Equivariant multiplicities of Coxeter arrangements and invariant bases

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November 2, 2010

Abstract

Let $\mathcal{A}$ be an irreducible Coxeter arrangement and W be its Coxeter group. Then W naturally acts on $\mathcal{A}$. A multiplicity $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$ is said to be equivariant when $\mathbf{m}$ is constant on each W-orbit of $\mathcal{A}$. In this article, we prove that the multi-derivation module $D(\mathcal{A}, \mathbf{m})$ is a free module whenever $\mathbf{m}$ is equivariant by explicitly constructing a basis, which generalizes the main theorem of [T2002]. The main tool is a primitive derivation and its covariant derivative. Moreover, we show that the W-invariant part $D(\mathcal{A}, \mathbf{m})^W$ for any multiplicity $\mathbf{m}$ is a free module over the W-invariant subring.

1 Introduction

Let V be an ℓ-dimensional Euclidean space with an inner product $I : V \times V \to \mathbb{R}$. Let S denote the symmetric algebra of the dual space V^* and F be its quotient field. Let Der_S be the S-module of $\mathbb{R}$-linear derivations from S to itself. Let Ω^1_S be the S-module of regular 1-forms. Similarly define Der_F and Ω^1_F over F. The dual inner product $I^* : V^* \times V^* \to \mathbb{R}$ naturally induces an F-bilinear form $I^* : \Omega^1_F \times \Omega^1_F \to F$. Then one has an F-linear bijection

$$I^* : \Omega^1_F \to \mathrm{Der}_F$$

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defined by $[I^*(\omega)](f) := I^*(\omega, df)$ for $f \in F$.

Let $\mathcal{A}$ be an irreducible Coxeter arrangement with its Coxeter group W. For each $H \in \mathcal{A}$, choose $\alpha_H \in V^*$ with $H = \ker(\alpha_H)$. Let $Q = \prod_{H\in\mathcal{A}} \alpha_H \in S$. Recall the S-module of logarithmic forms

$$\Omega^1(\mathcal{A}, \infty) = \{\omega \in \Omega^1_F \mid Q^N\omega \text{ and } (Q/\alpha_H)^N\omega \wedge d\alpha_H \text{ are both regular for any } H \in \mathcal{A} \text{ and } N \gg 0\}$$

and the S-module of logarithmic derivations

$$D(\mathcal{A}, -\infty) = I^*(\Omega^1(\mathcal{A}, \infty))$$

from [AT2010]. A map $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$ is called a multiplicity. For an arbitrary multiplicity, let

$$\begin{aligned} D(\mathcal{A}, \mathbf{m}) &= \{\theta \in D(\mathcal{A}, -\infty) \mid \theta(\alpha_H) \in \alpha_H^{\mathbf{m}(H)} S_{(\alpha_H)} \text{ for all } H \in \mathcal{A}\}, \\ \Omega^1(\mathcal{A}, \mathbf{m}) &= (I^*)^{-1} D(\mathcal{A}, -\mathbf{m}), \end{aligned}$$

where $S_{(\alpha_H)}$ is the localization of S at the prime ideal (α_H). These two modules were introduced in [Sa1980] (when $\mathbf{m}$ is constantly equal to one), in [Z1989] (when $\mathrm{im}(\mathbf{m}) \subset \mathbb{Z}_{>0}$), and in [A2008, AT2010, AT2009] (when $\mathbf{m}$ is arbitrary). A derivation $0 \neq \theta \in \mathrm{Der}_F$ is said to be **homogeneous of degree** d, or $\deg\theta = d$, if $\theta(\alpha) \in F$ is homogeneous of degree d whenever $\theta(\alpha) \neq 0$ ($\alpha \in V^*$). A multiarrangement $(\mathcal{A}, \mathbf{m})$ is called to be **free** with **exponents** $\exp(\mathcal{A}, \mathbf{m}) = (d_1, \ldots, d_\ell)$ if $D(\mathcal{A}, \mathbf{m}) = \oplus_{i=1}^{\ell} S \cdot \theta_i$ with a homogeneous basis θ_i such that $\deg(\theta_i) = d_i$ $(i = 1, \ldots, \ell)$. A multiplicity $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$ is said to be **equivariant** when $\mathbf{m}(H) = \mathbf{m}(wH)$ for any $H \in \mathcal{A}$ and any $w \in W$, i.e., $\mathbf{m}$ is constant on each orbit. In this article we prove

Theorem 1.1
For any irreducible Coxeter arrangement $\mathcal{A}$ and any equivariant multiplicity $\mathbf{m}$, the multiarrangement $(\mathcal{A}, \mathbf{m})$ is free.

For a fixed arrangement $\mathcal{A}$, we say that a multiplicity $\mathbf{m}$ is **free** if $(\mathcal{A}, \mathbf{m})$ is free. Although we have a limited knowledge about the set of all free multiplicities for a fixed irreducible Coxeter arrangement $\mathcal{A}$, it is known that there exist infinitely many non-free multiplicities unless $\mathcal{A}$ is either one- or two-dimensional [ATY2009]. Theorem 1.1 claims that any equivariant multiplicity is free for any irreducible Coxeter arrangement.

When the W-action on $\mathcal{A}$ is transitive, an equivariant multiplicity is constant and a basis was constructed in [SoT1998, T2002, AY2007, AT2010]. So we may assume, in order to prove Theorem 1.1, that the W-action on $\mathcal{A}$ is not transitive. In other words, we may only study the cases when $\mathcal{A}$ is of the type either B_ℓ, F_4, G_2 or $I_2(2n)$ $(n \geq 4)$. In these cases, $\mathcal{A}$ has exactly two W-orbits: $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. The orbit decompositions are explicitly given by: $B_\ell = A_1^\ell \cup D_\ell$, $F_4 = D_4 \cup D_4$, $G_2 = A_2 \cup A_2$ or $I_2(2n) = I_2(n) \cup I_2(n)$ $(n \geq 4)$. Note that A_1^ℓ is not irreducible.

When $\mathcal{A}$ is irreducible, the **primitive derivations** play the central role to define the Hodge filtration introduced by K. Saito. (See [Sa2003] for example.) For $R := S^W$, let D be an element of the lowest degree in Der_R, which is called a primitive derivation corresponding to $\mathcal{A}$. Then D is unique up to a nonzero constant multiple. A theory of primitive derivations in the case of non-irreducible Coxeter arrangements was introduced in [AT2009]. Thus we may consider a primitive derivation D_i corresponding with the orbit $\mathcal{A}_i$ $(1 \leq i \leq 2)$. We only use D_1 because of symmetricity. Note that D_1 is not unique up to a nonzero multiple when $\mathcal{A}_1 = A_1^\ell$ (non-irreducible). Denote the reflection groups of $\mathcal{A}_i$ by W_i $(i = 1, 2)$. The Coxeter arrangements B_ℓ, F_4, G_2 and $I_2(2n)$ $(n \geq 4)$ are classified into two cases, that is, (1) the primitive derivation D_1 can be chosen to be W-invariant for B_ℓ and F_4 (the first case) while (2) D_1 is W_2-antiinvariant for G_2 and $I_2(2n)$ $(n \geq 4)$ (the second case) as we will see in Section 4. Since the second cases are two-dimensional, Theorem 1.1 holds true. Thus the first case is the only remaining case to prove Theorem 1.1.

Let

$$\begin{array}{rcl} \nabla : \mathrm{Der}_F \times \mathrm{Der}_F & \longrightarrow & \mathrm{Der}_F \\ (\theta, \delta) & \mapsto & \nabla_\theta \delta \end{array}$$

be the **Levi-Civita connection** with respect to the inner product I on V. We need the following theorem for our proof of Theorem 1.1:

Theorem 1.2
([AT2010, AT2009]) Let $D(\mathcal{A}, -\infty)^W$ be the W-invariant part of $D(\mathcal{A}, -\infty)$. Then

$$\nabla_D : D(\mathcal{A}, -\infty)^W \xrightarrow{\sim} D(\mathcal{A}, -\infty)^W$$

is a T-linear automorphism where $T := \{f \in R \mid Df = 0\}$. When $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is the orbit decomposition,

$$\nabla_{D_1} : D(\mathcal{A}_1, -\infty)^{W_1} \xrightarrow{\sim} D(\mathcal{A}_1, -\infty)^{W_1}$$

is a T_1-linear automorphism where

$$R_1 := S^{W_1}, \ T_1 := \{f \in R_1 \mid D_1 f = 0\}.$$

Let E be the **Euler derivation** characterized by the equality $E(\alpha) = \alpha$ for every $\alpha \in V^*$. Suppose that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is the orbit decomposition and that the primitive derivation D_1 is W-invariant. Define

$$E^{(p,q)} := \nabla_D^{-q} \nabla_{D_1}^{q-p} E$$

for $p, q \in \mathbb{Z}$. Here, thanks to Theorem 1.2, we may interpret $\nabla_D^m = (\nabla_D^{-1})^{-m}$ and $\nabla_{D_1}^m = (\nabla_{D_1}^{-1})^{-m}$ when m is negative. Denote the equivariant multiplicity $\mathbf{m}$ by (m_1, m_2) when $\mathbf{m}(H) = m_1$ $(H \in \mathcal{A}_1)$ and $\mathbf{m}(H) = m_2$ $(H \in \mathcal{A}_2)$. Let $x_1, \dots, x_\ell$ be a basis for V^* and $P_1, \dots, P_\ell$ be a set of basic invariants with respect to W: $R = \mathbb{R}[P_1, \dots, P_\ell]$. Let $P_1^{(i)}, \dots, P_\ell^{(i)}$ be a set of basic invariants with respect to W_i: $R_i = \mathbb{R}[P_1^{(i)}, \dots, P_\ell^{(i)}]$ $(i = 1, 2)$. We use the notation

$$\partial_{x_j} := \partial/\partial x_j, \ \partial_{P_j} := \partial/\partial P_j, \ \partial_{P_j^{(i)}} := \partial/\partial P_j^{(i)} \quad (1 \le j \le \ell, 1 \le i \le 2).$$

The following theorem gives an explicit construction of a basis:

Theorem 1.3
Let $\mathcal{A}$ be an irreducible Coxeter arrangement. Suppose that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is the orbit decomposition and that the primitive derivation D_1 is W-invariant. Then

(1) the S-module $D(\mathcal{A}, (2p-1, 2q-1))$ is free with W-invariant basis

$$\nabla_{\partial_{P_1}} E^{(p,q)}, \dots, \nabla_{\partial_{P_\ell}} E^{(p,q)},$$

(2) the S-module $D(\mathcal{A}, (2p-1, 2q))$ is free with basis

$$\nabla_{\partial_{P_1^{(1)}}} E^{(p,q)}, \dots, \nabla_{\partial_{P_\ell^{(1)}}} E^{(p,q)},$$

(3) the S-module $D(\mathcal{A}, (2p, 2q-1))$ is free with basis

$$\nabla_{\partial_{P_1^{(2)}}} E^{(p,q)}, \dots, \nabla_{\partial_{P_\ell^{(2)}}} E^{(p,q)},$$

(4) the S-module $D(\mathcal{A}, (2p, 2q))$ is free with basis

$$\nabla_{\partial_{x_1}} E^{(p,q)}, \dots, \nabla_{\partial_{x_\ell}} E^{(p,q)}.$$

The existence of the **primitive decomposition** of $D(\mathcal{A}, (2p-1, 2q-1))^W$ is proved by the following theorem:

Theorem 1.4
Under the same assumption of Theorem 1.3 define

$$\theta_i^{(p,q)} : = \nabla_{\partial_{P_i}} E^{(p,q)} = \nabla_{\partial_{P_i}} \nabla_D^{-q} \nabla_{D_1}^{q-p} E \quad (1 \le i \le \ell)$$

for $p, q \in \mathbb{Z}$. *Then the set*

$$\{\theta_i^{(p+k,q+k)} \mid k \ge 0, 1 \le i \le \ell\}$$

is a T*-basis for* $D(\mathcal{A}, (2p-1, 2q-1))^W$. *Put*

$$\mathcal{G}^{(p,q)} := \bigoplus_{i=1}^{\ell} T \cdot \theta_i^{(p,q)}.$$

Then we have a T*-module decomposition (called the primitive decomposition)*

$$D(\mathcal{A}, (2p-1, 2q-1))^W = \bigoplus_{k \ge 0} \mathcal{G}^{(p+k,q+k)}.$$

We will also prove

Theorem 1.5
For any irreducible Coxeter arrangement $\mathcal{A}$ *and any multiplicity* $\mathbf{m}$*, the* R*-module* $D(\mathcal{A}, \mathbf{m})^W$ *is free.*

The organization of this article is as follows. In Section 2 we prove Thereom 1.3 when $q \ge 0$. In Section 3 we prove Theorem 1.4 to have the primitive decomposition, which is a key to complete the proof of Theorem 1.3 at the end of Section 3. In Section 4 we verify that the primitive derivation D_1 can be chosen to be W-invariant when $\mathcal{A}$ is a Coxeter arrangement of either the type B_ℓ or F_4. We also review the cases of G_2 and $I_2(2n)$ $(n \ge 4)$ and find that the primitive derivation D_1 is W_2-antiinvariant. In Section 5, combining Theorem 1.3 with earlier results in [T2002, AT2010, W2010], we finally prove Theorems 1.1 and 1.5.

2 Proof of Theorem 1.3 when $q \ge 0$

In this section we prove Theorem 1.3 when $q \ge 0$.

Recall $R = S^W = \mathbb{R}[P_1, \dots, P_\ell]$ is the invariant ring with basic invariants $P_1, \dots, P_\ell$ such that $2 = \deg P_1 < \deg P_2 \le \cdots \le \deg P_{\ell-1} < \deg P_\ell = h$, where h is the Coxeter number of W. Put $D = \partial_{P_\ell} \in \operatorname{Der} R$ which is a primitive derivation. Recall $T = \ker(D : R \to R) = \mathbb{R}[P_1, \dots, P_{\ell-1}]$. Then the covariant derivative ∇_D is T-linear. For $\mathbf{P} := [P_1, \ldots, P_\ell]$, the Jacobian matrix $J(\mathbf{P})$ is defined as the matrix whose (i,j)-entry is $\frac{\partial P_j}{\partial x_i}$. Define $A := [I^*(dx_i, dx_j)]_{1 \le i,j \le \ell}$ and $G := [I^*(dP_i, dP_j)]_{1 \le i,j \le \ell} = J(\mathbf{P})^T A J(\mathbf{P})$.

Definition 2.1
([Y2002, W2010]) Let $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$ *and* $\zeta \in D(\mathcal{A}, -\infty)^W$. *We say that* ζ *is* **m-universal** *when* ζ *is homogeneous and the* S*-linear map*

$$\begin{aligned} \Psi_\zeta : \mathrm{Der}_S &\longrightarrow D(\mathcal{A}, 2\mathbf{m}) \\ \theta &\longmapsto \nabla_\theta \zeta \end{aligned}$$

is bijective.

Example 2.2
The Euler derivation E *is* $\mathbf{0}$*-universal because* $\Psi_E(\delta) = \nabla_\delta E = \delta$ *and* $D(\mathcal{A}, \mathbf{0}) = \mathrm{Der}_S$.

Recall the T-automorphisms

$$\nabla_D^k \colon D(\mathcal{A}, -\infty)^W \xrightarrow{\sim} D(\mathcal{A}, -\infty)^W \ (k \in \mathbb{Z})$$

from Theorem 1.2. Recall the following two results concerning the $\mathbf{m}$-universality:

Theorem 2.3
[W2010, Theorem 2.8] If ζ *is* $\mathbf{m}$*-universal, then* $\nabla_D^{-1}\zeta$ *is* $(\mathbf{m}+\mathbf{1})$*-universal.*

Proposition 2.4
[W2010, Proposition 2.7] Suppose that ζ *is* $\mathbf{m}$*-universal. Let* $\mathbf{k} \colon \mathcal{A} \to \{+1, 0, -1\}$. *Then an* S*-homomorphism*

$$\Phi_\zeta : D(\mathcal{A}, \mathbf{k}) \to D(\mathcal{A}, \mathbf{k} + 2\mathbf{m})$$

defined by

$$\Phi_\zeta(\theta) := \nabla_\theta \zeta$$

gives an S*-module isomorphism.*

We require that assumption of Theorem 1.3 is satisfied in the rest of this section: Suppose that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is the orbit decomposition and that D_1, a primitive derivation with respect to $\mathcal{A}_1$ in the sense of [AT2009, Definition 2.4], is W-invariant. Let W_i, R_i, $P_j^{(i)}$, T_i, D_i $(i = 1, 2)$ are defined as in Section 1. Even when $\mathcal{A}_1$ is not irreducible, we may consider a T_1-isomorphism

$$\nabla_{D_1}^k \colon D(\mathcal{A}_1, -\infty)^{W_1} \xrightarrow{\sim} D(\mathcal{A}_1, -\infty)^{W_1} \ (k \in \mathbb{Z})$$

from Theorem 1.2.

Proposition 2.5
Suppose $q \geq 0$. The derivation $E^{(p,q)} := \nabla_D^{-q} \nabla_{D_1}^{q-p} E$ is (p,q)-universal.

Proof. When $\mathcal{A}_1$ is irreducible, [AY2007] and [AT2010] imply that $\nabla_{D_1}^{q-p} E$ is $(p-q,0)$-universal. When $\mathcal{A}_1$ is not irreducible, $\nabla_{D_1}^{q-p} E$ is $(p-q,0)$-universal because of [AT2009]. Thus $E^{(p,q)} = \nabla_D^{-q} \nabla_{D_1}^{q-p} E$ is (p,q)-universal by Theorem 2.3. □

Since $E^{(p,q)}$ is (p,q)-universal, Proposition 2.4 yields the following:

Proposition 2.6
Let $q \geq 0$ and $\mathbf{m} \colon \mathcal{A} \to \{+1, 0, -1\}$. Then an S-homomorphism

$$\Phi_{p,q} : D(\mathcal{A}, \mathbf{m}) \to D(\mathcal{A}, (2p, 2q) + \mathbf{m})$$

defined by

$$\Phi_{p,q}(\theta) := \nabla_\theta E^{(p,q)}$$

gives an S-module isomorphism.

Proof of Theorem 1.3 ($q \geq 0$). We may apply Proposition 2.6 because
(1) $\partial_{P_1}, \ldots, \partial_{P_\ell}$ form a basis for $D(\mathcal{A}, (-1,-1))$,
(2) $\partial_{P_1^{(1)}}, \ldots, \partial_{P_\ell^{(1)}}$ form a basis for $D(\mathcal{A}, (-1, 0))$,
(3) $\partial_{P_1^{(2)}}, \ldots, \partial_{P_\ell^{(2)}}$ form a basis for $D(\mathcal{A}, (0, -1))$, and
(4) $\partial_{x_1}, \ldots, \partial_{x_\ell}$ form a basis for $D(\mathcal{A}, (0, 0))$. □

3 Primitive decompositions

In this section we first prove Theorem 1.4 to define the primitive decomposition of $D(\mathcal{A}, (2p-1, 2q-1))^W$. Next we prove Theorem 1.3.

Proposition 3.1
Let ζ be $\mathbf{m}$-universal. Then
(1) the set $\{\nabla_{\partial_{P_j}} \nabla_D^{-k} \zeta \mid 1 \leq j \leq \ell, k \geq 0\}$ is linearly independent over T.
(2) Define $\mathcal{G}^{(k)}$ to be the free T-module with basis $\{\nabla_{\partial_{P_j}} \nabla_D^{-k} \zeta \mid 1 \leq j \leq \ell\}$ for $k \geq 0$. Then the Poincaré series $\mathrm{Poin}(\bigoplus_{k \geq 0} \mathcal{G}^{(k)}, t)$ satisfies:

$$\mathrm{Poin}(\bigoplus_{k \geq 0} \mathcal{G}^{(k)}, t) \quad = \quad \left(\prod_{i=1}^{\ell} \frac{1}{1 - t^{d_i}} \right) (\sum_{j=1}^{\ell} t^{p-d_j}),$$

where $p = \deg \zeta$ and $d_j = \deg P_j$ $(1 \leq j \leq \ell)$.

(3)

$$D(\mathcal{A}, 2\mathbf{m}-1)^W = \bigoplus_{k\geq 0} \mathcal{G}^{(k)}.$$

Proof. Let $k \in \mathbb{Z}_{\geq 0}$. By Theorem 2.3, $\zeta^{(k)} := \nabla_D^{-k}\zeta$ is $(\mathbf{m}+k)$-universal, where the "k" in the $(\mathbf{m}+k)$ stands for the constant multiplicity k by abuse of notation. Thus by Proposition 2.4 we have the following two bases:

$$\nabla_{\partial_{P_1}}\zeta^{(k)}, \dots, \nabla_{\partial_{P_\ell}}\zeta^{(k)},$$

for the S-module $D(\mathcal{A}, 2\mathbf{m}+2k-1)$ and

$$\nabla_{\partial_{I^*(dP_1)}}\zeta^{(k)}, \dots, \nabla_{\partial_{I^*(dP_\ell)}}\zeta^{(k)},$$

for the S-module $D(\mathcal{A}, 2\mathbf{m}+2k+1)$. Note that the two bases are also R-bases for $D(\mathcal{A}, 2\mathbf{m}+2k-1)^W$ and $D(\mathcal{A}, 2\mathbf{m}+2k+1)^W$ respectively. Since the T-automorphism

$$\nabla_D \colon D(\mathcal{A}, -\infty)^W \xrightarrow{\sim} D(\mathcal{A}, -\infty)^W$$

in Theorem 1.2 induces a T-linear bijection

$$\nabla_D \colon D(\mathcal{A}, 2\mathbf{m}+2k+1)^W \xrightarrow{\sim} D(\mathcal{A}, 2\mathbf{m}+2k-1)^W$$

as in [AT2009, Theorem 4.4], we may find an $\ell \times \ell$-matrix $B^{(k)}$ with entries in R such that

$$\begin{aligned} \nabla_D\left(\left[\nabla_{\partial_{P_1}}\zeta^{(k)}, \dots, \nabla_{\partial_{P_\ell}}\zeta^{(k)}\right] G\right) &= \nabla_D\left[\nabla_{\partial_{I^*(dP_1)}}\zeta^{(k)}, \dots, \nabla_{\partial_{I^*(dP_\ell)}}\zeta^{(k)}\right] \\ &= \left[\nabla_{\partial_{P_1}}\zeta^{(k)}, \dots, \nabla_{\partial_{P_\ell}}\zeta^{(k)}\right] B^{(k)}. \end{aligned}$$

The degree of (i,j)-th entry of $B^{(k)}$ is $m_i + m_j - h \leq h-2 < h$. In particular, the degree of $B^{(k)}_{i,\ell+1-i}$ is 0 and $B^{(k)}_{i,j} = 0$ if $i+j < \ell+1$. Hence each entry of $B^{(k)}$ lies in T and $\det B^{(k)} \in \mathbb{R}$. Since D is a derivation of the minimum degree in Der_R, one gets $[D, \partial_{P_i}] = 0$. Thus $\nabla_D \nabla_{\partial_{P_i}} = \nabla_{\partial_{P_i}} \nabla_D$. Operate ∇_D^{-1} on the both sides of the equality above, and get

$$\left[\nabla_{\partial_{P_1}}\zeta^{(k)}, \dots, \nabla_{\partial_{P_\ell}}\zeta^{(k)}\right] G = \left[\nabla_{\partial_{P_1}}\zeta^{(k+1)}, \dots, \nabla_{\partial_{P_\ell}}\zeta^{(k+1)}\right] B^{(k)}.$$

This implies that $\det B^{(k)} \in \mathbb{R}^\times$ because $\nabla_{\partial_{P_1}}\zeta^{(k)}, \dots, \nabla_{\partial_{P_\ell}}\zeta^{(k)}$ are linearly independent over S. Inductively we have

$$\begin{aligned} \left[\nabla_{\partial_{P_1}}\zeta^{(k+1)}, \dots, \nabla_{\partial_{P_\ell}}\zeta^{(k+1)}\right] &= \left[\nabla_{\partial_{P_1}}\zeta^{(k)}, \dots, \nabla_{\partial_{P_\ell}}\zeta^{(k)}\right] G(B^{(k)})^{-1} \\ &= \left[\nabla_{\partial_{P_1}}\zeta, \dots, \nabla_{\partial_{P_\ell}}\zeta\right] G(B^{(0)})^{-1} G(B^{(1)})^{-1} \cdots G(B^{(k)})^{-1} \\ &= \left[\nabla_{\partial_{P_1}}\zeta, \dots, \nabla_{\partial_{P_\ell}}\zeta\right] G_{k+1}, \end{aligned}$$

where $G_i = G(B^{(0)})^{-1}G(B^{(1)})^{-1}\cdots G(B^{(i-1)})^{-1}$ $(i \geq 0)$. Note that G appears i times in the definition of G_i. For $M = (m_{ij}) \in M_\ell(F)$, define $D[M] = (D(m_{ij})) \in M_\ell(F)$. Then $D^j[G_i] = O$ when $j > i$ and $\det D^i[G_i] \neq 0$ because $\det D[G] \neq 0$ and $D^2[G] = O$ (e.g., see [Sa1993, AT2009]).

(1) Suppose that $\{\nabla_{\partial_{P_j}}\zeta^{(k)} \mid 1 \leq j \leq \ell, k \geq 0\}$ is linearly dependent over T. Then there exist ℓ-dimensional column vectors $\mathbf{g}_0, \mathbf{g}_1, \ldots, \mathbf{g}_q \in T^\ell (q \geq 0)$ with $\mathbf{g}_q \neq \mathbf{0}$ such that

$$\mathbf{0} = \sum_{i=0}^{q}\left[\nabla_{\partial_{P_1}}\zeta^{(i)}, \ldots, \nabla_{\partial_{P_\ell}}\zeta^{(i)}\right]\mathbf{g}_i = \left[\nabla_{\partial_{P_1}}\zeta, \ldots, \nabla_{\partial_{P_\ell}}\zeta\right]\left(\sum_{i=0}^{q} G_i\mathbf{g}_i\right).$$

Since $\nabla_{\partial_{P_1}}\zeta, \ldots, \nabla_{\partial_{P_\ell}}\zeta$ are linearly independent over R, one has

$$\mathbf{0} = \sum_{i=0}^{q} G_i\mathbf{g}_i.$$

Applying the operator D on the both sides q times, we get $D^q[G_q]\mathbf{g}_q = \mathbf{0}$. Thus $\mathbf{g}_q = \mathbf{0}$ which is a contradiction. This proves (1).

(2) Compute

$$\begin{aligned}\mathrm{Poin}(\bigoplus_{k\geq 0}\mathcal{G}^{(k)}, t) &= \sum_{k\geq 0}\left(\prod_{i=1}^{\ell-1}\frac{1}{1-t^{d_i}}\right)(\sum_{j=1}^{\ell} t^{p-d_j+kd_\ell}) \\ &= \left(\prod_{i=1}^{\ell-1}\frac{1}{1-t^{d_i}}\right)(\sum_{k\geq 0} t^{kd_\ell})(\sum_{j=1}^{\ell} t^{p-d_j}) \\ &= \left(\prod_{i=1}^{\ell}\frac{1}{1-t^{d_i}}\right)(\sum_{j=1}^{\ell} t^{p-d_j}).\end{aligned}$$

(3) We have

$$D(\mathcal{A}, 2\mathbf{m}-1)^W \supseteq \bigoplus_{k\geq 0}\mathcal{G}^{(k)}$$

by (1). So it suffices to prove

$$\mathrm{Poin}(D(\mathcal{A}, 2\mathbf{m}-1)^W, t) = \mathrm{Poin}(\bigoplus_{k\geq 0}\mathcal{G}^{(k)}, t).$$

Since $D(\mathcal{A}, 2\mathbf{m}-1)^W$ is a free R-module with a basis

$$\nabla_{\partial_{P_1}}\zeta, \ldots, \nabla_{\partial_{P_\ell}}\zeta,$$

we obtain

$$\mathrm{Poin}(D(\mathcal{A},2\mathbf{m}-1)^W,t) = \left(\prod_{i=1}^{\ell}\frac{1}{1-t^{d_i}}\right)(\sum_{i=1}^{\ell}t^{p-d_j}) = \mathrm{Poin}(\bigoplus_{k\geq 0}\mathcal{G}^{(k)},t),$$

which completes the proof. □

We require that the assumption of Theorem 1.3 is satisfied in the rest of this section.

Proof of Theorem 1.4. Suppose $q \geq 0$ to begin with. Then, by Proposition 3.4, $E^{(p,q)}$ is (p,q)-universal. Apply Proposition 3.1 for $\zeta = E^{(p,q)}$ and $\mathbf{m} = (p,q)$, and we have Theorem 1.4:

$$D(\mathcal{A},(2p-1,2q-1))^W = \bigoplus_{k\geq 0}\mathcal{G}^{(p+k,q+k)}$$

when $q \geq 0$. Send the both handsides by ∇_D, and we get

$$D(\mathcal{A},(2p-3,2q-3))^W = \bigoplus_{k\geq 0}\mathcal{G}^{(p+k-1,q+k-1)}$$

because $\nabla_D\left(D(\mathcal{A},(2p-1,2q-1))^W\right) = D(\mathcal{A},(2p-3,2q-3))^W$ as in [AT2009, Theorem 4.4] and $\nabla_D(\theta_i^{(p,q)}) = \theta_i^{(p-1,q-1)}$. Apply ∇_D repeatedly to complete the proof for all $q \in \mathbb{Z}$. □

Note that we do not assume $p \geq 0$ in the following proposition:

Proposition 3.2
For $p,q \in \mathbb{Z}$, the S-module $D(\mathcal{A},(2p-1,2q-1))$ has a W-invariant basis.

Proof. Recall that

$$\nabla_{\partial_{P_1}}E^{(p,q)}, \nabla_{\partial_{P_2}}E^{(p,q)}, \dots \nabla_{\partial_{P_\ell}}E^{(p,q)},$$

which are W-invariant, form an S-basis for $D(\mathcal{A},(2p-1,2q-1))$ when $q \geq 0$ by Theorem 1.3 (1). It is then easy to see that they are also an R-basis for $D(\mathcal{A},(2p-1,2q-1))^W$ for $q \geq 0$. By [A2008] [AT2010], there exists a W-equivariant nondegenerate S-bilinear pairing

$$(\ ,\ ) : D(\mathcal{A},(2p-1,2q-1)) \times D(\mathcal{A},(-2p+1,-2q+1)) \longrightarrow S,$$

characterized by

$$(I^*(\omega),\theta) = \langle\omega,\theta\rangle$$

where $\omega \in \Omega^1(\mathcal{A}, (-2p+1, -2q+1))$ and $\theta \in D(\mathcal{A}, (-2p+1, -2q+1))$. Let $\theta_1, \ldots, \theta_\ell$ denote the dual basis for $D(\mathcal{A}, (-2p+1, -2q+1))$ satisfying

$$\left(\nabla_{\partial_{P_i}} E^{(p,q)}, \theta_j\right) = \delta_{ij}$$

for $1 \leq i, j \leq \ell$. Then $\theta_1, \ldots, \theta_\ell$ are W-invariant because the pairing $(\ ,\ )$ is W-equivariant. $\square$

Although the following lemma is standard and easy, we give a proof for completeness.

Lemma 3.3
Let M be an S-submodule of Der_F. The following two conditions are equivalent:

(1) M has a W-invariant basis Θ over S.

(2) The W-invariant part M^W is a free R-module with a basis Θ and there exists a natural S-linear isomorphism

$$M^W \otimes_R S \simeq M.$$

Proof. It suffices to prove that (1) implies (2) because the other implication is obvious. Suppose that $\Theta = \{\theta_\lambda\}_{\lambda \in \Lambda}$ is a W-invariant basis for M over S. Since it is linearly independent over S, so is over R. Let $\theta \in M^W$. Express

$$\theta = \sum_{i=1}^{n} f_i \theta_i$$

with $f_i \in S$ and $\theta_i \in \Theta$ $(i = 1, \ldots, n)$. Let $w \in W$ act on the both handsides. Then we get

$$\theta = \sum_{i=1}^{n} w(f_i)\theta_i.$$

This implies $f_i = w(f_i)$ for every $w \in W$. Hence $f_i \in R$ for each i. Therefore Θ is a basis for M^W over R. This is (2). $\square$

Proposition 3.4
For any $p, q \in \mathbb{Z}$, $E^{(p,q)}$ is (p,q)-universal.

Proof. By Theorem 1.4 we have the decomposition:

$$D(\mathcal{A}, (2p-1, 2q-1))^W = \bigoplus_{k \geq 0} \mathcal{G}^{(p+k, q+k)}$$

for $p, q \in \mathbb{Z}$. As we saw in Proposition 3.1 (2), we have

$$\text{(3.1)} \quad \operatorname{Poin}(D(\mathcal{A}, (2p-1, 2q-1))^W, t) = \operatorname{Poin}(\bigoplus_{k^g e0} \mathcal{G}^{(p+k,q+k)}, t)$$
$$= \left(\prod_{i=1}^{\ell} \frac{1}{1-t^{d_i}} \right) (\sum_{i=1}^{\ell} t^{m-d_j}),$$

where $m := \deg E^{(p,q)}$. Recall that the S-module $D(\mathcal{A}, (2p-1, 2q-1))$ has a W-invariant basis $\theta_1, \ldots, \theta_\ell$ by Proposition 3.2. By Lemma 3.3, we know that $\theta_1, \ldots, \theta_\ell$ form a basis for the R-module $D(\mathcal{A}, (2p-1, 2q-1))^W$. Thanks to (3.1) we may assume that $\deg \theta_j = m - d_j = \deg \nabla_{\partial_{P_j}} E^{(p,q)}$. Therefore there exists $M \in M_\ell(R)$ such that

$$[\theta_1, \ldots, \theta_\ell] M = [\nabla_{\partial_{P_1}} E^{(p,q)}, \ldots, \nabla_{\partial_{P_\ell}} E^{(p,q)}]$$

with $\det M \in \mathbb{R}$. Since

$$\max_{1 \leq i,j \leq \ell} \left| \deg \theta_i - \deg \nabla_{\partial_{P_j}} E^{(p,q)} \right| = d_\ell - d_1 < \deg P_\ell,$$

we get $M \in M_\ell(T)$. Since $\nabla_{\partial_{P_1}} E^{(p,q)}, \ldots, \nabla_{\partial_{P_\ell}} E^{(p,q)}$ are linearly independent over T by Proposition 3.1 (1), we have $\det M \in \mathbb{R}^\times$. Thus

$$\nabla_{\partial_{P_1}} E^{(p,q)}, \ldots, \nabla_{\partial_{P_\ell}} E^{(p,q)}$$

form an S-basis for $D(\mathcal{A}, (2p-1, 2q-1))$. Since

$$\left[\nabla_{\partial_{P_1}} E^{(p,q)}, \ldots, \nabla_{\partial_{P_\ell}} E^{(p,q)} \right] J(\mathbf{P})^T = \left[\nabla_{\partial_{x_1}} E^{(p,q)}, \ldots, \nabla_{\partial_{x_\ell}} E^{(p,q)} \right],$$

we may apply the multi-arrangement version of Saito's criterion [Sa1980, Z1989, A2008] to prove that $\nabla_{\partial_{x_1}} E^{(p,q)}, \ldots, \nabla_{\partial_{x_\ell}} E^{(p,q)}$ form an S-basis for $D(\mathcal{A}, (2p, 2q))$ for any $p, q \in \mathbb{Z}$. This shows that $E^{(p,q)}$ is (p, q)-universal for any $p, q \in \mathbb{Z}$. □

Proof of Theorem 1.3 ($q \in \mathbb{Z}$). Theorem 2.3 and Proposition 3.4 complete the proof by the same argument as that in Section 2 for $q \geq 0$. □

4 The cases of B_ℓ, F_4, G_2 and $I_2(2n)$

• **The case of B_ℓ**

The roots of the type B_ℓ are:

$$\pm x_i,\ \pm x_i \pm x_j \quad (1 \le i < j \le \ell)$$

in terms of an orthonormal basis $x_1, \ldots, x_\ell$ for V^*. Altogether there are $2\ell^2$ of them. Define

$$Q_1 := \prod_{i=1}^{\ell} x_i,\ \ Q_2 := \prod_{1 \le i < j \le \ell} (x_i \pm x_j),\ \ Q = Q_1 Q_2.$$

Then the arrangement $\mathcal{A}_1$ defined by Q_1 is of the type $A_1 \times \cdots \times A_1 = A_1^\ell$. The arrangement $\mathcal{A}_2$ defined by Q_2 is of the type D_ℓ. The arrangement $\mathcal{A}$ defined by Q is of the type B_ℓ and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is the orbit decomposition. Note that A_1^ℓ is not irreducible. Define

$$D_1 := \sum_{i=1}^{\ell} \frac{1}{x_i} \partial_{x_i}$$

which is a primitive derivation in the sense of [AT2009]. Obviously D_1 is W-invariant. Let $P_j = \sum_{i=1}^{\ell} x_i^{2j}$ $(j \ge 1)$. Then $P_1, \ldots, P_\ell$ form a set of basic invariants under W while $Q_1, P_1, \ldots, P_{\ell-1}$ form a set of basic invariants under W_2. Define a primitive derivation D_2 with respect to $\mathcal{A}_2$ so that

$$D_2(Q_1) = D_2(P_j) = 0 \ (j = 1, \ldots, \ell - 2),\ \ D_2(P_{\ell-1}) = 1.$$

Thus

$$(wD_2)(P_{\ell-1}) = D_2(w^{-1} P_{\ell-1}) = D_2(P_{\ell-1}) = 1 \ \ (w \in W).$$

This implies that D_2 is W-invariant.

- **The case of F_4**

The roots of the type F_4 are:

$$\pm x_i,\ (\pm x_1 \pm x_2 \pm x_3 \pm x_4)/2,\ \pm x_i \pm x_j \quad (1 \le i < j \le 4)$$

in terms of an orthonormal basis x_1, x_2, x_3, x_4 for V^*. Altogether there are 48 of them. Define

$$Q_1 := \prod_{1 \le i < j \le 4} (x_i \pm x_j),\ Q_2 := \prod_{i=1}^{4} x_i \prod (x_1 \pm x_2 \pm x_3 \pm x_4),\ Q = Q_1 Q_2.$$

The arrangement $\mathcal{A}_i$ defined by Q_i is of the type D_4 $(i = 1, 2)$. Then the arrangement $\mathcal{A}$ defined by Q is of the type F_4 and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ is the orbit decomposition. Define

$$P_1^{(1)} = \sum_{i=1}^{4} x_i^2,\ P_2^{(1)} = \sum_{i=1}^{4} x_i^4,\ P_3^{(1)} = x_1x_2x_3x_4,\ P_4^{(1)} = \sum_{i=1}^{4} x_i^6 + 5\sum_{i\neq j} x_i^2 x_j^4.$$

Compute

$$P_4^{(1)} = -4\sum_{i=1}^{4} x_i^6 + 5P_1^{(1)}P_2^{(1)}.$$

Thus $P_1^{(1)}, P_2^{(1)}, P_3^{(1)}, P_4^{(1)}$ are a set of basic invariants under W_1. The reflection τ with respect to $x_1 + x_2 + x_3 + x_4 = 0$ is given by

$$\tau(x_i) = \frac{2x_i - \sum_{j=1}^{4} x_j}{2}\ (i = 1, 2, 3, 4).$$

A calculation shows that $P_4^{(1)}$ is τ-invariant. Let s_i denote the reflection with respect to $x_i = 0$ $(1 \le i \le 4)$. Since the Coxeter group W_2 is generated by τ and s_i $(1 \le i \le 4)$, we know that $P_4^{(1)}$ is W_2-invariant thus W-invariant. Define a primitive derivation D_1 with respect to $\mathcal{A}_1$ so that

$$D_1(P_j^{(1)}) = 0\ (j = 1, 2, 3),\ \ D_1(P_4^{(1)}) = 1.$$

Thus

$$(wD_1)(P_4^{(1)}) = D_1(w^{-1}P_4^{(1)}) = D_1(P_4^{(1)}) = 1\ \ (w \in W).$$

This implies that D_1 is W-invariant. We conclude that D_2 is also W-invariant because an orthonormal coordinate change

$$x_1 = \frac{y_1 - y_2}{\sqrt{2}},\ x_2 = \frac{y_1 + y_2}{\sqrt{2}},\ x_3 = \frac{y_3 - y_4}{\sqrt{2}},\ x_4 = \frac{y_3 + y_4}{\sqrt{2}}$$

switches $\mathcal{A}_1$ and $\mathcal{A}_2$.

- **The cases of G_2 and $I_2(2n)$ $(n \ge 4)$**

The arrangement $\mathcal{A}$ of the type G_2 has exactly two orbits $\mathcal{A}_1$ and $\mathcal{A}_2$, each of which is of the type A_2. Let $n \ge 4$. Then the arrangement $\mathcal{A}$ of the type $I_2(2n)$ has exactly two orbits $\mathcal{A}_1$ and $\mathcal{A}_2$, each of which is of the type $I_2(n)$. In both cases, by [W2010], one may choose

$$D_1 = Q_2D,\ \ D_2 = Q_1D.$$

Since Q_2 is W_2-antiinvariant and D is W-invariant, D_1 is W_2-antiinvariant. Similarly D_2 is W_1-antiinvariant.

5 Proofs of Theorems 1.1 and 1.5

Assume that $\mathcal{A}$ is an irreducible Coxeter arrangement in the rest of the article.

Proof of Theorem 1.1. If $\mathcal{A}$ has the single orbit, then the result in [T2002, AY2007, AT2010] completes the proof. If not, then $\mathcal{A}$ has exactly two orbits. If $\mathcal{A}$ is of the type either G_2 or $I_2(2n)$ with $n \geq 4$, then $D(\mathcal{A}, \mathbf{m})$ is a free S-module because $\mathcal{A}$ lies in a two-dimensional vector space. For the remaining cases of the type B_ℓ and F_4, Section 4 allows us to apply Theorem 1.3 to complete the proof. □

A multiplicity $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$ is said to be **odd** if its image lies in $1 + 2\mathbb{Z}$.

Proposition 5.1
If $\mathbf{m}$ is equivariant and odd, then $D(\mathcal{A}, \mathbf{m})$ has a W-invariant basis over S.

Proof. When $\mathcal{A}$ has the single orbit, $\mathbf{m}$ is constant. In this case Proposition was proved in [T2002, AY2007, AT2010]. If $\mathcal{A}$ is of the type either G_2 or $I_2(2n)$ $(n \geq 4)$, then Proposition was verified in [W2010]. For the remaining cases of B_ℓ and F_4, Proposition 3.2 completes the proof. □

Recall the W-action on $\mathcal{A}$:

$$W \times \mathcal{A} \longrightarrow \mathcal{A}$$

by sending (w, H) to wH $(w \in W,\ H \in \mathcal{A})$. For any multiplicity $\mathbf{m} : \mathcal{A} \to \mathbb{Z}$, define a new multiplicity $\mathbf{m}^*$ by

$$\mathbf{m}^*(H) := \max_{w \in W} \left(2 \cdot \lfloor \mathbf{m}(wH)/2 \rfloor + 1\right),$$

where $\lfloor a \rfloor$ stands for the greatest integer not exceeding a. Then $\mathbf{m}^*$ is obviously equivariant and odd.

Proposition 5.2
For any irreducible Coxeter arrangement $\mathcal{A}$ and any multiplicity $\mathbf{m}$,

$$D(\mathcal{A}, \mathbf{m})^W = D(\mathcal{A}, \mathbf{m}^*)^W.$$

Proof. Since $\mathbf{m}(H) \leq \mathbf{m}^*(H)$ for any $H \in \mathcal{A}$, we have

$$D(\mathcal{A}, \mathbf{m})^W \supseteq D(\mathcal{A}, \mathbf{m}^*)^W.$$

We will show the other inclusion. Let $H \in \mathcal{A}$ and $\theta \in D(\mathcal{A}, \mathbf{m})^W$. It suffices to verify the following two statements:

(A) $\theta(\alpha_H) \in \alpha_H^{\mathbf{m}(wH)} S_{(\alpha_H)}$ for any $w \in W$,
(B) $\theta(\alpha_H) \in \alpha_H^{2m} S_{(\alpha_H)}$ implies $\theta(\alpha_H) \in \alpha_H^{2m+1} S_{(\alpha_H)}$ for any $m \in \mathbb{Z}$.

For $w \in W$ let w^{-1} act on the both sides of

$$\theta(\alpha_{wH}) \in \alpha_{wH}^{\mathbf{m}(wH)} S_{(\alpha_{wH})}$$

to get

$$\theta(\alpha_H) \in \alpha_H^{\mathbf{m}(wH)} S_{(\alpha_H)}.$$

This verifies (A).

Fix $H \in \mathcal{A}$. Let s be the orthogonal reflection through H. Then $s(\alpha_H) = -\alpha_H$. Suppose that $\theta(\alpha_H) = \alpha_H^{2m} p$ with $p \in S_{(\alpha_H)}$. Let s act on the both handsides and we have $\theta(-\alpha_H) = (-\alpha_H)^{2m} s(p)$. This implies $-p = s(p)$. Since $s(p) = p$ on H, one has $p = 0$ on H, which implies $p \in \alpha_H S_{(\alpha_H)}$. This verifies (B). □

Proof of Theorem 1.5. Thanks to Proposition 5.2 we may assume that $\mathbf{m}$ is equivariant and odd. Apply Proposition 5.1 and Lemma 3.3. □

Corollary 5.3

$$D(\mathcal{A}, \mathbf{m})^W \otimes_R S \simeq D(\mathcal{A}, \mathbf{m}^*).$$

Proof. Apply Proposition 5.1 and Lemma 3.3 to get

$$D(\mathcal{A}, \mathbf{m}^*)^W \otimes_R S \simeq D(\mathcal{A}, \mathbf{m}^*).$$

Then Proposition 5.2 completes the proof. □

The following corollary shows that the converse of Proposition 5.1 is true.

Corollary 5.4
The S-module $D(\mathcal{A}, \mathbf{m})$ has a W-invariant basis if and only if $\mathbf{m}$ is odd and equivariant.

Proof. Assume that $D(\mathcal{A}, \mathbf{m})$ has a W-invariant basis over S. Then, by Lemma 3.3, we get

$$D(\mathcal{A}, \mathbf{m})^W \otimes_R S \simeq D(\mathcal{A}, \mathbf{m}).$$

Compare this with Corollary 5.3 and we know that there exsits a common S-basis for both $D(\mathcal{A}, \mathbf{m})$ and $D(\mathcal{A}, \mathbf{m}^*)$. By the multi-arrangement version of Saito's criterion [Sa1980, Z1989, A2008], we have $\mathbf{m} = \mathbf{m}^*$. □

References

[A2008] T. Abe, A generalized logarithmic module and duality of Coxeter multiarrangements. arXiv.0807.2552v1.

[AT2010] T. Abe and H. Terao, A primitive derivation and logarithmic differential forms of Coxeter arrangements. *Math. Z.* **264** (2010), 813–828.

[AT2009] T. Abe and H. Terao, Primitive filtrations of the modules of invariant logarithmic forms of Coxeter arrangements. arXiv:0910.2506v1.

[ATY2009] T. Abe, H. Terao and M. Yoshinaga, Totally free arrangements of hyperplanes. *Proc. AMS* **137** (2009), 1405–1410.

[AY2007] T. Abe and M. Yoshinaga, Coxeter multiarrangements with quasi-constant multiplicities. *J. Algebra* **322** (2009), 2839–2847.

[OT1992] P. Orlik and H. Terao, *Arrangements of hyperplanes.* Grundlehren der Mathematischen Wissenschaften, **300**. Springer-Verlag, Berlin, 1992.

[Sa1980] K. Saito, Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **27** (1980), 265-291.

[Sa1993] K. Saito, On a linear structure of the quotient variety by a finite reflection group. *Publ. RIMS, Kyoto Univ.* **29** (1993), 535–579.

[Sa2003] K. Saito, Uniformization of the orbifold of a finite reflection group. *RIMS preprint* **1414** (2003).

[SoT1998] L. Solomon and H. Terao, The double Coxeter arrangements. *Comment. Math. Helv.* **73** (1998), 237–258.

[T2002] H. Terao, Multiderivations of Coxeter arrangements. *Invent. Math.* **148** (2002), 659–674.

[W2010] A. Wakamiko, Bases for the derivation modules of two-dimensional multi-Coxeter arrangements and universal derivations. arXiv.1010.5266v1. to appear in *Hokkaido Math. J.*

[Y2002] M. Yoshinaga, The primitive derivation and freeness of multi-Coxeter arrangements. *Proc. Japan Acad. Ser. A* **78** (2002), no. 7, 116–119.

[Z1989] G. Ziegler, Multiarrangements of hyperplanes and their freeness. Singularities (Iowa City, IA, 1986), 345–359, Contemp. Math., **90**, Amer. Math. Soc., Providence, RI, 1989.