THE ISENTROPIC EULER SYSTEM ADMITS SOME PLANE WAVE SUPERPOSITIONS

ROBERT E. TERRELL*

Abstract. A class of differentiable solutions is proved for the isentropic Euler equations in two and three space dimensions. The solutions are explicitly given in terms of solutions to inviscid Burgers equations, and several directions of propagation. The relative orientation of the directions is critical. Within the directional constraints, the Burgers solutions are arbitrary. The several velocities add, and the pressures combine nonlinearly. These solutions cannot exist beyond the time when shocks develop in any of the Burgers solutions.

Key words. Euler equations, Burgers equation, plane wave, isentropic, shock

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1. Main result.

Consider the isentropic Euler equations

$$u_t + u \cdot \nabla u + \rho^{-1}\nabla p = 0, \quad \rho_t + \operatorname{div}(\rho u) = 0, \quad p = k\rho^\gamma$$

We assume $1 < \gamma < 3$, k is constant, and set $a = \frac{\gamma-1}{2}$.

Theorem 1.1. *Let v_j be unit vectors in $\mathbb{R}^d$, $d = 2$ or 3, for which the dot products*

$$v_i \cdot v_j = -a, \qquad i \neq j. \tag{$*$}$$

The number N of such vectors is indicated in the table below.

Further suppose that $f_j(s,t)$ are differentiable solutions to Burgers equation

$$f_t + (1+a) f f_s = 0, \qquad s \in \mathbb{R}, \quad 0 \le t < T$$

Define

$$u(x,t) = \sum_{j=1}^{N} f_j(x\cdot v_j, t) v_j, \qquad \text{and } \rho = \Big(\frac{a}{\sqrt{k\gamma}} \sum_{j=1}^{N} f_j(x\cdot v_j, t)\Big)^{\frac{1}{a}}$$

Then u and ρ satisfy the isentropic Euler equations on this time interval and while $\sum f_j > 0$.

Note that the case $\gamma = 2$ corresponds to the shallow water model and you have three vectors v_k coplanar at 120 degrees, while $\gamma = \frac{5}{3}$ corresponds to the monatomic gas with four v_k having the symmetry of a regular tetrahedron.

Proof. We will write out the $d = 3$ case, and the case $d = 2$ can be obtained by deleting the third component of all vectors. Inspired by the treatment in Lax [3], we work with the symmetric hyperbolic form

$$q_t + A_1 q_{x_1} + A_2 q_{x_2} + A_3 q_{x_3} = 0, \qquad q = \begin{bmatrix} u \\ w \end{bmatrix}$$

of the Euler equations where q is a 4×1 vector consisting of the velocities together with $w = a^{-1}\sqrt{\gamma p/\rho}$, which is proportional to the sound speed. That gives in the isentropic case $\rho = (\frac{a}{\sqrt{k\gamma}} w)^{\frac{1}{a}}$. The two forms of the Euler equations are equivalent

*Mathematics Department, Cornell University, Ithaca, New York, 14853 (ret7@cornell.edu).

for differentiable solutions with $\rho > 0$. Here $A_j = u_j I + awL_j$, where I is the 4×4 identity matrix and

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad L_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We abbreviate $\partial f_j / \partial s$ evaluated at $(x \cdot v_j, t)$ by f_{js}. Component i of vector v_j is written v_{ji}. Also abbreviate $\partial f_j / \partial t(x \cdot v_j, t)$ by f_{jt}, and $f_j(x \cdot v_j, t)$ by f_j. Sums are from 1 to $N = 2$, 3, or 4, depending on the number of vectors v_k.

We will need to know the eigenvectors of linear combinations of the L_j. These eigenvectors may be read from the calculation

$$\begin{bmatrix} 0 & 0 & 0 & h \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & m \\ h & k & m & 0 \end{bmatrix} \begin{bmatrix} h \\ k \\ m \\ \pm 1 \end{bmatrix} = \pm \begin{bmatrix} h \\ k \\ m \\ \pm 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 & h \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & m \\ h & k & m & 0 \end{bmatrix} \begin{bmatrix} h_0 \\ k_0 \\ m_0 \\ 0 \end{bmatrix} = 0$$

whenever $h^2 + k^2 + m^2 = 1$ and $hh_0 + kk_0 + mm_0 = 0$.

Now look for solutions of the form $q(x,t) = \sum_k f_k(x \cdot v_k, t) z_k$ where constant vectors $v_k \in \mathbb{R}^3$ and $z_k \in \mathbb{R}^4$ are to be found. Then

$$q_t + \sum_j (u_j I + awL_j) q_{x_j} = \sum_k \Big(f_{kt} + \sum_j (u_j I + awL_j) f_{ks} v_{kj} \Big) z_k$$

$$= \sum_k \Big(f_{kt} + f_{ks} u \cdot v_k + aw f_{ks} \sum_j (v_{kj} L_j) \Big) z_k$$

Now suppose we are looking for eigenvectors $\sum_j (v_{kj} L_j) z_k = \lambda_k z_k$. As displayed above, we may either choose $z_k = \begin{bmatrix} v_k \\ \lambda_k \end{bmatrix}$ with $\lambda_k = \pm 1$, or if $\lambda_k = 0$ then the first three components of z_k must be orthogonal to v_k.

With any such choices of eigenvectors then

$$q_t + \sum_j A_j q_{x_j} = \sum_k \Big(f_{kt} + f_{ks} \cdot (u \cdot v_k + aw\lambda_k) \Big) z_k$$

$$= \sum_k \Big(f_{kt} + f_{ks} \cdot (q \cdot \begin{bmatrix} v_k \\ a\lambda_k \end{bmatrix}) \Big) z_k = \sum_k \Big(f_{kt} + f_{ks} \cdot \Big(\sum_m f_m z_m \cdot \begin{bmatrix} v_k \\ a\lambda_k \end{bmatrix} \Big) \Big) z_k$$

We choose to make the dot products $z_m \cdot \begin{bmatrix} v_k \\ a\lambda_k \end{bmatrix} = 0$ for $k \neq m$, which decouples the system into the equations

$$f_{kt} + \Big(z_k \cdot \begin{bmatrix} v_k \\ a\lambda_k \end{bmatrix} \Big) f_k f_{ks} = 0.$$

If $\lambda_k = \pm 1$ we have $z_k \cdot \begin{bmatrix} v_k \\ a\lambda_k \end{bmatrix} = v_k \cdot v_k + a\lambda_k^2 = 1 + a$. If $\lambda_k = 0$ then $z_k = \begin{bmatrix} v_k^\perp \\ 0 \end{bmatrix}$ where $v_k^\perp$ is some vector perpendicular to v_k, and $z_k \cdot \begin{bmatrix} v_k^\perp \\ 0 \end{bmatrix} = 0$, so we need f_k independent of t, as well as $v_m^\perp \cdot v_k = 0$ for $k \neq m$.

Now we analyze the several cases of dot products and eigenvalues. In the cases where some $\lambda_k = -1$, we replace v_k by $-v_k$ and $f_k(s,t)$ by $-f_k(-s,t)$. This effectively replaces -1 by $+1$, and we can assume from now on that all $\lambda_k \geq 0$.

The most important case, and the one stated in the Theorem, is when all eigenvalues are $+1$. Let the v_k be N unit vectors with all $v_k \cdot v_m = -a$ for $k \neq m$, and all $\lambda_k = 1$. The N is given in the table. The decoupled equations for the f_k are the inviscid Burgers [1] equation $f_{kt} + (1+a) f_k f_{ks} = 0$. The solutions are of the form $q = \sum_{k=1}^{N} f_k(x \cdot v_k, t) \begin{bmatrix} v_k \\ 1 \end{bmatrix}$. This completes the proof. □

Another possibility is that some eigenvalue is 0. Corresponding to each 0 eigenvalue you may replace the term $f_k(x \cdot v_k, t) \begin{bmatrix} v_k \\ 1 \end{bmatrix}$ by $g_k(x \cdot v_k) \begin{bmatrix} v_k^{\perp} \\ 0 \end{bmatrix}$ where g_k is any differentiable function, provided that $v_k^{\perp}$ is perpendicular to v_k and *all* the other v_m.

$\mathbb{R}^d$	$1 < \gamma < \frac{5}{3}$	$\gamma = \frac{5}{3}$	$\frac{5}{3} < \gamma < 2$	$\gamma = 2$	$2 < \gamma < 3$
$\mathbb{R}^2$	2	2	2	3	2
$\mathbb{R}^3$	3	4	3	3	2

FIG. 1.1. *The table shows the number N of vectors v_k available for various d and γ.*

Remark on the time of existence. Such configurations cannot generally live beyond the time when shocks develop in any of the f_k. For example, suppose a shock of speed σ develops in f_1, and assume $\gamma = 1.4$. The jump condition on density is $[\rho]\sigma = [\rho u] \cdot v_1$ or

$$\left[\left(\frac{a}{\sqrt{k\gamma}} \sum_k f_k \right)^{\frac{1}{a}} \right] \sigma = \left[\left(\frac{a}{\sqrt{k\gamma}} \sum_k f_k \right)^{\frac{1}{a}} (f_1 - a f_2 - a f_3) \right].$$

But this is not possible. Consider a line segment lying in a plane level set of f_3 and within the shock plane. Along this segment, f_2 will in general take a continuous range of values, while f_3 is constant and f_1 has different one-sided limits depending on the side of the shock plane from which the segment is approached. Since $1/a = 5$, the jump condition is a polynomial identity in the values of f_2. This contradicts the fundamental theorem of algebra.

Preliminary calculations done using `clawpack` [2], [4] suggest that there is a distinction in the appearance of pressure contours in two cases of crossing wave fronts shortly after breaking occurs, depending on whether the angles between the fronts match equation $(*)$.

REFERENCES

[1] J. M. BURGERS, *A mathematical model illustrating the theory of turbulence*, Adv. Appl. Mech., 1 (1948) pp. 171-179
[2] `http://www.amath.washington.edu/~claw/`
[3] P. D. LAX, *Hyperbolic Partial Differential Equations,* Providence, RI, American Mathematical Society, 2006.
[4] R. J. LEVEQUE, *Finite Volume Methods for Hyperbolic Problems,* Cambridge University Press, 2002.