ON SELF-SIMILARITIES OF ERGODIC FLOWS

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Abstract. Given an ergodic flow $T = (T_t)_{t\in\mathbb{R}}$, let $I(T)$ be the set of reals $s \neq 0$ for which the flows $(T_{st})_{t\in\mathbb{R}}$ and T are isomorphic. It is proved that $I(T)$ is a Borel subset of $\mathbb{R}^*$. It carries a natural Polish group topology which is stronger than the topology induced from $\mathbb{R}$. There exists a mixing flow T such that $I(T)$ is an uncountable meager subset of $\mathbb{R}^*$. For a generic flow T, the transformations T_{t_1} and T_{t_2} are spectrally disjoint whenever $|t_1| \neq |t_2|$. A generic transformation (i) embeds into a flow T with $I(T) = \{1\}$ and (ii) does not embed into a flow with $I(T) \neq \{1\}$. For each countable multiplicative subgroup $S \subset \mathbb{R}^*_+$, it is constructed a Poisson (and Gaussian) flow T with simple spectrum such that $I(T) \cap \mathbb{R}^*_+ = S$. If S is without rational relations then there is a rank-one weakly mixing rigid flow T with $I(T) = S$.

0. Introduction

The isomorphism problem for measure preserving group actions is one of the central problems in ergodic theory. Even within the framework of a singe action this problem raises some interesting and difficult questions. For instance, consider an action of $\mathbb{R}^n$, i.e. a multidimensional flow. Then the automorphisms of $\mathbb{R}^n$ generate (via linear time changes in the flow) a continuum of new flows with possibly different classical invariants such as entropy and spectrum. We also note that an $\mathbb{R}^n$-flow has a rich structure of subactions corresponding to lower dimensional subgroups and co-compact lattices. A natural problem is to investigate (i) when these subactions are isomorphic, (ii) which invariants can distinguish non-isomorphic subactions, etc. Our paper is devoted to the simplest particular case of this general problem. We will only consider flows with one-dimensional time. However even in this case there are a lot of open problems.

Let $T = (T_t)_{t\in\mathbb{R}}$ be an ergodic free measure preserving flow on a standard non-atomic probability space $(X, \mathfrak{B}, \mu)$. Given $s \in \mathbb{R}^*$, we denote by $T\circ s$ a flow $(T_{st})_{t\in\mathbb{R}}$. Let

$$I(T) := \{s \in \mathbb{R}^* \mid T \circ s \text{ is isomorphic to } T\}.$$

It is easy to see that $I(T)$ is a multiplicative subgroup of $\mathbb{R}^*$. If $I(T) \not\subset \{-1, 1\}$ then T is called *self-similar*. We are also interested in a closely related invariant

$$E(T) := \{t \in \mathbb{R}^* \mid T_t \text{ is isomorphic to } T_1\}.$$

Of course, $I(T) \subset E(T)$. In the present paper we investigate properties of the invariants $I(T)$ and $E(T)$.

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The property of *total* self-similarity for flows, i.e. when $I(T) \supset \mathbb{R}^*_+$ is well known in ergodic theory. If T possesses this property then the maximal spectral type of T is Lebesgue [KaT, Proposition 1.23]. In [Ma], it was shown that the total self-similarity implies mixing of all orders for the horocycle flow on any surface of constant negative curvature (in fact, for any flow acting by translations on a homogeneous space of a semisimple Lie group by a lattice). A generalization of that result was obtained in [Ry1]: given an arbitrary weakly mixing flow T, if $E(T)$ has positive Lebesgue measure then T is mixing of all orders. However it is unknown so far whether $\mathrm{Leb}(E(T)) > 0$ implies that $I(T) \supset \mathbb{R}^*_+$. Moreover, the following version of D. Ornstein's question [Th] is open:

— is there an ergodic flow T such that $E(T) = \mathbb{R}^*$ but $I(T) = \{1\}$?

We note however that if T acts by translations on a homogeneous space of a Lie group by a lattice and $E(T)$ is uncountable then $I(T) \supset \mathbb{R}^*_+$ [St].

In [dJR] it is proved that if T is weakly mixing simple prime flow, then $I(T) = E(T)$ and $T \circ s \perp^{\mathrm{F}} T$ whenever $s \notin I(T)$. The symbol $\perp^{\mathrm{F}}$ denotes the disjointness in the sense of Furstenberg [Fu]. It was stated in [Ry4] that for a rank-one mixing flow T we have $T \circ s \perp^{\mathrm{F}} T$ for all $s \in \mathbb{R}^* \setminus \{1\}$. It is interesting to note that if T, in addition, has a simple Lebesgue spectrum (see [Pr] in this connection) then the Koopman representations of $T \circ s$ and T are unitarily equivalent for all $s \in \mathbb{R}^*$. This illustrates a drastic difference between the disjointness in the sense of Furstenberg and the spectral disjointness.

In the case when the maximal spectral type of a flow T is singular, the transformation T_t is spectrally disjoint with T_1 for Leb.-a.a. $t \in \mathbb{R}$ [Ry3]. An example of a non-mixing flow T with minimal self-joinings and $I(T) = \{1\}$ was constructed in [dJP] (see also [dJR, Proposition 6.8]).

It is easy to construct a non-mixing ergodic flow T with $I(T)$ infinite. Consider, for instance, an infinite Cartesian product

$$\mathbb{R} \ni t \mapsto T_t := \cdots \times S_{\alpha^{-1}t} \times S_t \times S_{\alpha t} \times S_{\alpha^2 t} \times \cdots ,$$

where S is a non-mixing weakly mixing flow. In this connection a natural question was posed in [Ry1]:

— is there a non-mixing weakly mixing flow T such that $E(T)$ is uncountable.

It remains open (cf. Theorem 2.1 below).

An extensive study of various self-similarity problems was undertaken in a recent paper [FrL] by K. Frączek and M. Lemańczyk. In particular, the following was done.

(i) Examples of non-self-similar ergodic special flows built over certain interval exchange transformations are given.

(ii) For each countable subgroup $G \subset \mathbb{R}^*$, a weakly mixing flow T is constructed with $I(T) = G$.

(iii) If G is independent as a subset of the $\mathbb{Q}$-linear space $\mathbb{R}$ then a weakly mixing Gaussian flow T with simple spectrum is constructed such that $I(T) = G \sqcup (-G)$ and $T \circ t$ is spectrally disjoint with T whenever $|t| \notin G$.

They also raised several questions as a some program for further investigation of self-similarity problems. These questions together with additional ones kindly sent to us by M. Lemańczyk stimulated our present work. We give a complete answer to the following.

— (Q1) Are the sets $I(T)$ and $E(T)$ Borel subsets of $\mathbb{R}$?

— (Q2) Is there a natural Polish topology on $I(T)$?
— (Q3) Is there a flow T for which the group $I(T)$ is uncountable and has zero Lebesgue measure?
— (Q4) Can we embed a typical transformation into a self-similar flow?
— (Q5) Is the absence of self-similarity generic in the set of all measure preserving flows on $(X, \mathfrak{B}, \mu)$?
— (Q6) Find weakly mixing rank-one self-similar flows.

Thus we solved Problems 1, 2, 5, 6 from the list in [FrL, Section 10] plus two additional ones (Q2) and (Q6) by M. Lemańczyk. We also remove a redundant "independence" condition on G from (iii). Moreover, (iii) is proved in the full generality for both Poisson and Gaussian flows. We now state precisely these and other main results of our work.

Proposition 1.3.

(i) *$I(T)$ and $E(T)$ are Borel subsets of $\mathbb{R}$.*
(ii) *There is a topology τ on $I(T)$ which is stronger than the topology induced from $\mathbb{R}^*$ and such that $(I(T), \tau)$ is a Polish topological group.*

It is possible to have $I(T) \neq E(T)$. The set $E(T)$ does not need to be a subgroup of $\mathbb{R}^*$ for an arbitrary T (see Example 1.1 and S. Tikhonov's Example 1.2).

Theorem 2.1. *There is a mixing (of all orders) flow T such that $I(T)$ is uncountable but $I(T) \not\supset \mathbb{R}^*_+$.*

By $\mathrm{Aut}(X, \mu)$ and $\mathrm{Flow}(X, \mu)$ we denote the group of μ-preserving transformations of (X, μ) and the set of μ-preserving flows on (X, μ) respectively. We endow these sets with the natural Polish topologies (see Section 1 below). As usual, we say that a property is *generic* in a Polish space P if the subset of elements satisfying this property is residual in P.

Corollary 3.3. *For a generic flow T, the group $I(T)$ is trivial. Moreover, $\sigma_T \perp \sigma_{T\circ t}$ if $|t| \neq 1$ and $T \perp^{\mathrm{F}} T \circ t$ if $t = -1$.*

Here and below σ_T denotes the measure of maximal spectral type of T.

Theorem 3.6.

(i) *A generic transformation from $\mathrm{Aut}(X, \mu)$ embeds into a flow T such that $I(T) = \{1\}$. Moreover, it embeds into a flow possessing all the properties listed in Corollary 3.3.*
(ii) *A generic transformation from $\mathrm{Aut}(X, \mu)$ does not embed into a flow T with $I(T) \neq \{1\}$.*

We note that (ii) does not follow directly from (i) because a generic transformation from $\mathrm{Aut}(X, \mu)$ embeds into continuum of pairwise non-isomorphic flows [ES].

Theorem 4.1. *Let S be a countable subgroup of $\mathbb{R}^*_+$ such that S considered as a subset of the $\mathbb{Q}$-linear space $\mathbb{R}$ is independent. Denote by T^S the Cartesian product flow $\bigotimes_{s\in S} T \circ s$ acting on the space $(X, \mu)^S$. For a generic flow $T \in \mathrm{Flow}(X, \mu)$,*

(i) *the flow T^S is rank one rigid and weakly mixing,*
(ii) *$I(T^S) = S$ and, moreover,*
(iii) *$T^S \perp^{\mathrm{F}} (T^S) \circ t$ for each real $t \notin S \cup \{0\}$.*

For arbitrary countable subgroups of $\mathbb{R}^*$ we prove the following.

Theorem 4.3. *Let S be a countable subgroup of $\mathbb{R}^*$. There is a weakly mixing rank-one rigid flow T such that $I(T) \supset S$.*

Theorem 4.4. *Let S be a countable subgroup of $\mathbb{R}^*_+$. There is a weakly mixing Poisson flow $\widetilde{W}$ with a simple spectrum such that $I(\widetilde{W}) \cap \mathbb{R}^*_+ = S$ and $\sigma_{\widetilde{W}} \perp \sigma_{\widetilde{W} \circ t}$ for each positive $t \notin S$. Hence there is also a weakly mixing Gaussian flow F with a simple spectrum such that $I(F) = S \sqcup (-S)$ and $\sigma_F \perp \sigma_{F \circ t}$ for each $t \notin S \sqcup (-S)$.*

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1. Topological and algebraic properties of $I(T)$ and $E(T)$

Let $T = (T_t)_{t \in \mathbb{R}}$ be an ergodic free measure preserving flow on a standard probability space $(X, \mathfrak{B}, \mu)$. In this section we study algebraic and topological properties of $I(T)$ and $E(T)$ and answer questions (Q1) and (Q2).

Denote by $\Lambda(T) \subset \mathbb{R}$ the discrete spectrum of T. Then $\Lambda(T)$ is a countable subgroup of $\mathbb{R}$. Denote by $\mathfrak{F}$ the Kroneker factor of T, i.e. $\mathfrak{F} \subset \mathfrak{B}$ is the sub-σ-algebra generated by all proper functions of T. It is easy to verify that

$$s\Lambda(T) = \Lambda(T) \qquad \text{for each} \quad s \in I(T). \tag{1-1}$$

We first give a simple example of a free ergodic flow T such that $I(T) \neq E(T)$. This flow has a non-trivial discrete spectrum. Weakly mixing flows with this property also exist but they are more involved (see Example 1.2 below).

Example 1.1. Let $B = (B_t)_{t \in \mathbb{R}}$ be a Bernoulli flow with infinite entropy and let $P = (P_t)_{t \in \mathbb{R}}$ be an ergodic flow with pure point spectrum $\mathbb{Z}$. Then the product flow $B \times P$ is free and ergodic. Since $\Lambda(B \times P) = \mathbb{Z}$, it follows from (1-1) that $I(B \times P) \subset \{-1, 1\}$. The converse inclusion is obvious. Hence $I(B \times P) = \{-1, 1\}$. On the other hand, it follows easily from the isomorphism theory for Bernoulli transformations that $E(B \times P) = \mathbb{Z}$. Thus, $I(B \times P) \neq E(B \times P)$.

Denote by $\mathrm{Aut}(X, \mu)$ the group of all μ-preserving invertible transformations of (X, μ). Endow it with the weak topology in which $R_n \to R$ if $\mu(R_n A \triangle RA) \to 0$ as $n \to \infty$. Then $\mathrm{Aut}(X, \mu)$ is a Polish group [Ha].

In the following example by S. Tikhonov, a weakly mixing flow V is constructed such that $I(V) \neq E(V)$ and $E(V)$ is not a subgroup of $\mathbb{R}^*$.

Example 1.2. It can be deduced easily from [dRdS] and [dJL] that there is a residual subset $\mathcal{F} \in \mathrm{Aut}(X, \mu)$ such that for each transformation $S \in \mathcal{F}$ the following holds.

(i) S is weakly mixing.
(ii) There exists a flow $T \in \mathrm{Flow}(X, \mu)$ such that $T_1 = S$,
(iii) $C(S) = C(T_q)$ for each $q \in \mathbb{Q}$ and
(iv) the centralizer of the infinite product transformation

$$\cdots \times T_{\frac{1}{4}} \times T_{\frac{1}{2}} \times T_1 \times T_2 \times T_4 \times \cdots$$

of the product space $(X, \mu)^{\mathbb{Z}}$ is the infinite product $C(S)^{\mathbb{Z}}$, i.e. this centralizer is as "small" as possible.

According to [Ti], a generic transformation in $\mathrm{Aut}(X,\mu)$ has a continuum of roots in each residual subset of $\mathrm{Aut}(X,\mu)$. Therefore there are transformations $S \neq \widehat{S}$ in $\mathcal{F}$ such that $S^2 = \widehat{S}^2$. Let T be a flow satisfying (ii)–(iv) and let $\widehat{T}$ be a flow satisfying (ii)–(iv) with $\widehat{S}$ instead of S. We now define a flow V on the space $(X,\mu)^{\mathbb{Z}}$ by setting

$$V = \cdots \times T \circ 2^{-2} \times T \circ 2^{-1} \times T \times \widehat{T} \circ 2 \times T \circ 2^2 \times \cdots$$

It follows from (i) that V is weakly mixing. It is straightforward that $RV_1R^{-1} = V_2$, where $R : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ denotes the shift. Therefore $2 \in E(V)$. We now show that $2^{-1} \notin E(V)$. Indeed, if $QV_1Q^{-1} = V_{2^{-1}}$ for some transformation Q of $(X,\mu)^{\mathbb{Z}}$ then $QV_2Q^{-1} = V_1$. Hence the transformation QR commutes with V_1 and

$$R^{-1}V_1R = (QR)^{-1}V_{2^{-1}}QR.$$

However then it follows from (iv) and (iii) that the transformations T_1 and $\widehat{T}_1$, i.e. S and $\widehat{S}$ in view of (ii), are conjugate by an element of the group $C(S)$. Hence $S = \widehat{S}$, a contradiction. Thus, $2^{-1} \notin E(V)$ and therefore $2 \notin I(V)$.

Let d_w be a complete metric on $\mathrm{Aut}(X,\mu)$ compatible with the weak topology. Denote by $\mathrm{Flow}(X,\mu)$ the set of all μ-preserving flows on (X,μ). Endow it with the topology of uniform weak convergence on the compact subsets in $\mathbb{R}$. This topology is compatible with the following metric d:

$$d(T,S) = \sup_{0\le t\le 1} d_w(T_t, S_t).$$

Then $(\mathrm{Flow}(X,\mu), d)$ is a Polish space (see [dRdS]).

Proposition 1.3.

(i) *$I(T)$ and $E(T)$ are Borel subsets of $\mathbb{R}$.*

(ii) *There is a topology τ on $I(T)$ which is stronger than the topology induced from $\mathbb{R}^*$ and such that $(I(T),\tau)$ is a Polish topological group.*

Proof. There are two commuting continuous actions of $\mathrm{Aut}(X,\mu)$ and $\mathbb{R}^*$ on $\mathrm{Flow}(X,\mu)$:

$$\text{(1-2)}\qquad \begin{aligned} &\mathrm{Aut}(X,\mu) \times \mathrm{Flow}(X,\mu) \ni (R,T) \mapsto R \bullet T \in \mathrm{Flow}(X,\mu) \text{ and}\\ &\mathbb{R}^* \times \mathrm{Flow}(X,\mu) \ni (s,T) \mapsto T \circ s \in \mathrm{Flow}(X,\mu),\end{aligned}$$

where the flow $R \bullet T$ is given by $(R \bullet T)_t := RT_tR^{-1}$ for all $t \in \mathbb{R}$. Of course, the two actions are continuous. It follows that the set

$$A_T := \{(R,s) \in \mathrm{Aut}(X,\mu) \times \mathbb{R}^* \mid R \bullet T = T \circ s\}$$

is a closed subgroup of $\mathrm{Aut}(X,\mu) \times \mathbb{R}^*$. The group $I(T)$ is the image of A_T under a continuous homomorphism $\pi : A_T \ni (R,s) \mapsto s \in \mathbb{R}^*$. We may view π as a one-to-one continuous homomorphism from the Polish group : $A_T/\mathrm{Ker}\,\pi$ onto $I(T)$. Thus $I(T)$ is a Borel subset of $\mathbb{R}$ since it is a one-to-one continuous image of the Polish space. Let τ be the topology on $I(T)$ in which the map : $A_T/\mathrm{Ker}\,\pi \to I(T)$ is a homeomorphism. Then (ii) holds.

Let $M_T := \{(R, s) \in \mathrm{Aut}(X, \mu) \times \mathbb{R}^* \mid T_s = RT_1R^{-1}\}$. Then M_T is a closed subset of $\mathrm{Aut}(X, \mu) \times \mathbb{R}^*$. Since the centralizer $C(T_1)$ of T_1, i.e. the group of all transformations commuting with T_1, is closed in $\mathrm{Aut}(X, \mu)$, there is a Borel subset $B \subset \mathrm{Aut}(X, \mu)$ such that every transformation $S \in \mathrm{Aut}(X, \mu)$ can be written uniquely as a product $S = WR$ with $R \in B$ and $W \in C(T_1)$. Now it is easy to verify that $E(T)$ is a one-to-one image of the Borel subset $M_T \cap (B \times \mathbb{R}^*)$ under a continuous map $\tau : M_T \ni (R, s) \mapsto s \in \mathbb{R}^*$. Hence $E(T)$ is Borel. $\square$

Since every Borel subgroup of $\mathbb{R}^*$ is either non-empty open or meager and of zero Lebesgue measure, we deduce from Proposition 1.3(ii) that

Corollary 1.4. *Either $I(T)$ contains $\mathbb{R}^*_+$ and then the maximal spectral type of T is Lebesgue (see* [KaT, Proposition 1.23]*) or $I(T)$ is meager and $Leb(I(T)) = 0$.*

Remark 1.5. It follows from the proof of Proposition 1.3 that the following sequence of Polish groups

$$1 \to \mathrm{Ker}\, \pi \xrightarrow{\mathrm{id}} M_T \xrightarrow{\pi} I(T) \to 1 \tag{1-3}$$

is exact. An interesting question is when it splits, i.e. there is a continuous homomorphism $R : I(T) \ni s \mapsto R_s \in \mathrm{Aut}(X, \mu)$ such that $T_{st} = R_s T_t R_s^{-1}$ for all $t \in \mathbb{R}$ and $s \in I(T)$. It splits in the case when T is a horocycle flow or when T is a Bernoulli flow with infinite entropy. Also, if $I(T)$ is countable and $I(T) \subset \mathbb{R}^*_+$ then (1-3) splits. We do not know the answer in the general case.

2. Mixing flow T with $I(T)$ meager and uncountable

Our main purpose in this section is to construct an ergodic flow T such that $I(T)$ is uncountable and meager. This answers (Q3). We construct such a flow as a 2-point extension of a horocycle flow. The extension is chosen in such a way to partially "destroy" self-similarities of the base flow. This means that uncountably many of elements of the corresponding geodesic flow in the base lift to the extension and some elements do not lift. Measurable orbit theory plays a key role in choosing such an extension.

We first observe that if T is ergodic and $I(T)$ is uncountable then T is weakly mixing. Indeed, this follows from (1-1).

Theorem 2.1. *There is a mixing (of all orders) flow T such that $I(T)$ is uncountable but $I(T) \not\supset \mathbb{R}^*_+$.*

The proof of this theorem is based heavily on the orbit theory of amenable dynamical systems. Therefore we begin this section with a preliminary material on the orbit theory.

If an equivalence relation $\mathcal{R}$ on $(X, \mathfrak{B}, \mu)$ is the orbit equivalence relation of a μ-preserving action T of a locally compact second countable group G then $\mathcal{R}$ is called *measure preserving*. If every $\mathcal{R}$-saturated measurable subset of X is either μ-null or μ-conull then $\mathcal{R}$ is called *ergodic*. We note that $\mathcal{R}$ is ergodic if and only if T is ergodic.

If the $\mathcal{R}$-class of a.e. point $x \in X$ is countable then $\mathcal{R}$ is called *discrete*. If the $\mathcal{R}$-class of a.e. point $x \in X$ is uncountable then $\mathcal{R}$ is called *continuous*. If $\mathcal{R}$ is ergodic then it is either discrete or continuous.

We do not give here the general definition of amenability for equivalence relations (see [Zi]) but just note that if G is amenable then $\mathcal{R}$ is *amenable*. Given a compact second countable group K, a Borel map $\alpha : \mathcal{R} \to K$ is called a *cocycle* of $\mathcal{R}$ if there is a μ-conull subset $Y \subset X$ such that

$$\alpha(x,y)\alpha(y,z) = \alpha(x,z) \quad \text{for all } (x,y),(y,z) \in \mathcal{R}_Y := \mathcal{R} \cap (Y \times Y).$$

We do not distinguish between cocycles which agree a.e. Recall that two cocycles $\alpha, \beta : \mathcal{R} \to K$ *agree a.e.* if there is a μ-conull subset $Z \subset X$ such that $\alpha(x,y) = \beta(x,y)$ for all $(x,y) \in \mathcal{R}_Z$. Two cocycles $\alpha, \beta : \mathcal{R} \to K$ are *cohomologous* (we will denote $\alpha \approx \beta$) if there is a Borel map $\phi : X \to K$ and a μ-conull subset Y such that

$$\alpha(x,x') = \phi(x)\beta(x,x')\phi(x')^{-1} \quad \text{for all } (x,x') \in \mathcal{R}_Y.$$

An invertible μ-preserving transformation S of X is called an *automorphism* of $\mathcal{R}$ if there are μ-conull subsets X_1 and X_2 such that $(S \times S)\mathcal{R}_{X_1} = \mathcal{R}_{X_2}$. Then we can define a cocycle $\alpha \circ S : \mathcal{R} \to K$ by setting $\alpha \circ S(x,x') := \alpha(Sx, Sx')$. It is easy to verify that $\alpha \approx \beta$ if and only if $\alpha \circ S \approx \beta \circ S$.

A Borel measure preserving action $D = (D_h)_{h \in H}$ of a locally compact second countable group H on (X,μ) is called *strictly $\mathcal{R}$-outer* if there is a conull subset $X' \subset X$ such that

(i) D_h is an automorphism of $\mathcal{R}$ for each $h \in H$ and
(ii) if $(D_h x, x) \in \mathcal{R}$ for some $x \in X'$ and $h \in H$ then $h = 1_H$.

A μ-preserving invertible transformation S of X is called *$\mathcal{R}$-inner* if (x, Sx) belongs to $\mathcal{R}$ for a.a. x. Of course, it is an automorphism of $\mathcal{R}$. It is straightforward that $\alpha \circ S \approx \alpha$ for each cocycle α of $\mathcal{R}$ and each $\mathcal{R}$-inner automorphism S. Consider a measure preserving transformation S^α of the product space $(X \times K, \mu \times \lambda_K)$ by setting

$$S^\alpha(x,k) = (Sx, \alpha(Sx,x)k),$$

where λ_K is the Haar measure on K. Then S^α is called the *α-skew product extension* of S. If $\mathcal{R}$ is generated by a G-action $T = (T_g)_{g \in G}$ then $((T_g)^\alpha)_{g \in G}$ is a measure preserving G-action on $(X \times K, \mu \times \lambda_K)$. Denote by $\mathcal{R}(\alpha)$ the orbit equivalence relation of this action. It does not depend on a particular choice of T generating $\mathcal{R}$. We note that $(x,k) \sim_{\mathcal{R}(\alpha)} (x',k')$ if and only if $x \sim_{\mathcal{R}} x'$ and $k' = \alpha(x',x)k$. If $\mathcal{R}(\alpha)$ is ergodic then α is called *ergodic*.

Proposition 2.2. *There are an amenable ergodic measure preserving continuous equivalence relation $\mathcal{T}$ on a standard probability space $(\widetilde{Y}, \widetilde{\mathfrak{B}}, \widetilde{\nu})$, a strictly $\mathcal{T}$-outer flow $\widetilde{F} = (\widetilde{F}_t)_{t \in \mathbb{R}}$ on $\widetilde{Y}$ and an ergodic cocycle $\widetilde{\beta} : \mathcal{T} \to \mathbb{Z}/2\mathbb{Z}$ such that the set $L := \{t \in \mathbb{R} \mid \widetilde{\beta} \circ \widetilde{F}_t \approx \widetilde{\beta}\}$ is a proper uncountable subgroup of $\mathbb{R}$.*

Proof. We use a 3-step construction.

(A) Let $F' = (F'_t)_{t \in \mathbb{R}}$ denote the following flow on $(\mathbb{T}, \lambda_{\mathbb{T}})$:

$$F'_t z = z + t.$$

We view $\mathbb{T}$ as an interval $[0,1)$. The addition is considered mod 1. We note that F' is transitive and periodic, $F'_{t+1} = F'_t$ for each t. Fix an irrational number θ_1. Denote by $\mathcal{R}_{\theta_1}$ the orbit equivalence relation of the transformation F'_{θ_1} on $(\mathbb{T}, \lambda_{\mathbb{T}})$.

Then $\mathcal{R}_{\theta_1}$ is discrete and ergodic. There is a bijection between the cocycles of $\mathcal{R}_{\theta_1}$ with values in $\mathbb{Z}/2\mathbb{Z}$ and the set $\mathcal{M}(\mathbb{T}, \mathbb{Z}/2\mathbb{Z})$ of measurable functions from $\mathbb{T}$ to $\mathbb{Z}/2\mathbb{Z}$. Such a bijection is defined in a highly non-unique way. For instance, it is established by the map $\beta \mapsto a_\beta$, where $a_\beta(z) := \beta(z, z + \theta_1)$. The set $\mathcal{M}(\mathbb{T}, \mathbb{Z}/2\mathbb{Z})$ endowed with the topology of convergence in measure is a Polish space. Therefore we will consider the set $\mathcal{Z}$ of $\mathbb{Z}/2\mathbb{Z}$-valued cocycles of $\mathcal{R}_{\theta_1}$ as a Polish space. The following properties of this topological space hold:

(i) the cohomology class of every cocycle is dense in $\mathcal{Z}$,
(ii) the subset of ergodic cocycles β is a dense G_δ in $\mathcal{Z}$,
(iii) the subset of cocycles β such that $(F'_{\theta_1})_\beta$ is rigid is a G_δ in $\mathcal{Z}$,
(iv) if $\beta(z, F'_{\theta_1} z) = 1$ for all z then $(F'_{\theta_1})_\beta$ is rigid,
(v) if t is rationally independent with θ_1 then the subset of cocycles β such that $\beta \circ F'_t \not\approx \beta$ is residual in $\mathcal{Z}$.

The properties (i)–(iii) are well known. If $\beta(z, F'_{\theta_1} z) = 1$ for all z then $(F'_{\theta_1})_\beta$ has pure point spectrum and hence (iv) follows. The property (v) follows from [GLS, Theorem 1.2] or [Da1, Theorem 4.2].

It follows from (i)–(v) that given $t_0 > 0$ which is rationally independent with θ_1, there exists a cocycle $\beta : \mathcal{R}_{\theta_1} \to \mathbb{Z}/2\mathbb{Z}$ such that the transformation $(F'_{\theta_1})_\beta$ is ergodic, rigid and

$$L := \{t \in \mathbb{R} \mid \beta \circ F'_t \approx \beta\} \not\ni t_0.$$

Of course, L is a subgroup of $\mathbb{R}$ and $L \supset \mathbb{Z}$. It is well known (for instance, see [New]) that for each transformation $A \in C((F'_{\theta_1})_\beta)$ there are $t \in L$ and a map $\phi \in \mathcal{M}(\mathbb{T}, \mathbb{Z}/2\mathbb{Z})$ such that

$$A(z, i) = (F'_t z, i + \phi(z)) \quad \text{for all} \quad (z, i) \in \mathbb{T} \times \mathbb{Z}/2\mathbb{Z}.$$

The map $A \mapsto t + \mathbb{Z}$ is a group homomorphism from $C((F'_{\theta_1})_\beta)$ onto the quotient group $L/\mathbb{Z}$. The kernel of this homomorphism is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Since $(F'_{\theta_1})_\beta$ is rigid, $C((F'_{\theta_1})_\beta)$ is uncountable. Hence L is uncountable.

(B) Consider the torus $(Y, \nu) := (\mathbb{T} \times \mathbb{T}, \lambda_\mathbb{T} \times \lambda_\mathbb{T})$. Let $Q := F'_{\theta_1} \times F'_{\theta_2}$, where θ_2 is an irrational such that the reals $1, \theta_1, \theta_2$ are rationally independent. Then Q is a transformation of (Y, ν) with pure point spectrum. Denote by $\mathcal{R}_Q$ the Q-orbit equivalence relation. Then $\mathcal{R}_Q$ is discrete and ergodic. Define a flow $F = (F_t)_{t \in \mathbb{R}}$ on (Y, ν) by setting: $F_t := F'_t \times F'_{\theta_3 t}$, where θ_3 is an irrational such that reals $1, \theta_3, \theta_1\theta_3 + \theta_2$ are rationally independent. It is straightforward to verify that F is strictly $\mathcal{R}_Q$-outer. Consider now an extension $\beta \otimes 1 : \mathcal{R}_Q \to \mathbb{Z}/2\mathbb{Z}$ of β given by

$$\beta \otimes 1(z, w, z', w') := \beta(z, z').$$

Then $\beta \otimes 1$ is a cocycle of $\mathcal{R}_Q$. We first claim that it is ergodic. Indeed, the $(\beta \otimes 1)$-skew product extension $Q_{\beta \otimes 1}$ of Q is isomorphic to the product $(F_{\theta_1})_\beta \times F_{\theta_2}$ (the corresponding isomorphism is given by a permutation of coordinates in the space $\mathbb{T} \times \mathbb{T} \times \mathbb{Z}/2\mathbb{Z}$ where $Q_{\beta \otimes 1}$ acts). The discrete spectrum of $(F_{\theta_1})_\beta$ is $\{n\theta_1 + \mathbb{Z} \in \mathbb{T} \mid n \in \mathbb{Z}\}$. Hence it intersects trivially with the discrete spectrum of F_{θ_2} which is $\{n\theta_2 + \mathbb{Z} \in \mathbb{T} \mid n \in \mathbb{Z}\}$. Therefore $(F_{\theta_1})_\beta \times F_{\theta_2}$ is ergodic, as desired.

Next we claim that

$$\{t \in \mathbb{R} \mid (\beta \otimes 1) \circ F_t \approx \beta \otimes 1\} = L. \tag{2-1}$$

The inclusion $\supset$ is obvious. To prove the converse, let $f : Y \to \mathbb{Z}/2\mathbb{Z}$ be a map such that

$$\beta \circ F'_t(z, z') = -f(z, w) + \beta(z, z') + f(z', w')$$

for some $t \in \mathbb{R}$ and all $(z, w, z', w') \in \mathcal{R}_Q \cap (Y' \times Y')$, where Y' is a ν-conull subset in Y. Without loss of generality we can think that Y' is $\mathrm{Id} \times F'_{\theta_2}$-invariant. It follows that

$$\beta \circ F'_t(z, z') = -f(z, F'_{\theta_2} w) + \beta(z, z') + f(z', F'_{\theta_2} w')$$

and hence the function $(z, w) \mapsto f(z, w) - f(z, F'_{\theta_2} w)$ is $\mathcal{R}_Q$-invariant. Since $\mathcal{R}_Q$ is ergodic, this function is constant. This implies that $f(z, w) = f(z, F'_{2\theta_2} w)$ for a.a. $(z, w) \in Y$, i.e. f is invariant under the transformation $\mathrm{Id} \times F'_{2\theta_2}$. Since the transformation $F'_{2\theta_2}$ of Y is ergodic, f does not depend on w. Thus $\beta \circ F'_t \approx \beta$ and hence the inclusion $\subset$ in (2-1) is established.

(C) Let $(\widetilde{Y}, \widetilde{\nu}) := (Y \times \mathbb{T}, \nu \times \lambda_{\mathbb{T}})$. Define an equivalence relation $\mathcal{T}$ on $(\widetilde{Y}, \widetilde{\nu})$ by setting

$$(y, z) \sim_{\mathcal{T}} (y', z') \quad \text{if} \quad y \sim_{\mathcal{R}_Q} y'.$$

Then $\mathcal{T}$ is an amenable ergodic continuous measure preserving equivalence relation on $\widetilde{Y}$. Define a flow $\widetilde{F} = (\widetilde{F}_t)_{t \in \mathbb{R}}$ on $\widetilde{Y}$ by setting $\widetilde{F}_t := F_t \times \mathrm{Id}$, $t \in \mathbb{R}$. Then $\widetilde{F}$ is strictly $\mathcal{T}$-outer. Next, consider the cocycle $\widetilde{\beta} := \beta \otimes 1 \otimes 1 : \mathcal{T} \to \mathbb{Z}/2\mathbb{Z}$ of $\mathcal{T}$. Then, of course, $\widetilde{\beta}$ is ergodic and

$$\{t \in \mathbb{R} \mid \widetilde{\beta} \circ \widetilde{F}_t \approx \widetilde{\beta}\} = \{t \in \mathbb{R} \mid (\beta \otimes 1) \circ F_t \approx \beta \otimes 1\} = L.$$

□

We will need an auxiliary fact which is a particular case of [VF, Theorem 1].

Lemma 2.3. *Let $\mathcal{R}_i$ be an amenable ergodic continuous measure preserving equivalence relation on a standard probability space $(X_i, \mathfrak{B}_i, \mu_i)$ and let $V^{(i)} = (V_h^{(i)})_{h \in H}$ be a strictly $\mathcal{R}_i$-outer action of an amenable locally compact second countable group H on X_i, $i = 1, 2$. Then there is a Borel isomorphism $R : (X_1, \mu_1) \to (X_2, \mu_2)$ and two conull subsets $Y_1 \subset X_1$ and $Y_2 \subset X_2$ such that*

$$(R \times R)(\mathcal{R}_1 \cap (Y_1 \times Y_1)) = \mathcal{R}_2 \cap (Y_2 \times Y_2)$$

and for each $h \in H$, there exists an $\mathcal{R}_2$-inner transformation S_h of X_2 with

$$R V_h^{(1)} R^{-1} = V_h^{(2)} S_h.$$

The following lemma is perhaps well known. However we were unable to find its proof in the literature. Therefore we provide our proof of it.

Lemma 2.4. *Let $H = (H_s)_{s \in \mathbb{R}}$ and $G = (G_t)_{t \in \mathbb{R}}$ be the horocycle flow and the geodesic flow on a surface X of constant negative curvature. Let μ denote the normalized volume on X. Then the joint action $\mathbb{R} \rtimes \mathbb{R} \ni (s, t) \mapsto H_s G_t$ of the semidirect product $\mathbb{R} \rtimes \mathbb{R}$ on (X, μ) is free (mod 0).*

Proof. We define multiplication on $\mathbb{R} \rtimes \mathbb{R}$ by setting

$$(s, t)(s', t') := (s + e^t \cdot s', t + t').$$

Without loss of generality we may assume that $X = \Gamma\backslash \mathrm{SL}_2(\mathbb{R})$ for a lattice $\Gamma \in \mathrm{SL}_2(\mathbb{R})$, μ is Haar measure on X and

$$H_s(\Gamma g) = \Gamma g \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, \qquad G_t(\Gamma g) = \Gamma g \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix},$$

$g \in \mathrm{SL}_2(\mathbb{R})$, $t, s \in \mathbb{R}$ [Ra]. Then the action $\mathbb{R} \rtimes \mathbb{R} \ni (s,t) \mapsto H_s G_t$ is well defined.

Since H is free, we only need to show that the subset

$$\left\{ g \in \mathrm{SL}_2(\mathbb{R}) \middle| \, \Gamma g \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} = \Gamma g \;\text{ for some } b \in \mathbb{R} \text{ and } 1 \neq a > 0 \right\}$$

is of zero Haar measure in $\mathrm{SL}_2(\mathbb{R})$. For this purpose, we will show that for each $\gamma \in \Gamma$, the subset

$$M_\gamma := \left\{ g \in \mathrm{SL}_2(\mathbb{R}) \middle| \, g \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} g^{-1} = \gamma \;\text{ for some } b \in \mathbb{R} \text{ and } 1 \neq a > 0 \right\}$$

is of zero measure in $\mathrm{SL}_2(\mathbb{R})$. Indeed, given $g_1, g_2 \in M_\gamma$, the product $g_1 g_2^{-1}$ commutes with the matrix $\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix}$. Since $a \neq 1$, it follows that $g_1 g_2^{-1}$ is a lower-triangular matrix. It remains to note that the Haar measure of the subgroup of lower-triangular matrices in $\mathrm{SL}_2(\mathbb{R})$ is zero. □

Let $T = (T_f)_{f \in F}$ be an action of a locally compact Abelian group F on (X, μ). A measure ν on $X \times X$ is a *2-fold self-joining* of T if ν is invariant under the diagonal action $(T_f \times T_f)_{f \in F}$ and the marginal projections of ν are both equal μ. If each ergodic 2-fold self-joining of T is either $\mu \times \mu$ or a measure supported by the graph of T_f for some $f \in F$ then T is said *to have property MSJ*$_2$ (two-fold minimal self-joinings) [dJR].

Proof of Theorem 2.1. Let $H = (H_s)_{s \in \mathbb{R}}$ and $G = (G_t)_{t \in \mathbb{R}}$ be the horocycle flow and geodesic flow on the surface (X, μ) of constant negative curvature. Suppose that H has the property of MSJ_2 (see [Ra]). It follows from Lemma 2.4 that there is a Borel H- and G-invariant μ-conull subset $X_0 \subset X$ such that

(•) if $G_t H_s x = x$ for some $t, s \in \mathbb{R}$ and $x \in X_0$ then $t = s = 0$.

Denote by $\mathcal{R}$ the H-orbit equivalence relation on X_0. Then $\mathcal{R}$ is amenable ergodic continuous and μ-preserving. It follows from (•) that G is strictly $\mathcal{R}$-outer.

Let $\widetilde{Y}, \widetilde{\nu}, \mathcal{T}, \widetilde{F}, \widetilde{\beta}, L, t_0$ denote the same objects as in Proposition 2.2. Then by Lemma 2.3, there is a Borel isomorphism $R : (\widetilde{Y}, \widetilde{\nu}) \to (X_0, \mu)$ and conull subsets $X_1 \subset X_0$ and $\widetilde{Y}_1 \subset \widetilde{Y}$ such that $(R \times R)(\mathcal{T} \cap (\widetilde{Y}_1 \times \widetilde{Y}_1)) = \mathcal{R} \cap (X_1 \times X_1)$ and for each $t \in \mathbb{R}$,

$$R \widetilde{F}_t R^{-1} = G_t S_t,$$

where S_t is an $\mathcal{R}$-inner transformation of X_0. Denote by α the cocycle $\widetilde{\beta} \circ R^{-1} : \mathcal{R} \to \mathbb{Z}/2\mathbb{Z}$. Since $\widetilde{\beta}$ is ergodic, so is α. For each $t \in \mathbb{R}$,

$$\alpha \circ G_t = \widetilde{\beta} \circ (\widetilde{F}_t R^{-1} S_t^{-1}). \tag{2-3}$$

It follows from Proposition 2.2 that the cocycle in the right-hand side of (2-3) is cohomologous to $\widetilde{\beta} \circ (R^{-1} S_t^{-1}) = \alpha \circ S_t^{-1}$ if and only if $t \in L$. Since S_t is $\mathcal{R}$-inner, we obtain $\alpha \circ S_t^{-1} \approx \alpha$. Thus,

$$\{t \in \mathbb{R} \mid \alpha \circ G_t \approx \alpha\} = L,$$

and L is a proper uncountable subgroup of $\mathbb{R}$.

Denote by H^α the α-skew product extension of H. We will show that

$$I(H^\alpha) \cap \mathbb{R}_+ = \{e^a \mid a \in L\}. \tag{2-4}$$

Given $a \in \mathbb{R}$, the real e^a belongs to $I(H^\alpha)$ if and only if there exists a transformation V of $X \times \mathbb{T}$ such that $V \bullet H^\alpha = H^\alpha \circ e^a$ (for the definition of $\bullet$ and $\circ$ we refer to Section 1). Now we are going to describe the "structure" of V. Denote by κ the corresponding graph-joining of H^α and $H^\alpha \circ e^a$, i.e. κ is supported on the graph of V. Thus κ is a measure on $X \times \mathbb{Z}/2\mathbb{Z} \times X \times \mathbb{Z}/2\mathbb{Z}$. Denote by κ' the projection of κ to $X \times X$. Then κ' is an ergodic joining of H and $H \circ e^a$. Hence $\kappa' \circ (\mathrm{Id} \times G_a)$ is an ergodic 2-fold self-joining of H. Since H has MSJ$_2$, it follows that either $\kappa' = \mu \times \mu$ or κ' is a graph-joining supported on the graph of $G_a H_s$ for some $s \in \mathbb{R}$. In the first case we get a contradiction to the fact that κ is a graph-joining. Therefore the second case holds. Then

$$V(x, \cdot) = (G_a H_s x, \cdot).$$

Replacing V with $V H^\alpha_{-s}$ we can assume without loss of generality that

$$V(x, \cdot) = (G_a x, \cdot).$$

It is a standard trick to show that such a V conjugates H^α with $H^\alpha \circ e^a$ if and only if $V(x, z) = (G_a x, i + \phi(x))$ for some Borel function $\phi : X \to \mathbb{Z}/2\mathbb{Z}$ such that the cocycles $\alpha \circ G_a$ and α are cohomologous, i.e. $a \in L$. Thus, (2-4) is established. Therefore $I(H^\alpha)$ is uncountable and $I(H^\alpha) \neq \mathbb{R}^*_+$. As we noted in the beginning of this section, the uncountability of $I(H^\alpha)$ implies that H^α is weakly mixing. Since H is mixing of all orders, we deduce from [Ru] that H^α is also mixing of all orders. □

Remark 2.5. (i) In a similar way, one can obtain the following generalization of Theorem 2.1. Let R be an irrational rotation on $(\mathbb{T}, \lambda_\mathbb{T})$ and let K be a compact second countable group. Denote by $\mathrm{Aut}\, K$ the group of continuous automorphisms of K. Fix an ergodic cocycle β of the R-orbital equivalence relation $\mathcal{D}$ to K. We let

$$L(\mathcal{D}, \alpha) := \{S \in C(R) \mid \beta \circ S \approx v \circ \beta, v \in \mathrm{Aut}\, K\}.$$

Since $C(R) = \mathbb{T}$, we denote by $\pi : \mathbb{R} \to C(R)$ the canonical projection $t \mapsto \pi(t) := t + \mathbb{Z}$. Then there is a mixing flow T such that

$$I(T) \cap \mathbb{R}^*_+ = \{e^t \mid \pi(t) \in L(\mathcal{D}, \beta)\}.$$

Thus we obtain a class of flows T for which the invariant $I(T)$ is of purely "cohomological" nature.

(ii) We also note that the groups like $L(\mathcal{D}, \beta)$ and their orbital analogues appear naturally when studying extensions of ergodic dynamical systems and equivalence relations. For more information about them we refer the reader to [Da1], [DaG] and references therein.

3. Flows without self-similarity

In this section we study the problem of existence of self-similarities from the Baire category point of view. Our purpose is to show that a generic flow has no self-similarities and a generic transformation does not embed into a flow with self-similarities. This answers (Q4) and (Q5).

Let $\mathcal{P}$ stand for the set of continuous probability measures on the one-point compactification $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$. Then $\mathcal{P}$ is a compact metric space in the $*$-weak topology. Denote by $\mathcal{P}_C \subset \mathcal{P}$ the subset of continuous, i.e. non-atomic, measures. It is well known that $\mathcal{P}_C$ is a dense G_δ in $\mathcal{P}$ [Na]. Hence it is Polish when endowed with the induced topology. Since $\sigma(\{\infty\}) = 0$ for each $\sigma \in \mathcal{P}_C$, we identify $\mathcal{P}_C$ with the space of non-atomic probability measures on $\mathbb{R}$.

Given $\sigma \in \mathcal{P}$ and $t \neq 0$, we define a measure σ_t by setting $\sigma_t(A) := \sigma(t \cdot A)$ for each Borel subset $A \subset \mathbb{R}$.

Lemma 3.1. *The set $\mathcal{S} := \{\sigma \in \mathcal{P}_C \mid \sigma_t \perp \sigma$ for all $t > 0, t \neq 1\}$ is a G_δ in $\mathcal{P}_C$.*

Proof. For each open subset $O \subset \mathbb{R}$, the map

$$\mathcal{P}_C \times \mathbb{R}_+^* \ni (\sigma, t) \mapsto \sigma_t(O) \in \mathbb{R}$$

is continuous. Therefore, given a segment $I \not\ni 1$ in $\mathbb{R}_+^*$ and an open subset $O \subset \mathbb{R}$, the map

$$f_{O,I} : \mathcal{P}_C \ni \sigma \mapsto (\sigma(O), \max_{t \in I} \sigma_t(O)) \in \mathbb{R}^2$$

is continuous. It is easy to verify that

$$\mathcal{S} = \bigcap_{I \not\ni 1} \bigcap_{n \in \mathbb{N}} \bigcup_{O} f_{O,I}^{-1}((1 - 1/n, +\infty) \times (-\infty, 1/n)),$$

where I runs over segments with positive rational endpoints and O runs over the collection of open subsets in $\mathbb{R}$. Hence $\mathcal{S}$ is a G_δ in $\mathcal{P}_C$. □

It is well known that the subset $\mathcal{W}$ of weakly mixing flows on $(X, \mathfrak{B}, \mu)$ is a dense G_δ in $\text{Flow}(X, \mu)$. Fix an orthonormal basis $(v_j)_{j \in \mathbb{N}}$ in $L_0^2(X, \mu)$. Given $T \in \text{Flow}(X, \mu)$, let $U_T = (U_T(t))_{t \in \mathbb{R}}$ denote the corresponding *Koopman* unitary representation of $\mathbb{R}$ in $L_0^2(X, \mu)$. Recall that $U_T(t)f := f \circ T_{-t}$. For each j, let $\sigma_{T,j}$ be the only probability measure on $\mathbb{R}$ such that

$$\langle U_T(t) v_j, v_j \rangle = \int_{\mathbb{R}} \exp(2\pi i \lambda t)\, d\sigma_{T,j}(\lambda).$$

We now let $\sigma_T := \sum_{j=1}^{\infty} 2^{-j} \sigma_{T,j}$. Then σ_T is a measure of maximal spectral type of U_T and the map

$$\mathcal{W} \ni T \mapsto \sigma_T \in \mathcal{P}_C$$

is continuous.

Theorem 3.2. *The subset $\mathcal{T} := \{T \in \mathcal{W} \mid \sigma_T \in \mathcal{S}\}$ is a dense G_δ in* $\text{Flow}(X, \mu)$.

Proof. It follows from Lemma 3.1 that $\mathcal{T}$ is G_δ in $\mathcal{W}$. In [FrL], a Gaussian flow T with a simple spectrum was constructed such that $T \in \mathcal{T}$ (see also another example in Proposition 3.4 below). It remains to use the fact that the $\mathrm{Aut}(X,\mu)$-orbit[1] of each ergodic flow in $\mathcal{W}$ is dense in $\mathrm{Flow}(X,\mu)$. As in the case of $\mathbb{Z}$-actions, this case follows easily from the Rokhlin lemma (see e.g. [DaSo], where a more general case was under consideration). □

We recall two concepts of disjointness for dynamical systems. Let we are given two actions $T=(T_a)_{a\in A}$ and $S=(S_a)_{a\in A}$ of a locally compact second countable Abelian group A on standard probability spaces (X,μ) and (Y,ν) respectively. The actions are called

- (i) *disjoint in the sense of Furstenberg* if $\mu\times\nu$ is the only $(T_a\times S_a)_{a\in A}$-invariant measure on $X\times Y$ with marginals μ and ν. We will denote this by $T\perp^{\mathrm{F}} S$.
- (ii) *spectrally disjoint* if the measures of maximal spectral type of T and S are mutually orthogonal.

If T and S are spectrally disjoint then $T\perp^{\mathrm{F}} S$. The converse is not true. We note also that if $t=-1$ then T and $T\circ t$ have the same maximal spectral type. If T and S are weakly mixing and A_0 is a cocompact subgroup in A then $T\perp^{\mathrm{F}} S$ if and only if $(T\restriction A_0)\perp^{\mathrm{F}}(S\restriction A_0)$ [dJR].

Corollary 3.3. *For a generic flow T, the group $I(T)$ is trivial. Moreover, $\sigma_T\perp\sigma_{T\circ t}$ if $|t|\neq 1$ and $T\perp^{\mathrm{F}} T\circ t$ if $t=-1$.*

Proof. (i) Take $T\in\mathcal{T}$. Since $\sigma_{T\circ t}=(\sigma_T)_t$ and $(\sigma_T)_{-t}\sim(\sigma_T)_t$, we obtain that the flows $T\circ t$ and T are spectrally disjoint for all and $t\in\mathbb{R}^*$, $t\neq -1$. Hence $I(T)\subset\{-1,1\}$.

(ii)The map $\mathrm{Flow}(X,\mu)\ni T\mapsto T_1\in\mathrm{Aut}(X,\mu)$ is continuous. By [dJ], the set of transformations S such that $S\perp^{\mathrm{F}} S^{-1}$ is a dense G_δ in $\mathrm{Aut}(X,\mu)$. An example a weakly mixing flow T with $T_1\perp^{\mathrm{F}} T_{-1}$ was given in [dJP]. It follows that the set $\mathcal{A}:=\{T\in\mathcal{W}\mid T_1\perp^{\mathrm{F}} T_{-1}\}$ is a dense G_δ in $\mathcal{W}$. If $T\in\mathcal{A}$ then $-1\notin I(T)$. □

We now give an explicit example of a rank-one flow $T\in\mathcal{T}$. For that we recall a classical cutting-and-stacking construction of rank-one flows. The construction process is inductive. Suppose we are given

- (a) a sequence of integers $r_n>1$, and
- (b) a sequence of mappings $s_n:\{1,\dots,r_n\}\to\mathbb{R}_+$.

On the n-th step we have a tower, say X_n, which is a rectangular of hight h_n and width w_n. We cut it into r_n subtowers of equal width w_n/r_n. Enumerate these subtowers from the left to the right by $1,\dots,r_n$. Then for each $j=1,\dots,r_n$, we put a rectangle of hight $s_n(j)$ and width $w_{n+1}:=w_n/r_n$ on the top of the j-th subcolumn. Thus we obtain a family of r_n enumerated towers of hight

$$h_n+s_n(1),h_n+s_n(2),\dots,h_n+s_n(r_n).$$

All of them have the same width w_{n+1}. We now stack these towers in the following way: put the second tower on the top of the first tower, the third tower on the top of the second one and so on. Then we obtain a new tower X_{n+1} of hight $h_{n+1}:=r_nh_n+\sum_{j=1}^{r_n}s_n(j)$ and width w_{n+1}. Since X_{n+1} is embedded into $\mathbb{R}^2$, we endow it with the induced Lebesgue measure, say μ_{n+1}.

Continuing this procedure infinitely many times we obtain a σ-finite standard non-atomic measure space (X,μ) as an inductive limit of the sequence of finite

[1]We mean the action defined by the formula (1-2).

measure spaces $(X_0,\mu_0)\subset(X_1,\mu_1)\subset\cdots$. It is easy to see that μ is finite if and only if

$$\sum_{n=1}^{\infty}h_n^{-1}r_n^{-1}\sum_{j=1}^{r_n}s_n(j)<\infty.$$

We will say that a function $f:X\to\mathbb{C}$ is X_n-*measurable* if f is supported on X_n and $f(x)=f(x')$ whenever x and x' are on the same hight in X_n.

We now define a flow $T=(T_t)_{t\in\mathbb{R}}$ on X by setting

$$T_t(y,z):=(y,t+z),\quad \text{whenever } (y,z),(y,t+z)\in X_n,$$

$n=0,1,\dots$. Geometrically this means that T_t moves a point in X_n up with a unit speed until the point reaches the top of X_n. It is easy to verify that T is well defined on the entire space (more precisely, on a μ-conull subset of) X when $n\to\infty$. This flow preserves μ. We call T the *rank-one flow* associated with $(r_n,s_n)_{n=1}^{\infty}$.

Proposition 3.4. *Let T be a finite measure preserving rank-one flow associated with a sequence $(r_n,s_n)_{n=1}^{\infty}$ and let $r_n=10^n$ for all n. Suppose that there are a sequence of positive integers $n_k\to\infty$ and a sequence of positive reals $u_k\to0$ such that $u_kr_{n_k}\to\infty$ and for each k,*

(i) *$s_{n_k-1}(j)=0$ for all $1\le j\le r_{n_k-1}$ and*
(ii) *$s_{n_k}(j)=(j-1)u_k$ for all $1\le j\le r_{n_k}$.*

Then $T\in\mathcal{T}$.

Proof. The conditions (i) and (ii) mean that infinitely many towers, numbered with n_k-1, have a *flat* roof with no spacers added at all while the subsequent towers, numbered with n_k, have a *staircase* roof.

Fix $l>0$. We claim that

$$U_T(-dh_{n_k})\to 10^{-l}I\quad\text{for } d=1-10^{-l}\text{ and}\tag{3-1}$$

$$U_T(-ch_{n_k})\to 0\qquad\text{uniformly in } c\in[1,10^l],\tag{3-2}$$

where the arrows mean the convergence in the weak operator topology as $k\to\infty$. It follows from (3-1) and (3-2) and the spectral theorem for U_T that $\sigma_{T\circ d}\perp\sigma_{T\circ c}$ for all $c\in[1,10^l]$. Hence $\sigma_T\perp(\sigma_T)_t$ for all $t\in(d^{-1},10^l)$. Since l is arbitrary, we obtain $\sigma_T\in\mathcal{S}$. This implies easily that $T\in\mathcal{W}$. Hence $T\in\mathcal{T}$.

It remains to prove (3-1) and (3-2). We need a notation. Given a function $f\in L^2(X,\mu)$, denote by $f_{k,i}$ the restriction of f to the i-th subtower of X_k, $1\le i\le r_k$, i.e. $f_{k,i}(x)=f(x)$ if x belongs to the i-th subtower and $f_{k,i}(x)=0$ otherwise.

First of all we verify $U_T(-h_{n_k})\to0$. Take f in the unit ball of $L_0^2(X,\mu)$. Then for each $\epsilon>0$, there is $k_0>0$ and $f'\in L_0^2(X,\mu)$ such that f' is X_{k_0}-measurable, $\|f-f'\|_2\le\epsilon$, and $|f'|<D$ for some real D. Take k such that $n_k>k_0$. Cross out from X_{n_k} the bottom layer of hight $(r_{n_k}-2)u_k$. Denote the rest of X_{n_k} by $X_{n_k}^0$. Since f' is bounded and $\mu(X_{n_k})-\mu(X_{n_k}^0)\to0$, we can assume without loss of generality that f' is supported on $X_{n_k}^0$. Then $f'=\sum_{j=1}^{r_{n_k}}f'_{n_k,j}$. It is easy to deduce from (ii) that $f'_{n_k,j} \circ T_{h_{n_k}} = f'_{n_k,j+1} \circ T_{-u_k j}$ for all $1 \le j < r_{n_k}$. We have

$$\begin{aligned}\langle U_T(-h_{n_k})f, f\rangle &= \sum_{j,q=1}^{r_{n_k}-1} \langle f'_{n_k,j+1} \circ T_{-ju_k}, f'_{n_k,q}\rangle \pm 2\epsilon \pm 2\|f'_{n_k,r_{n_k}}\|_2 \\ &= \sum_{j=1}^{r_{n_k}-1} \langle (f' \circ T_{-ju_k})_{n_k,j+1}, f'_{n_k,j+1}\rangle \pm 2\epsilon \pm \frac{2D}{r_{n_k}} \\ &= \sum_{j=1}^{r_{n_k}-1} \frac{1}{r_{n_k}} \langle U_T(ju_k)f', f'\rangle \pm 2\epsilon \pm \frac{2D}{r_{n_k}} \\ &= \left\langle \left(\frac{1}{r_{n_k}} \sum_{j=1}^{r_{n_k}-1} U_T(ju_k)\right) f, f \right\rangle \pm 4\epsilon \pm \frac{2D}{r_{n_k}}.\end{aligned}$$

Applying the mean ergodic we obtain that $\langle U_T(-h_{n_k})f, f\rangle \to 0$, as desired. Only a slight modification of the above argument is needed to prove the following fact: for each integer $p > 0$,

$$\sup_g |\langle U_T(-ph_{n_k})f, g\rangle| \to 0, \tag{3-3}$$

where the supremum is taken over all X_{n_k}-measurable functions g with $\|g\|_2 \le 1$.

Now let f' be an X_{n_k-1}-measurable bounded function. Since $h_{n_k} = h_{n_k-1} r_{n_k-1}$, it follows from (i) that

$$f'_{n_k-1,j} \circ T_{dh_{n_k}} = f'_{n_k-1,j+dr_{n_k-1}} \quad \text{for all } 1 \le j \le 10^{-l} r_{n_k-1}.$$

We let $f^\bullet := \sum_{j=1}^{10^{-l} r_{n_k-1}} f'_{n_k-1,j}$ and $f^\circ := f' - f^\bullet - f'_{n_k-1,r_{n_k-1}}$. Then

$$\begin{aligned}\langle U_T(-dh_{n_k})f^\bullet, f'\rangle &= \sum_{j=1}^{10^{-l} r_{n_k-1}} \langle f'_{n_k-1,j+dr_{n_k-1}}, f'\rangle \\ &= \sum_{j=1}^{10^{-l} r_{n_k-1}} \frac{\|f'\|_2^2}{r_{n_k-1}} = 10^{-l}\|f'\|_2^2\end{aligned} \tag{3-4}$$

and

$$\langle U_T(-dh_{n_k})f^\circ, f'\rangle = \langle f^\circ \circ T_{(d-1)h_{n_k}}, U_T(h_{n_k})f'\rangle. \tag{3-5}$$

It is easy to verify that the function $f^\circ \circ T_{(d-1)h_{n_k}}$ is X_{n_k}-measurable. Therefore it follows from (3-3) and (3-5) that $\langle U_T(-dh_{n_k})f^\circ, f'\rangle \to 0$. This fact plus (3-4) imply (3-1).

To show (3-2) we take $c \in [1, 10^l]$ and write ch_{n_k} as $ch_{n_k} = c_k h_{n_k} + c'_k$ with $c_k \in \mathbb{N}$ and $0 \le c'_k < h_{n_k}$. Partition X_{n_k} by a horizontal line on the hight $h_{n_k} - c'_k$ into two subsets $X^0_{n_k}$ (bottom part) and $X^1_{n_k}$ (upper part). Take a bounded X_{n_k}-measurable function f'. Then

$$\begin{aligned}&\langle U_T(-ch_{n_k})f', f'\rangle \\ &= \langle U_T(-c_k h_{n_k})(f' 1_{X^0_{n_k}}) \circ T_{c'_k}, f'\rangle + \langle U_T(-(c_k+1)h_{n_k})(f' 1_{X^1_{n_k}}) \circ T_{c'_k - h_{n_k}}, f'\rangle \\ &= \langle (f' 1_{X^0_{n_k}}) \circ T_{c'_k}, U_T(c_k h_{n_k})f'\rangle + \langle (f' 1_{X^1_{n_k}}) \circ T_{c'_k - h_{n_k}}, U_T((c_k+1)h_{n_k})f'\rangle.\end{aligned}$$

Since the functions $f'1_{X^0_{n_k}} \circ T_{c'_k}$ and $(f'1_{X^1_{n_k}}) \circ T_{c'_k - h_{n_k}}$ are X_{n_k}-measurable, we can apply (3-3) to obtain $\sup_{1\le c\le 10^l} |\langle U_T(-ch_{n_k})f', f'\rangle| \to 0$ as $k \to \infty$. $\square$

Remark 3.5. We note that it follows directly from Proposition 3.4 that $\mathcal{T}$ is residual. Indeed, the subset $\mathcal{L}$ of flows T such that for each $l > 0$, the limits (3-1) and (3-2) exist along a common subsequence of $(n_k)_{k>1}$ is a G_δ in $\mathrm{Flow}(X,\mu)$. This subset is invariant under the action of $\mathrm{Aut}(X,\mu)$ by conjugation. Hence if it is non-empty then it is dense. As was shown in the proof of Proposition 3.4, $\emptyset \ne \mathcal{L} \subset \mathcal{T}$. Thus $\mathcal{T}$ is residual. On the other hand, the statement of Theorem 3.2 (which uses Lemma 3.1) is sharper: $\mathcal{T}$ is G_δ itself.

Theorem 3.6.

(i) *A generic transformation from* $\mathrm{Aut}(X,\mu)$ *embeds into a flow* T *such that* $I(T) = \{1\}$. *Moreover, it embeds into a flow possessing all the properties listed in Corollary 3.3.*

(ii) *A generic transformation from* $\mathrm{Aut}(X,\mu)$ *does not embed into a flow* T *with* $I(T) \ne \{1\}$.

Proof. (i) It was shown in [dRdS] that the image of a non-meager subset in $\mathrm{Flow}(X,\mu)$ under the map $T \mapsto T_1$ is non-meager in $\mathrm{Aut}(X,\mu)$. If a non-meager subset of $\mathrm{Aut}(X,\mu)$ is invariant under the conjugacy then it is residual in $\mathrm{Aut}(X,\mu)$ [GK]. In view of that, (i) follows from Theorem 3.2 and Corollary 3.3.

(ii) A similar reasoning yields that the set $\{T_1 \mid T \in \mathcal{L}\}$ is residual in $\mathrm{Aut}(X,\mu)$. See Remark 3.5 for the definition of $\mathcal{L}$. Hence the intersection

$$\mathcal{J} := \{T_1 \mid T \in \mathcal{L}\} \cap \{S \in \mathrm{Aut}(X,\mu) \mid S \perp^{\mathrm{F}} S^{-1} \text{ and } S \text{ has a simple spectrum}\}$$

is also residual in $\mathrm{Aut}(X,\mu)$. Take $J \in \mathcal{J}$ and suppose that $J = Q_1$ for a flow $Q \in \mathrm{Flow}(X,\mu)$. Since $J = T_1$ for a flows $T \in \mathcal{L}$ and J has a simple spectrum, the flows T and Q commute. Hence a flow $P : \mathbb{R} \ni t \mapsto P_t := T_t Q_t^{-1}$ is well defined. This flow is periodic, i.e. $P_{t+1} = P_t$ for all $t \in \mathbb{R}$. Since $T \in \mathcal{L}$, we have that for each $l > 0$, (3-1) and (3-2) hold along a common subsequence of $(n_k)_{k=1}^\infty$. Therefore utilizing the fact that the group $\{P_t \mid t \in \mathbb{R}\}$ is compact we can pass to a further subsequence, say $(m_{l,k})_{k=1}^\infty$, such that

(a) $U_Q(-dh_{m_{l,k}}) \to 10^{-l} U_P(\xi)$ for some $0 \le \xi < 1$ and for $d = 1 - 10^{-l}$ and

(b) $U_Q(-ch_{m_{l,k}}) \to 0$ uniformly in $c \in [1, 10^l]$.

The conditions (a) and (b) imply that $Q \in \mathcal{T}$ in the same way as (3-1) and (3-2) imply $T \in \mathcal{T}$ in the proof of Proposition 3.4. Hence $I(Q) \subset \{-1, 1\}$. It follows from the definition of $\mathcal{J}$ that $Q_1 \perp Q_{-1}$. Therefore $-1 \notin I(Q)$. Thus $I(Q) = \{1\}$. $\square$

4. Countable groups of self-similarities

In this section we construct flows T with a prescribed countable group $I(T)$. We solve (Q6) and remove a redundant condition from [FrL, Theorem 9.4].

I. Rank one and self-similarities. Let S be a countable subgroup of $\mathbb{R}^*_+$ such that S considered as a subset of the $\mathbb{Q}$-linear space $\mathbb{R}$ is independent. It was shown in [FrL] that there is a Gaussian flow T with a simple spectrum such that $I(T) = S \sqcup (-S)$ and $\sigma_T \perp (\sigma_T)_t$ for each $t \notin S \sqcup (-S)$. We prove the existence of a rank-one flow with similar (but not identical) properties.

Theorem 4.1. *Let S be a countable subgroup of $\mathbb{R}_+^*$ such that S considered as a subset of the $\mathbb{Q}$-linear space $\mathbb{R}$ is independent. Let T^S denote the Cartesian product flow $\bigotimes_{s\in S} T\circ s$ acting on the space $(X,\mu)^S$. For a generic flow $T\in \mathrm{Flow}(X,\mu)$,*

(i) *the flow T^S is rank one rigid and weakly mixing,*
(ii) *$I(T^S)=S$ and, moreover,*
(iii) *$T^S \perp^{\mathrm{F}} (T^S)\circ t$ for each real $t\notin S\cup\{0\}$.*

Proof. Since the set of transformation of rank one is a G_δ in $\mathrm{Aut}(X,\mu)$, it follows that the following sets

$$\mathcal{O} := \{T\in \mathrm{Flow}(X,\mu) \mid T_1 \text{ is rank one and rigid}\},$$
$$\mathcal{O}_S := \{T\in \mathrm{Flow}(X,\mu) \mid T^S \in O\}$$

are G_δ in $\mathrm{Flow}(X,\mu)$. The two sets are non-empty because they contain any flow with pure point rational spectrum. Hence they are dense in $\mathrm{Flow}(X,\mu)$. Take a flow $T\in\mathcal{O}_S\cap\mathcal{T}$. Since the transformation T_1 is of rank one, so is T. It is obvious that $I(T^S)\supset S$. Now take $r\notin S\cup\{0\}$. Suppose that $T^S \not\perp^{\mathrm{F}} T^S\circ r$. Then there is an ergodic joining $\rho\neq\mu^S\times\mu^S$ of T^S and $T^S\circ r$. In other words, ρ is a measure on $X^{S\sqcup rS}$ which is invariant under $\bigotimes_{s\in S\sqcup rS} T\circ s$, and the projections of ρ on X^S and X^{rS} are both μ^S. Since the spectral disjointness implies the disjointness in the sense of Furstenberg, Theorem 3.2 yields that ρ is pairwise independent, i.e. the projection of ρ on any "coordinate plane" $X\times X$ is $\mu\times\mu$. The measure $\sigma_{T\circ z}$ is singular to Lebesgue measure for each $z\in\mathbb{R}^*$. Hence the maximal spectral type of the transformation $(T\circ z)_1 = T_z$ is also singular. Therefore we may apply Host theorem [Ho] to the dynamical system $(X^{S\sqcup rS},\rho,\bigotimes_{s\in S\sqcup rS} T_s)$. It yields $\rho=\mu^{S\sqcup rS}$, a contradiction. Thus the claims (i)—(iii) are all proved. $\square$

Remark 4.2. We note that the condition on S can not be removed from the statement of Theorem 4.1. The theorem does not hold whenever S contains a pair of rationally dependent reals. This follows from the fact that if n is a positive integer and T is an ergodic flow then the product flow $T\times T\circ n$ is never of rank one.[2] We will show more: *the weak closure theorem* (see [Ry2]) does not hold for this flow, i.e. the centralizer $C(T\times T\circ n)$ of this flow is not the weak closure of the group $\{T_t\times T_{nt} \mid t\in\mathbb{R}\}$ in $\mathrm{Aut}(X\times X,\mu\times\mu)$. Indeed, suppose that the weak closure theorem holds for $T\times T\circ n$. Fix $t>0$. Since the transformation $\mathrm{Id}\times T_t$ commutes with $T\times T\circ n$, it follows that there is a sequence $t_i\to\infty$ such that $T_{t_i}\times T_{nt_i}\to \mathrm{Id}\times T_t$ as $i\to\infty$. Then on the one hand $T_{t_i}\to\mathrm{Id}$ and hence $T_{nt_i}=T_{t_i}^n\to\mathrm{Id}$ but on the other hand $T_{nt_i}\to T_t\neq\mathrm{Id}$, a contradiction.

For arbitrary countable subgroups $S\subset\mathbb{R}^*$ we are unable to find a rank-one weakly mixing flow T with $I(T)=S$. However we can prove the following (weaker) assertion.

Theorem 4.3. *Let S be a countable subgroup of $\mathbb{R}^*$. There is a weakly mixing rank-one rigid flow T such that $I(T)\supset S$.*

Proof. Let $\mathbb{R}\rtimes S$ denote the semidirect product $\mathbb{R}$ with S with the multiplication as follows:

$$(r,s)(r',s') := (r+s\cdot r', ss').$$

[2]An analogous assertion for $\mathbb{Z}$-actions was proved by the second named author in [Ry6].

We furnish $\mathbb{R} \rtimes S$ with the natural (product) locally compact second countable topology. Let $\mathcal{A}$ stand for the set of all measure preserving actions of $\mathbb{R} \rtimes S$ on $(X, \mathfrak{B}, \mu)$. We endow $\mathcal{A}$ with the topology of uniform convergence on the compacts in $\mathbb{R} \rtimes G$. Then $\mathcal{A}$ is a Polish space. One can show in a standard way that each of the following subsets is a G_δ in $\mathcal{A}$:

(a) $\mathcal{A}_1 := \{W \in \mathcal{A} \mid \text{the action } \mathbb{R} \ni t \mapsto W_{(t,0)} \text{ is weakly mixing}\}$,

(b) $\mathcal{A}_2 := \{W \in \mathcal{A} \mid \text{the transformation } W_{(1,0)} \text{ is rigid and of rank one}\}$,

The two sets are invariant under the action of $\mathrm{Aut}(X, \mu)$ on $\mathcal{A}$ by conjugacy. Again, using the Rokhlin lemma for $(\mathbb{R} \rtimes S)$-actions one can show that the conjugacy class of each free $(\mathbb{R} \rtimes S)$-action is dense in $\mathcal{A}$. Therefore $\mathcal{A}_1$ and $\mathcal{A}_2$ are dense G_δ if they contain at least one free $(\mathbb{R} \rtimes S)$-action. Of course, $\mathcal{A}_1$ contains such an action.

It remains to construct a free $(\mathbb{R} \rtimes S)$-action belonging to $\mathcal{A}_2$. Let Γ be a dense countable subgroup in $\mathbb{R}$ such that $S \cdot \Gamma = \Gamma$. Let T be an ergodic flow with pure point spectrum Γ. Denote by $\widehat{\Gamma}$ the Abelian group dual to Γ. Then $\widehat{\Gamma}$ is compact and connected. We note that T is defined on $(\widehat{\Gamma}, \lambda_{\widehat{\Gamma}})$ in the following way:

$$T_t x := x + h(t), \tag{4-1}$$

where $h : \mathbb{R} \to \widehat{\Gamma}$ is a continuous one-to-one homomorphism with dense range in $\widehat{\Gamma}$. Of course, S also acts on $\widehat{\Gamma}$ as follows

$$s \cdot x(\gamma) := x(s^{-1} \cdot \gamma), \qquad \gamma \in \Gamma. \tag{4-2}$$

The two actions (4-1) and (4-2) generate a measure preserving action, say W, of $\mathbb{R} \rtimes S$. It is easy to verify that W is free and $W \in \mathcal{A}_2$. $\square$

We note that Theorem 4.3 refines [Ag] and [Da2, Theorem 1.3].

II. Poisson and Gaussian flows with coutable self-similarities. Let T be a measure preserving flow on an infinite σ-finite measure space $(X, \mathfrak{B}, \mu)$. Since the non-trivial constant functions are not integrable, the associated Koopman representation U_T is defined on the entire space $L^2(X, \mu)$. We will always assume that T has no non-trivial invariant subsets of finite positive measure. Then the maximal spectral type σ_T of T is continuous. For $t \in \mathbb{R}$, we denote by $\widetilde{T}_t$ the Poisson suspension of T_t (see [Ne] and [Ro]). Then $\widetilde{T} := (\widetilde{T}_t)_{t \in \mathbb{R}}$ is a weakly mixing finite measure preserving flow. As we noted in [DaR], if T has a simple spectrum then the *Gaussian* flow associated with σ_T is spectrally equivalent to $\widetilde{T}$, i.e. the Koopman representations generated by the two flows are unitarily equivalent.

Theorem 4.4. *Let S be a countable subgroup of $\mathbb{R}^*_+$. There is a weakly mixing Poisson flow $\widetilde{W}$ with a simple spectrum such that $I(\widetilde{W}) \cap \mathbb{R}^*_+ = S$ and $\sigma_{\widetilde{W}} \perp (\sigma_{\widetilde{W}})_t$ for each positive $t \notin S$. Hence there is also a weakly mixing Gaussian flow F with a simple spectrum such that $I(F) = S \sqcup (-S)$ and $\sigma_F \perp (\sigma_F)_t$ for each $t \notin S \sqcup (-S)$.*

Given a unitary operator V in a Hilbert space $\mathcal{H}$, we denote by $\mathrm{WCP}(V)$ the *weak closure of the powers* of V, i.e. the closure of the group $\{V^n \mid n \in \mathbb{Z}\}$ in the weak operator topology. The unitary operator $\bigoplus_{n \geq 0} V^{\odot n}$ acting in a Hilbert space $\bigoplus_{n \geq 0} \mathcal{H}^{\odot n}$ is called the *exponent* of V. It is denoted by $\exp(V)$.

The following two lemmata are well known. For their proof we refer the reader to, e.g., [Ry5] and [DaR] respectively.

Lemma 4.5. *Let V has a simple spectrum. If*

$$\mathrm{WCP}(V) \supset \{\alpha_n I + \beta_n V \mid n \in \mathbb{N} \text{ and } \alpha_i/\beta_i \neq \alpha_j/\beta_j \text{ whenever } i \neq j\}$$

then $\exp(V)$ *has a simple spectrum.*

Lemma 4.6. *Let U, V be two unitary operators in a Hilbert space $\mathcal{H}$. If U and V have a simple spectrum and $\mathrm{WCP}(U \otimes V) \ni aI \otimes V$ for some $a > 0$ then the tensor product $U \otimes V$ has a simple spectrum.*

We also need the following lemma.

Lemma 4.7. *Let $U = (U(t))_{t \in \mathbb{R}}$ be a weakly continuous unitary representation of $\mathbb{R}$ in a Hilbert space $\mathcal{H}$. If U has a simple spectrum and $\mathrm{WCP}(U(c)) \ni U(a)$ for some $c, a > 0$ with $c/a \notin \mathbb{Q}$ then the operator $U(c)$ has a simple spectrum.*

Proof. Let $h \in \mathcal{H}$ be a cyclic vector for U. Denote by $\mathcal{Z}$ the $U(c)$-cyclic space generated by h. Since $\mathcal{Z}$ is invariant under any operator from $\mathrm{WCP}(U(c))$, it follows that $U(a)h \in \mathcal{Z}$. By the same reason, $U(na + mc)h \in \mathcal{Z}$ for all $n, m \in \mathbb{Z}$. Since $c/a \notin \mathbb{Q}$, the subgroup $\{na + mc \mid n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Therefore, $U(t)h \in \mathcal{Z}$ for all $t \in \mathbb{R}$. It follows that $\mathcal{Z} = \mathcal{H}$. □

Proof of Theorem 4.4. It is enough to consider the Poissonian case only since the Gaussian case follows from it.

Suppose that we have a measure preserving flow T on a σ-finite infinite measure space $(X, \mathfrak{B}, \mu)$ such that the following hold.

(i) $\exp(U_T(s))$ has a simple spectrum for each $s \in S$.

(ii) For each finite sequence $s_1 < s_2 < \cdots < s_k$ of elements in S and each $1 \le l_0 \le k$, there is a sequence of integers $t_j \to \infty$ such that

$$U_T(t_j s_l) \to \frac{1}{2k} I \quad \text{if } 1 \le l \le k,\ l \neq l_0, \tag{4-3}$$

$$U_T(t_j s_{l_0}) \to \frac{1}{2k} U_T(s_{l_0}) \quad \text{if } j \to \infty \text{ and} \tag{4-4}$$

$$U_T(t_j b) \to 0 \quad \text{for each positive } b \notin S. \tag{4-5}$$

We first show how to use this flow to prove the theorem and after that we will explain how to construct such a flow.

Given a finite sequence of reals $0 < z_1 < \cdots < z_k$ and an integer vector $(n_1, \dots, n_k) \in \mathbb{N}^k$, we let $O_{z_1,\dots,z_k}^{n_1,\dots,n_k} := U_T(z_1)^{\odot n_1} \otimes \cdots \otimes U_T(z_k)^{\odot n_k}$.

Claim A. For each finite sequence $s_1 < \cdots < s_k$ of elements of S and each $(n_1, \dots, n_k) \in \mathbb{N}^k$, the operator $O_{s_1,\dots,s_k}^{n_1,\dots,n_k}$ has a simple spectrum.

We verify this claim by induction in k. If $k = 1$ then the claim is true by (i). Suppose it is true for some k. Take a sequence $s_1 < \cdots < s_{k+1}$ and $(n_1, \dots, n_{k+1}) \in \mathbb{N}^{k+1}$. By the inductive hypothesis, $O_{s_1,\dots,s_k}^{n_1,\dots,n_k}$ has a simple spectrum. The operator $U_T(s_{k+1})^{\odot n_{k+1}}$ has a simple spectrum by (i). Letting $l_0 = k + 1$ we deduce from (4-3) and (4-4) that

$$\mathrm{WCT}(O_{s_1,\dots,s_k}^{n_1,\dots,n_k} \otimes U_T(s_{k+1})^{\odot n_{k+1}}) \ni \frac{1}{(2k+2)^{n_1 + \cdots + n_{k+1}}} I \otimes U_T(s_{k+1})^{\odot n_{k+1}}.$$

Now Lemma 4.6 yields that the operator $O_{s_1,\dots,s_{k+1}}^{n_1,\dots,n_{k+1}} = O_{s_1,\dots,s_k}^{n_1,\dots,n_k} \otimes U_T(s_{k+1})^{\odot n_{k+1}}$ has a simple spectrum.

Claim B. Given two finite sequences $s_1 < \cdots < s_k$ and $s'_1 < \cdots < s'_d$ of elements from S and two integer vectors $(n_1, \dots, n_k) \in \mathbb{N}^k$ and $(m_1, \dots, m_d) \in \mathbb{N}^d$, if $\{s_1, \dots, s_k\} \neq \{s'_1, \dots, s'_d\}$ then $O^{n_1,\dots,n_k}_{s_1,\dots,s_k}$ is spectrally disjoint with $O^{m_1,\dots,m_d}_{s'_1,\dots,s'_d}$.

Without loss of generality we may assume that there is $1 \le l_0 \le k$ such that $s_{l_0} \notin \{s'_1, \dots, s'_d\}$. Then we deduce from (4-3) and (4-4) that there is a sequence of integers $t_j \to \infty$ such that

$$\left(O^{m_1,\dots,m_d}_{s'_1,\dots,s'_d}\right)^{t_j} \to \frac{1}{(2r)^{m_1+\cdots+m_d}} I^{\otimes(m_1+\cdots+m_d)} \quad \text{and}$$

$$\left(O^{n_1,\dots,n_k}_{s_1,\dots,s_k}\right)^{t_j} \to \frac{1}{(2r)^{n_1+\cdots+n_k}} I^{\otimes(n_1+\cdots+n_{l_0-1})} \otimes U_T(s_{l_0})^{\odot n_{l_0}} \otimes I^{\otimes(n_{l_0+1}+\cdots+n_k)}$$

as $j \to \infty$, where r is the cardinality of the set $\{s_1, \dots, s_k, s'_1, \dots, s'_d\}$. Hence the unitary operators $O^{n_1,\dots,n_k}_{s_1,\dots,s_k}$ and $O^{m_1,\dots,m_d}_{s'_1,\dots,s'_d}$ are spectrally disjoint, as claimed.

Claim C. Let $0 < b \notin S$. Given two finite sequences $s_1 < \cdots < s_k$ and $s'_1 < \cdots < s'_d$ of elements from S and two integer vectors $(n_1, \dots, n_k) \in \mathbb{N}^k$ and $(m_1, \dots, m_d) \in \mathbb{N}^d$, the operators $O^{n_1,\dots,n_k}_{s_1,\dots,s_k}$ and $O^{m_1,\dots,m_d}_{bs'_1,\dots,bs'_d}$ are spectrally disjoint.

Indeed, we deduce from (4-3) and (4-5) that there is a sequence of integers $t_j \to \infty$ such that

$$\left(O^{n_1,\dots,n_k}_{s_1,\dots,s_k}\right)^{t_j} \to \frac{1}{(2r)^{n_1+\cdots+n_k}} I^{\otimes(n_1+\cdots+n_k)} \quad \text{and} \quad \left(O^{m_1,\dots,m_d}_{ts'_1,\dots,ts'_d}\right)^{t_j} \to 0$$

as $j \to \infty$, where r is the cardinality of the set $\{s_1, \dots, s_k, s'_1, \dots, s'_d\}$. The claim follows.

Now let $(Y, \nu) = (X, \mu) \times (S, \kappa)$, where κ is the *counting* measure on S. We define a flow $W = (W_t)_{t \in \mathbb{R}}$ on (Y, ν) by setting

$$W_t(x, s) := (T_{st}, s), \qquad x \in X, s \in S.$$

Then W preserves the σ-finite measure ν. This flow is not ergodic but every invariant subset is of either infinite or zero measure. The Koopman representation of $\mathbb{R}$ associated with W is $U_W = \bigoplus_{s \in S} U_T \circ c$. Let $\widetilde{W}$ denote the Poisson suspension of W. Since $U_{\widetilde{W}_1} = \exp(U_W(1)) \ominus \mathbb{C}$ (see e.g., [Ne]), we have

$$\begin{aligned} U_{\widetilde{W}_1} &= \left(\bigotimes_{s \in S} \exp(U_T(s)) \right) \ominus \mathbb{C} \\ &= \bigoplus_{k=1}^{\infty} \bigoplus_{s_1 < \cdots < s_k} \bigoplus_{n_1=1}^{\infty} \cdots \bigoplus_{n_k=1}^{\infty} O^{n_1,\dots,n_k}_{s_1,\dots,s_k}, \end{aligned} \tag{4-6}$$

where $s_1, \dots, s_k$ run over S. It now follows from Claims A and B that the operator $U_{\widetilde{W}_1}$ has a simple spectrum. Hence the flow $\widetilde{W}$ also has a simple spectrum. It is obvious that $S \subset I(\widetilde{W})$.

Now take a positive $b \notin S$. We are going to show that the Poisson flow $\widetilde{W} \circ b$ is spectrally disjoint with $\widetilde{W}$. For that it is enough to prove that the transformations $\widetilde{W}_b$ and $\widetilde{W}_1$ are spectrally disjoint. We have

$$\begin{aligned} U_{\widetilde{W}_b} &= \left(\bigotimes_{s \in S} \exp(U_T(bs)) \right) \ominus \mathbb{C} \\ &= \bigoplus_{k=1}^{\infty} \bigoplus_{s_1 < \cdots < s_k} \bigoplus_{n_1=1}^{\infty} \cdots \bigoplus_{n_k=1}^{\infty} O^{n_1,\dots,n_k}_{bs_1,\dots,bs_k}, \end{aligned} \tag{4-7}$$

where $s_1, \ldots, s_k$ run over S. It remains to compare (4-6) and (4-7) and apply Claim C.

To complete the proof of Theorem 4.4 we need to construct the dynamical system (X, μ, T) satisfying (i) and (ii). For that we use the cutting-and-stacking inductive construction of rank-one flows. The flow T will be a rank-one flow associated with a sequence $(r_n, \sigma_n)_{n=1}^{\infty}$. Thus our purpose is to define the sequence of cuts r_n and spacer maps $\sigma_n : \{1, \ldots, r_n\} \to \mathbb{R}_+$. For that we partition $\mathbb{N}$ into infinite subsets:

$$\mathbb{N} = \Big(\bigsqcup_{s \in S} \bigsqcup_{q \in \mathbb{N}} \bigsqcup_{i=1}^{2} \mathcal{L}_{s,q}^{i} \Big) \sqcup \Big(\bigsqcup_{k=1}^{\infty} \bigsqcup_{s_1 < \cdots < s_k} \bigsqcup_{l_0=1}^{k} \mathcal{M}_{s_1,\ldots,s_k}^{l_0} \Big),$$

where $s_1, \ldots, s_k$ run over S.

If for each $s \in S$ and $q \in \mathbb{N}$,

$$\text{(4-8)} \qquad \mathrm{WCP}(U_T(s)) \ni U_T(\sqrt{2}s) \quad \text{and} \quad \mathrm{WCP}(U_T(s)) \ni \frac{1}{q} I + \frac{q-1}{q} U_T(s)$$

then $U_T(s)$ has a simple spectrum by Lemma 4.7 and $\exp(U_T(s))$ has a simple spectrum by Lemma 4.5, i.e. (i) is satisfied. To achieve this, we put

- $r_n = n!$ and $\sigma_n(i) = \sqrt{2}s$ for all $1 \le i \le r_n$ for all $n \in \mathcal{L}_{s,q}^1$,
- $r_n = n!$ and $\sigma_n(i) = 0$ if $1 \le i < r_n/q$ and $\sigma_n(i) = s$ if $(1-q)r_n/q \le i \le r_n$ for all $n \in \mathcal{L}_{s,q}^2$.

A standard verification implies that $U_T(h_n) \to U_T(\sqrt{2}s)$ if $\mathcal{L}_{s,q}^1 \ni n \to \infty$ and $U_T(h_n) \to \frac{1}{q} I + \frac{q-1}{q} U_T(s)$ if $\mathcal{L}_{s,q}^2 \ni n \to \infty$, where h_n as usual denotes the hight of the n-th tower. We note that though T has not yet been defined entirely, these limits are well defined because they do not depend on the choice of r_n, σ_n when $n \notin \bigsqcup_{s \in S} \bigsqcup_{q \in \mathbb{N}} (\mathcal{L}_{s,q}^1 \sqcup \mathcal{L}_{s,q}^2)$. Thus, (4-8), and hence (i), is satisfied.

To realize (ii) we fix a finite sequence $s_1 < s_2 < \cdots < s_k$ of elements in S and $1 \le l_0 \le k$. Enumerate the elements of $\mathcal{M}_{s_1,\ldots,s_k}^{l_0}$ in ascending order: $n_1 < n_2 < \cdots$. We now let $r_{n_j} := 2k$ for all j. Instead of writing precise formulas for the spacer maps $\sigma_{n_j} : \{1, \ldots, 2k\} \to \mathbb{R}_+$ we illustrate the idea of the construction with the following picture of the (n_j+1)-th tower in this subsequence (see Figure 4.1). To be specific, we choose $k = 3$ and $l_0 = 2$. Since the tower is very "high", we place it horizontally.

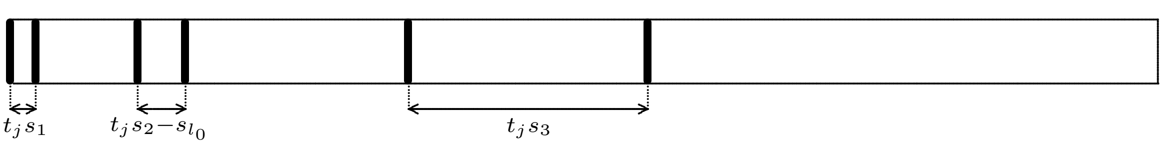

FIGURE 4.1. $(n_j + 1)$-th tower.

The black stripes here are the copy of the n_j-th tower. They are very "thing" because we choose the parameter $t_j \in \mathbb{N}$ very large. It is easy to see that (4-3) and (4-4) hold. Denote by $a_{j,i}$ the distances between the $2i$-th and $(2i+1)$-th copies of the n_j-th tower in the (n_j+1)-th tower, $j = 1, 2$. Let $a_{j,3}$ be the distance between the 6-th copy and the top of the (n_j+1)-th tower. We arrange the spacers in the (n_j+1)-th tower in such a way that $t_j s_1 \ll a_{j,1} \ll a_{j,2} \ll a_{j,3}$, where the sign "$\ll$" means *grows much faster* as $j \to \infty$. Then (4-5) follows. $\square$

Remark 4.8. It is also possible to construct a mixing Poisson (and a mixing Gaussian) flow $\widetilde{W}$ satisfying the conditions of Theorem 4.4. For that one should adapt the technique of *forcing of mixing* from our previous paper [DaR] (see also [Ry5]), devoted to spectral multiplicities of mixing infinite measure preserving $\mathbb{Z}$-actions, to the setup of infinite measure preserving flows. We do not see any principal difficulty in such an adaptation. However we do not go into details here because this technique is a bit laborious.

We now sharpen Lemma 3.1.

Corollary 4.9. *Let $\mathcal{P}_C^0$ denote the set of continuous, fully supported measures σ on $\mathbb{R}$ such that $(\sigma^p)_t \perp \sigma^q$ for all $t > 0$, $t \neq 1$, and $p, q \in \mathbb{N}$, where the upper indices p, q denote the convolution powers. Then $\mathcal{P}_C^0$ is a dense G_δ in $\mathcal{P}$.*

Proof. We first recall a well known fact that the fully supported continuous measures on $\mathbb{R}$ form a G_δ subset in $\mathcal{P}$ (see, e.g. [Na]). Since the maps $\mathcal{P} \ni \sigma \mapsto \sigma^p \in \mathcal{P}$ are continuous for all $p \in \mathbb{N}$, we can argue as in the proof of lemma 3.1 to show that $\mathcal{P}_C^0$ is a G_δ. It follows from the proof of Theorem 4.4 that a measure of maximal spectral type of U_W belongs to $\mathcal{P}_C^0$. It remains to note that the equivalence class of every fully supported non-atomic measure is dense in $\mathcal{P}$. □

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