Dynamical control of quantum state transfer within hybrid open systems

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We analyze quantum state-transfer optimization within hybrid open systems, from a "noisy" (write-in) qubit to its "quiet" counterpart (storage qubit). Intriguing interplay is revealed between our ability to avoid bath-induced errors that profoundly depend on the bath-memory time and the limitations imposed by leakage out of the operational subspace. Counterintuitively, under no circumstances is the fastest transfer optimal (for a given transfer energy).

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Commonly, manipulations of quantum information can be schematically divided into three stages: "writing-in", "storage", and "reading-out" [1]. Realistically, some systems are better suited for writing-in or reading-out than for storage, and vice versa. This has prompted the suggestion of *hybrid*, composite quantum systems [2–5]: quantum operations are rapidly performed and efficiently written in a qubit susceptible to decoherence, e.g., a Josephson (superconducting) qubit; then the quantum information is transferred (directly or via a link) to a storage qubit resilient to decoherence (encoded in an ensemble of, e.g., ultracold atoms); then, on demand, transferred back and read-out from the same fragile qubit. Our aim is to examine a strategy for *maximizing* the average fidelity of quantum state-transfer in such hybrids, from a subspace fragile against decoherence to a robust subspace, by choosing an appropriate dynamical control field.

To this end, we resort to a novel general approach to the control of *arbitrary* multidimensional quantum operations in open systems described by the reduced density matrix $\hat{\rho}(t)$: if the desired operation is disturbed by linear couplings to a bath, via operators $\hat{S} \otimes \hat{B}$ (where $\hat{S}$ is the traceless system operator, and $\hat{B}$ is the bath operator), one can choose controls to maximize the operation fidelity according to the following recipe, which holds to second order in the system-bath coupling (Suppl. Info.): (i) The control (modulation) transforms the system-bath coupling operators to the time-dependent form $\hat{S}(t) \otimes \hat{B}(t)$ in the interaction picture, via the rotation matrix $\varepsilon_i(t)$: a set of time dependent coefficients in the operator basis $\hat{\sigma}_i$ (Pauli matrices in the case of a qubit), such that:

$$\hat{S}(t) = \sum_i \varepsilon_i(t)\hat{\sigma}_i. \tag{1}$$

(ii) This allows to write the time-independent *score matrix*, describing how the fidelity scores (changes) for each pair of basis operators applied:

$$\Gamma_{ij} \equiv \overline{\langle\psi|\,[\sigma_i, \sigma_j\,|\psi\rangle\,\langle\psi|]\,|\psi\rangle}, \tag{2}$$

where the overline is an average over all possible initial states. (iii) Using Γ_{ij} one arrives at a simple expression for the average fidelity of *any* desired operation (within the stipulated second-order accuracy):

$$\begin{aligned} f_{\rm avr}(t) =& 1 - \int_{-\infty}^{\infty} d\omega G(\omega) F(t,\omega), \\ F(t,\omega) \equiv& t^{-1}\varepsilon_{t,i}(\omega)\Gamma_{ij}\varepsilon^*_{t,j}(\omega), \end{aligned} \tag{3}$$

where $\varepsilon_{t,i}(\omega)$ is the finite-time Fourier-transform of the rotation matrix $\varepsilon_i(t)$, and the coupling spectrum $G(\omega)$ is the Fourier-transform of the bath memory (correlation) function $\left\langle \hat{B} e^{\imath \hat{H}_B t} \hat{B} e^{-\imath \hat{H}_B t} \right\rangle$. Namely, the modulation (control) spectrum $F(t,\omega)$ is defined according to the operation, via the Γ_{ij} score matrix. (iv) This fidelity is maximized by the variational Euler-Lagrange method [6], which minimizes the overlap between $G(\omega)$ and $F(t,\omega)$ under the constraint of a given control energy or action.

We use this general approach to optimize a reliable transfer of a quantum state from a fragile qubit to a robust qubit. We choose to focus on the case of two resonant qubits with temporally controlled coupling strength. The free Hamiltonian without decoherence is then

$$\begin{aligned} \hat{H}_S(t) =& \frac{\omega_0}{2}\left(\hat{\sigma}_z^{(1)} + \hat{\sigma}_z^{(2)}\right) + H_c(t), \\ \hat{H}_c(t) =& V(t)\hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_x^{(2)} \end{aligned} \tag{4}$$

where $\hat{H}_c(t)$ is the Hamiltonian for the controlled interaction between the qubits, $V(t)$ describing the ajustable amplitude of the interaction (see Fig. 1-inset, for an example where the interaction amplitude is ajustable using an external laser field, as in [2]). The system-bath interaction Hamiltonian is taken to be

$$\hat{H}_I = \hat{S} \otimes \hat{B}(t) = \hat{\sigma}_z^{(1)} \otimes \hat{B}(t), \tag{5}$$

where $\hat{B}(t)$ is the bath operator $\hat{B}$ rotating with the free bath Hamiltonian $\hat{H}_B$. This model represents proper dephasing in the source qubit 1 due to the bath operator $\hat{B}$, whereas the target qubit 2 is robust against decoherence. This model can be generalized to *any degree of asymmetry* between the decoherence properties of the two qubits.

Equations (4)-(5) conserve the parity of the number of excitations. Hence, the full two-qubit system can be split into two subsystems $\mathcal{O} = \text{span}\{|g_1e_2\rangle, |e_1g_2\rangle\}$ and $\mathcal{E} = \text{span}\{|g_1g_2\rangle, |e_1e_2\rangle\}$, $\mathcal{O}$ and $\mathcal{E}$ standing for *odd* and *even* excitation numbers, respectively:

$$\begin{aligned}\hat{H}_S + \hat{H}_I =& \hat{H}_{\mathcal{O}} + \hat{H}_{\mathcal{E}} \\ \hat{H}_{\mathcal{O}} =& V(t)\hat{\sigma}_x^{\mathcal{O}} + \hat{\sigma}_z^{\mathcal{O}} \otimes \hat{B}(t) \\ \hat{H}_{\mathcal{E}} =& \omega_0\hat{\sigma}_z^{\mathcal{E}} + V(t)\hat{\sigma}_x^{\mathcal{E}} + \hat{\sigma}_z^{\mathcal{E}} \otimes \hat{B}(t)\end{aligned} \tag{6}$$

where the appropriate Pauli matrices in the $\mathcal{O}$ ($\mathcal{E}$) subsystems are: $\hat{\sigma}_x^{\mathcal{O}} = |g_1e_2\rangle\langle e_1g_2| + \text{H.C.}$, $\hat{\sigma}_z^{\mathcal{O}} = |e_1g_2\rangle\langle e_1g_2| - |g_1e_2\rangle\langle g_1e_2|$, $\hat{\sigma}_x^{\mathcal{E}} = |g_1g_2\rangle\langle e_1e_2| + \text{H.C.}$ and $\hat{\sigma}_z^{\mathcal{E}} = |e_1e_2\rangle\langle e_1e_2| - |g_1g_2\rangle\langle g_1g_2|$. In essence we have one resonant and one non-resonant two-level system, both coupled to the same dephasing bath, which renders them inseparable. Both are subject to the same $\hat{\sigma}_x$ control, which must be chosen to maximize the fidelity of a rotation in the $\mathcal{O}$ subsystem while keeping the $\mathcal{E}$ subsystem unchanged.

The accumulated phase

$$\phi(t) = \int_0^t V(t')dt' \tag{7}$$

is our control function. In the ideal case, without decoherence or leakage, the state transfer from qubit 1 to qubit 2 can be perfectly realized if at the final time, t_f, the phase $\phi(t)$ satisfies $\phi(t_f) = \frac{\pi}{2}$, whence any initial state of qubit 1 is mapped onto that of qubit 2 (initially in the ground state)

$$(\alpha|g_1\rangle + \beta|e_1\rangle)|g_2\rangle \to |g_1\rangle(\alpha|g_2\rangle - i\beta|e_2\rangle), \tag{8}$$

for any normalized α, β. Here the states $|g_1\rangle$ ($|g_2\rangle$) and $|e_1\rangle$ ($|e_2\rangle$) are respectively the ground and the excited states of the source qubit 1 and the target qubit 2.

There are two conflicting noise (error) considerations for the transfer, each affecting a different subsystem: (i) In the presence of interaction between the source qubit 1 and the bath, the longer the information stays in qubit 1 the lower the fidelity of the transfer (manifest in subsystem $\mathcal{O}$). (ii) On the other hand, if we make the transfer extremely fast, it may result in population from $|g_1\rangle|g_2\rangle$ leaking into $|e_1\rangle|e_2\rangle$, thus lowering the fidelity of transfer (manifested in subsystem $\mathcal{E}$). Such leakage[6, 7] signifies the violation of the rotating wave approximation (RWA). Namely, fast modulation $V(t)$ may incur unwanted, off-resonant transitions if the transfer rate is comparable to the energy difference (level distance) of the qubits, ω_0.

We first focus on bath-related errors (i), assuming that the RWA is valid, i.e., there is no leakage because of the RWA violation. This may be the case if the transfer time is much slower than the energy separation ω_0. This is also true when the non-RWA terms simply do not exist, such as in $2D$ or $3D$ Heisenberg interactions (of the form $\hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_x^{(2)} + \hat{\sigma}_y^{(1)} \otimes \hat{\sigma}_y^{(2)}$ or $\hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_x^{(2)} + \hat{\sigma}_y^{(1)} \otimes \hat{\sigma}_y^{(2)} + \hat{\sigma}_z^{(1)} \otimes \hat{\sigma}_z^{(2)}$, respectively) where only number-conserving terms exist. The control Hamiltonian $H_C(t)$ then has the RWA form [8]:

$$\hat{H}_c(t) \equiv V(t)\left(|e_1g_2\rangle\langle g_1e_2| + |g_1e_2\rangle\langle e_1g_2|\right) \tag{9}$$

The general expression (3) derived from the score matrix (2) for the average fidelity of the transfer, completed at t_f, is then (see Suppl.):

$$\overline{f(t_f)} = 1 - \int_{-\infty}^{\infty} d\omega G(\omega)F(t_f,\omega), \tag{10}$$

$$\begin{aligned}F(t,\omega) =& \frac{2}{3}\left|\int_0^t d\tau \cos^2(\phi(\tau))^2 e^{-i\omega\tau}\right|^2 \\ &+ \frac{1}{2}\left|\int_0^t d\tau \sin(2\phi(\tau))e^{-i\omega\tau}\right|^2,\end{aligned} \tag{11}$$

being the transfer-oriented modulation control spectrum.

Thus, the average fidelity of the transfer has an involved dependence on the modulation $V(t)$ and the transfer time t_f. The problem at hand is to find the *optimal* transfer that minimizes the average infidelity at time t_f, $1 - \overline{F(t_f)}$. Obviously, zero infidelity is obtainable for infinitely fast (zero-time) transfer, if we allow infinitely strong control. Since this is unphysical, we add a constraint on the total energy E of the transfer process

$$\int_0^{t_f} dt\,(V(t))^2 = \int_0^{t_f} dt\left(\frac{d\phi(t)}{dt}\right)^2 = E. \tag{12}$$

As discussed below, this constraint can prevent leakage to levels out of the operational qubit subspace[6, 7]. The constraint defines the minimum possible time for the transfer $t_{\min} = \frac{\pi^2}{4E}$.

We illustrate the general expressions (10)-(12) for a typical non-Markovian Lorentzian bath spectrum, i.e. an exponentially decaying correlation function $\Phi(t) = \frac{\gamma}{t_c}e^{-|t|/t_c}$, t_c being the correlation (memory) time. One might expect that for such a simple bath the best strategy is the fastest possible transfer under the energy constraint, i.e. when the modulation is given by $V(0 \leq t \leq t_{\min}) = 2E/\pi$. Surprisingly, a *slower* transfer ($t_f > t_{\min}$) with an appropriate modulation $\phi(t)$ (detailed below) can improve the average fidelity even for a *purely Markovian* bath, with negligible correlation (memory) time $t_c/t_{\min} \to 0$, and more so for baths with memory times longer than the transfer time, $t_c \gtrsim t_{\min}$.

When the bath is memoryless, i.e. Markovian, this improvement is limited, as shown in Fig. 1, to about 12%. By comparing the "best" solution to the "fastest" one (Fig. 1 (a), (b)), one can see the the "best" solution starts off faster and then slows down, being overtaken by the "fastest" solution only at $t \approx 0.9t_{\min}$. This illustrates the source of the Markovian noise reduction: the "best" solution starts off faster, so as to transfer more of the information while it is still nearly untainted by the bath. Obviously, towards the end it must slow down so as to comply with the energy constraint, thus resulting in total transfer time t_f that is longer than the fastest time $t_{\min}$ for the given energy.

However, when the memory-time t_c of the bath is comparable to or larger than the characteristic transfer time $t_c \gtrsim t_{\min}$, a much larger improvement can be achieved (see Fig. 1). Remarkably, the best solution actually performs a full transfer, $\phi(t) = \pi/2$, well within the modulation time, but rather than stop at $\phi = \pi/2$ it then "*overshoots*" the transfer, so that $\phi(t) > \pi/2$, and then returns slowly to $\pi/2$. This can explain the source of the noise reduction — when "overshooting", the information partially returns from the target (storage) qubit to the source (noisy) qubit, but with a negative sign. Hence, similarly to the "echo" method, the noise now operates in the reverse direction, correcting itself, i.e., the non-Markov bath effect is undone. This requires transfer times *significantly* larger than the minimal transfer time $t_{\min}$, ranging from $3t_{\min}$ to even $10t_{\min}$ or more, yet the fidelity increases substantially (up to 50% in Fig. 1).

Using the Euler-Lagrange variational method one can find an analytical solution for the optimal modulation phase $\phi(t)$, given a Markovian bath at long times (see Suppl. Info.). This yields:

$$\frac{d\phi_M(x)}{dx} = \sqrt{\frac{\sin^2(2\phi_M(x))}{2} + 2\frac{\cos^4(\phi_M(x))}{3}}, \tag{13}$$

with $\phi_M(0) = 0$. Eq. (13) determines the shape of $\phi_M(x)$ and its formal "energy" $e_M = \int_0^\infty |\phi_M'(x)|^2 dx = 1.038\ldots$ (where both x and e_M are dimensionless). The general Markovian optimal modulation at infinite time for *any* energy E is then $\phi(t) = \phi_M(\frac{E}{e_M}t)$, with an infidelity of $\gamma\frac{e_M^2}{E} = \gamma\frac{1.077\ldots}{E}$, γ being the dephasing rate of the source qubit 1. The fastest modulation with energy E has an infidelity of $\gamma\frac{\pi^2}{8E} = \gamma\frac{1.233\ldots}{E}$. This means that the optimal modulation has about 12% less infidelity than the fastest modulation for the same energy.

Let us now take into account the breakdown of the RWA as a noise source. A realistic coupling of the form $\sigma_x^{(1)} \otimes \sigma_x^{(1)}$, as in (4), has non-RWA terms of the form $|g_1 g_2\rangle \langle e_1 e_2|$ which do not conserve excitation. In all control scenarios such terms are either discarded or, at best, any transfer of population via non-RWA terms is considered alongside all other forms of leakage. However, there is a drastic difference in timescale between the non-RWA terms in (4) and leakage to higher levels: the qubit level separation ω_0 is often orders of magnitude smaller than the separation to the next (non-qubit) level. Hence, leakage to higher levels requires timescales that are orders of magnitude smaller than those breaking the RWA.

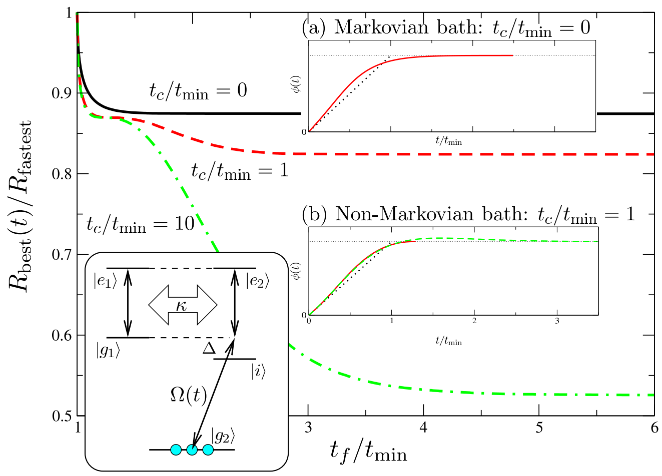

FIG. 1. Inset - scheme of coupling between "noisy" source qubit 1 and quiet target qubit 2: 2-photon transfer off-resonantly through $|i\rangle$ gives effective $\sigma_x^1 \otimes \sigma_x^2$ coupling with a controllable strength $V(t) = \frac{\kappa\Omega(t)}{\Delta}$, where $\Omega(t)$ is the Rabi frequency of an external laser field. Main panel: dependence of the lowest achievable average infidelity on the transfer time t_f normalized to the fastest transfer time $t_{\min}$ at a given transfer energy (Eq. 12). This function is plotted for various bath memory times: (black, solid) $t_c = 0$ (Markovian);(red, dash) $t_c = t_{\min}$; (green, dash-dot) $t_c = 10t_{\min}$. Even for Markovian baths ($t_c \ll t_{\min}$) the best solution is not the fastest one. For non-Markovian baths ($t_c \gtrsim t_{\min}$) two plateaux (regions of insensitivity to t_f) can be seen. The first plateau is independent of the memory time, and matches the Markovian plateau. The second plateau is lower the longer the memory of the bath. In (a) and (b) the transfer phase $\phi(t)$ is plotted versus $t/t_{\min}$. The fastest modulation (black, dotted) with the Markovian optimal modulation (red, solid) and the non-Markovian optimal modulation (green, dashed), in Markovian (a) and non-Markovian (b) baths. In the Markovian bath the optimal modulation transfer starts off faster than the "fastest" transfer (when the information is still "fresh"), and slows down subsequently. For the non-Markovian bath, optimal modulation achieves full transfer ($\phi(t) = \pi/2$) well within the modulation, but then "overshoots" ($\phi(t) > \pi/2$), and eventually returns to $\phi(t) = \pi/2$.

The RWA in (4) breaks down if the transfer time is similar to or less than the inverse energy separation of the qubit, $t_{\min} \lesssim \omega_0^{-1}$. This is a common case for qubits whose resonance frequency is in the microwave (GHz) or RF (MHz) range. In such cases, the optimization process must take into account both the dephasing due to the bath as in (10)-(12) and the error due to the non-RWA terms when minimizing the infidelity.

If one of these errors is vastly larger than the other it is the only one to consider. The problem changes only if the bath-induced and the non-RWA errors are similar. In this case we find that dephasing of the doubly excited state caused by $\hat{\sigma}_x^{\mathcal{E}} \otimes \hat{B}$ in (6) is a fourth order effect and hence can be ignored in the present second order treatment. The result of this approximation is that the system

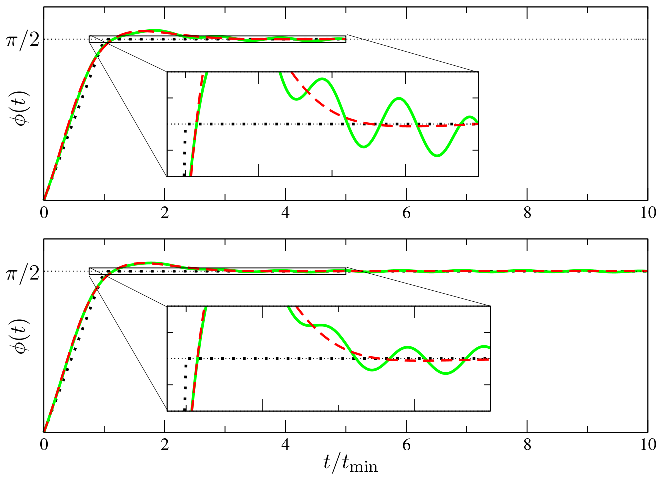

FIG. 2. Fastest (black, dots) vs. best modulation for RWA (red, dash) and non-RWA (green) transfer with a final time of $t_f = 5t_{\min}$ (top) and $t_f = 10t_{\min}$ (bottom) in a non-Markovian bath. The level separation ω_0 satisfies $\omega_0 t_{\min} = \pi$, giving $\sim 2.5\%$ population leakage for fastest modulation and $\sim 10\%$ loss of fidelity from decoherence. The sinusoidal "wiggling" in the non-RWA solution is larger when the final time is shorter (top).

can be split into two completely separate subsystems $\mathcal{O}$ and $\mathcal{E}$, the former suffering only from dephasing and the latter only from unwanted population of the doubly excited level $|e_1 e_2\rangle$. The Hamiltonians of these systems are

$$\begin{aligned} \hat{H}_{\mathcal{O}} =& V(t)\hat{\sigma}_x^{\mathcal{O}} + \hat{\sigma}_z^{\mathcal{O}} \otimes \hat{B}, \\ \hat{H}_{\mathcal{E}} =& \omega_0 \hat{\sigma}_z^{\mathcal{E}} + V(t)\hat{\sigma}_x^{\mathcal{E}}. \end{aligned} \tag{14}$$

The goal of our optimization is to find a control $V(t)$, shared by both subsystems, which maximizes the fidelity of the transfer in subsystem $\mathcal{O}$ (as per (10)-(12)) while at the same time minimizing the doubly-excitation in subsystem $\mathcal{E}$.

The optimal modulation for Markovian and non-Markovian baths, for different final times t_f, is given in Fig. 2. The result shows that the optimal modulation resembles the solution for dephasing in Fig. 1, but with added "wiggles". The "wiggling" takes up the entire time allowed for the transfer, but, given more time for the total transfer, the amplitude of the "wiggle" diminish. This can be understood as follows: first you should complete the transfer assuming the RWA so as to minimize the information lost to the bath. Once the transfer is complete, and decoherence is minimized, we can use whatever energy is left to return the "leaked" excitation from $|e_1 e_2\rangle$ back to $|g_1 g_2\rangle$. This can be done by a weak sinusoidal modulation of frequency $2\omega_0$, inducing a Rabi coupling between the doubly excited and zero-excited levels of $\mathcal{E}$. In practice this is not a two-stage modulation, as the weak oscillation is superimposed on top of the transfer modulation.

The energy needed to "undo" the non-RWA effect is inversely proportional to the allowed time — $E \approx \frac{|\psi_{ee}|^2}{2t_f}$ (where ψ_{ee} is the amplitude of the doubly excited state $|e_1 e_2\rangle$). Hence, given enough time, the correction of the non-RWA effects requires negligible energy, yielding the same results as in the RWA case. If, however, time is limited — a larger proportion of the energy of the transfer must be reserved for the correction of the non-RWA effect, resulting in smaller reduction of the bath-induced noise.

To conclude, our analysis of state-transfer optimization within hybrid open systems, from a "noisy" qubit to its "quiet" counterpart, has revealed an intriguing interplay between our ability to avoid both bath-induced errors that profoundly depend on the bath-memory time and the limitations imposed by leakage out of the operational subspace. Counterintuitively, under no circumstances is the fastest transfer optimal (for a given transfer energy). Generalizations to higher-dimensional cases are expected to follow analogous trends.

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[1] Michael A. Nielsen and Isaac L. Chuang, *Quantum computation and quantum information.* (Cambridge University Press, Cambridge, 2001).
[2] D. Petrosyan, G. Bensky, G. Kurizki, I. Mazets, J. Majer, J. Schmiedmayer, Phys. Rev. A **79**, 040304(R) (2009).
[3] A. Imamoğlu, Phys. Rev. Lett. **102**, 083602 (2009)
[4] P. Rabl, D. DeMille, J. M. Doyle, M. D. Lukin, R. J. Schoelkopf, and P. Zoller, Phys. Rev. Lett. **97**, 033003 (2006)
K. Tordrup and K. Mølmer, Phys. Rev. A **77**, 020301 (2008)
[5] J. Verdú, H. Zoubi, Ch. Koller, J. Majer, H. Ritsch and J. Schmiedmayer, Phys. Rev. Lett. **103**, 043603 (2009)
[6] J. Clausen, G. Bensky and G. Kurizki, Phys. Rev. Lett. **104**, 040401 (2010)
G. Gordon, G. Kurizki and D. Lidar, Phys. Rev. Lett. **101**, 010403 (2008)
[7] L-A. Wu, G. Kurizki and P. Brumer, Phys. Rev. Lett. **102**, 080405 (2009)
L-A. Wu, M.S. Byrd and D.A. Lidar, Phys. Rev. Lett. **89**, 127901 (2002)
M.S. Byrd, L-A. Wu and D.A. Lidar, J. Mod. Optics **51**, 2449 (2004)
M.S. Byrd, D.A. Lidar, L-A. Wu and P. Zanardi, Phys. Rev. A **71**, 052301 (2005)
D.A. Lidar, Phys. Rev. Lett. **100**, 160506 (2008)
[8] A. G. Kofman and G. Kurizki, Phys. Rev. Lett. **93**, 130406 (2004)
G. Gordon, N. Erez and G. Kurizki, J. Phys. B **40**, S75

(2007)