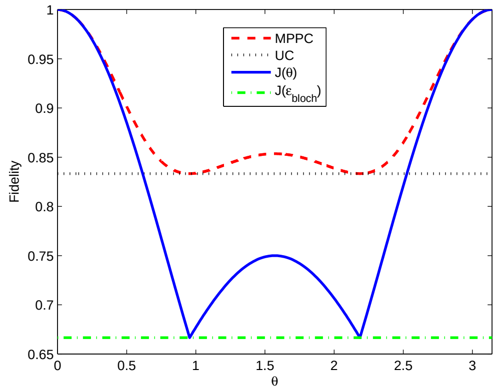

Figure 1. (Color online)The θ dependence of the classicalities and the cloning fidelilties for $\mathcal{E}(\theta)$ and $\mathcal{E}_{bloch}$: $J(\theta)$ (solid), $J(\mathcal{E}_{bloch})$ (dashdot), the MPPC (dashed), and the UC (dotted).

It can be seen that when $\theta = \arcsin(\sqrt{2/3})$ or $\pi - \arcsin(\sqrt{2/3})$, $J(\theta)$ reaches its minimal value 2/3 which is equal to the classicality of $\mathcal{E}_{bloch}$. When $\theta = \pi/2$, the states of the ensemble are equiprobably on the $x - y$ equator, and the classicality of this ensemble is 3/4.

An unknown state cannot be perfectly cloned, but can be approximately cloned. The approximate cloning theories have been established and developed very well [6, 19–21]. In Fig. 1, the classicality $J(\theta)$ is depicted, together with the classicality $J(\mathcal{E}_{bloch})$, the fidelity of the optimal mirror phase-covariant cloning (MPCC) [21] and the fidelity of universal cloning (UC) [19]. From Fig. 1, we can see that at the points $\theta = \arcsin(\sqrt{2/3})$ and $\theta = \pi - \arcsin(\sqrt{2/3})$, both the MPCC fidelity $F(\theta)$ and the $J(\theta)$ reach the minimal value 5/6 (the same as the UC fidelity [19]) and 2/3 (the same as $J(\mathcal{E}_{bloch})$) respectively. Roughly speaking, Fig. 1 shows that the more classical an ensemble is, the more perfectly the states in it can be cloned. The classicalities of the ensembles used in the BB84 protocol and that in the six-state protocol are the same as $J(\pi/2)$ and $J(\mathcal{E}_{bloch})$ respectively; it is more interesting to note that the cloning strategies for the BB84 ensemble and the six-state ensemble are equivalent to the strategies for the phase-covariant cloning and the universal cloning respectively [6]. However, it must be pointed out that the optimal cloning fidelities of two ensembles could be different, even if the classicality of them are the same.

In conclusion, we have constructed a quantity J to measure the classicality of a given ensemble. The quantity J can tell how classical the ensemble is. When $J = 1$ the ensemble behaves like a purely classical ensemble; and when $J < 1$ the ensemble cannot be considered as a classical ensemble anymore. We have revealed that the more classical an ensemble is, the better an unknown state from the ensemble can be cloned. The quantity of classicality provides us with a tool to evaluate how well classical tasks such as cloning, deleting, and distinguishing could be accomplished for quantum ensembles. We also define the quantumness of an ensemble and we surprisingly find that the quantumness of an ensemble used in quantum key distribution is exactly the attainable lower bound of error rate. Our work may be useful for further investigation of classical and quantum features of an ensemble and it provides a quantitative framework for various tasks in quantum communication.

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