RUMOUR PROCESSES ON ℕ

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ABSTRACT. We study four discrete time stochastic systems on ℕ modeling processes of rumour spreading. The involved individuals can either have an active or a passive role, speaking up or asking for the rumour. The appetite in spreading or hearing the rumour is represented by a set of random variables whose distributions may depend on the individuals. Our goal is to understand - based on those random variables distribution - whether the probability of having an infinite set of individuals knowing the rumour is positive or not.

1. INTRODUCTION

Until a few decades ago, epidemic and rumour models where treated under the same class of models. While there is a clear similitude among the status of the individuals in the models (susceptible are ignorants, immunes are stiflers and infected are spreaders) the rates at which individuals change their status might be qualitatively different (Pearce [16]). Generally speaking, the production of stiflers is definitely more complex than the production of immune individuals.

Lately the mathematics of rumors has observed a good deal of interest. The focus used to be at deterministic or stochastic models,

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modeling homogeneously mixed populations living on spaces with no structure as the Maki-Thompson (Maki and Thompson [15] and Sudbury [18]) and Daley-Kendall (Daley and Kendal [5] and Pittel [17]) models. Among the possible variations one can find in recent literature are competing rumours (Kostka *et al* [11]), more than two people meeting at a time (Kesten and Sidoravicius [10]), moving agents (Kurtz *et al* [12]) and rumours through tree-like graphs (Lebensztayn and Rodriguez [14] and Lebensztayn *et al* [13]), complex networks (Isham *et al* [9]), grids (Roy *et al* [1]) and multigraphs (Bertachi and Zucca [2]).

Still, the most important question for both models, epidemic and rumour, is in terms of a rumour model, if a spreader (an individual who wants to see the rumour spread) is introduced into a reservoir of ignorants under what conditions the rumour will spread to a large proportion of the population, instead of dying out quickly without having done so. Another important question is, if it does not dies out quickly, what is the final proportion of individuals hit by the rumour?

We study discrete time stochastic systems on $\mathbb{N} = \{0, 1, 2, \dots\}$ which dynamic is as follows. First, consider that at time zero all vertices of $\mathbb{N}$ are declared inactive, except for the origin, which is active. It instantly exerts influence on its neighbors vertices, activating a contiguous random set of them placed on its right. In general, that is the behavior of every vertex in case it is activated.

We take into account an *homogeneous* and an *heterogeneous* versions for what we call the *radius of influence* of a vertex. In the *homogeneous* version, as a rule, the next moment to what it has been activated, each active vertex carries the same (random) behavior of the origin, independent of it and of everything else. We also deal with an *heterogeneous* version where each vertex, if activated, has a distinct distribution for its radius of influence.

We say that the process *survives* if the amount of vertices activated is infinite. Otherwise we say the process *dies out*. We call this the *Firework Process*, associating the activation dynamic of a vertex to a rumour process. Vertices become spreaders as soon as they are activated. Next time, they propagate the rumour and immediately become stiflers.

A possible variation is what we call *Reverse Firework Process*. In this variation a vertex, instead of being hit by a rumour, defines a set of neighbors on its left to which it asks once someone in this set hears the rumour. We call this variation *Reverse Firework Process*. We also deal with an *homogeneous* and an *heterogeneous* versions of this variation. The models are shown to be qualitatively different in some pertinent cases.

Our main interest is to establish whether each process has positive probability of survival which is equivalent to a rumour propagation.

This is done according to the distribution of the random variable that defines the radius of influence of each active vertex.

The paper is organized as follows. Section 2 presents the main results. Section 3 brings the proofs for the main results together with auxiliary lemmas and handy inequalities. In Section 4 we present examples where some conditions can be verified.

2. MAIN RESULTS

2.1. **Firework Process.** Consider $\{u_i\}_{i\in\mathbb{N}}$ a set of vertices of $\mathbb{N}$ such that $0 = u_0 < u_1 < u_2 < \cdots$ and a set of independent random variables $\{R_i\}_{i\in\mathbb{N}}$ assuming values in $\mathbb{R}_+$ whose joint distribution is $\mathbb{P}$. The *Firework Process* can be formally defined in the following way. At time 0, an explosion of size R_0 comes from the origin, activating all vertices $u_i \leq R_0$. As a rule, at every discrete time t all vertices u_j activated at time $t - 1$ generate their explosions (whose *radius of influence* is R_j), and they do this just once, activating the vertices u_i (only those which has not been activated before) such that $u_j < u_i \leq u_j + R_j$. Observe that except for the set of vertex $\{u_i\}$, all others vertices are non-actionable, meaning that the random variable associated to them is 0 almost surely.

If for all u_j activated at time $t - 1$ there are no vertices u_i in this latter condition the process *dies out*. That means the rumour reaches only a finite amount of individuals. If, on the contrary the process

never stops, we say it *survives*, meaning that the rumour reaches an infinity number of individuals. We call the process *homogeneous* if all R_i have the same distribution and $u_i = i$ for all i. Otherwise we call it *heterogeneous*. We focus to the cases $\mathbb{P}(R_i < 1) \in (0, 1)$ for all i.

Let us consider the following events

- $V_n = $ *the vertex u_n is hit by an explosion,*
- $V = \lim_{n \to \infty} V_n.$

2.1.1. *The Homogeneous case.*

Theorem 2.1. *For the Homogeneous Firework Process, consider*

$$a_n = \prod_{i=0}^{n} \mathbb{P}(R < i + 1).$$

Then

$$\sum_{n=1}^{\infty} a_n = \infty \ \text{if and only if } \mathbb{P}[V] = 0.$$

Besides

$$\mathbb{P}(V) \ \geq \prod_{j=0}^{\infty} \left[1 - \prod_{i=0}^{j} \mathbb{P}(R < i + 1) \right], \tag{2.1}$$

$$\mathbb{P}(V) \ \leq 1 - \mathbb{P}(R = 0) - \sum_{k=1}^{n} \left[\mathbb{P}(R = k) \prod_{j=0}^{k-1} \mathbb{P}[R \leq j] \right]. \tag{2.2}$$

Corollary 2.2. *For the Homogeneous Firework Process, consider*

$$L = \lim_{n \to \infty} n \mathbb{P}(R \geq n).$$

We have that

(I) If $L > 1$ then $\mathbb{P}[V] > 0$.

(II) If $L < 1$ then $\mathbb{P}[V] = 0$.

(III) If $L = 1$ and there exists N such that for all $n \geq N$

$$\mathbb{P}(R \geq n) \leq \frac{1}{n-1}, \;\; then \; \mathbb{P}[V] = 0.$$

Remark 2.3. Consider a *Homogeneous Firework Process* with R assuming values on $\mathbb{N}$. Observe that, in this case, if $\mathbb{E}[R] < \infty$ then $L = 0$. Consequently for R assuming values on $\mathbb{N}$,

$$\mathbb{E}[R] < \infty \Rightarrow \mathbb{P}[V] = 0.$$

Next result gives a criteria for the case when the distribution of the random variable R is a power law.

Corollary 2.4. *Let $\alpha > 1$ and Z_α be an appropriate constant. Consider the Homogeneous Firework Process such that*

$$\mathbb{P}(R = k) = \frac{Z_\alpha}{(k+1)^\alpha} \; for \; k \in \mathbb{N}. \tag{2.3}$$

(I) If $\alpha < 2$ then $\mathbb{P}[V] > 0$.

(II) If $\alpha \geq 2$ then $\mathbb{P}[V] = 0$.

Remark 2.5. Observe that for the *Homogeneous Firework Process* if R has a power law distribution as in (2.3), with $\alpha = 2$, we have that

$$\mathbb{E}[R] = \infty \text{ and } \mathbb{P}[V] = 0.$$

2.1.2. *The Heterogeneous case.*

Remark 2.6. Consider the *Heterogeneous Firework Process.* One can get a sufficient condition for $\mathbb{P}[V] = 0$ ($\mathbb{P}[V] > 0$) by a coupling argument. Consider $\mathbb{P}(R_i \geq k) \leq \mathbf{P}(R \geq k)$ ($\mathbb{P}(R_i \geq k) \geq \mathbf{P}(R \geq k)$) for some random variable R which distribution $\mathbf{P}$ satisfies $\lim_{n\to\infty} n\mathbf{P}(R \geq n) < 1$ ($\lim_{n\to\infty} n\mathbf{P}(R \geq n) > 1$). Finally use part (II) (part (I)) of Corollary 2.2.

Theorem 2.7. *Consider a Heterogeneous Firework Process which actionable vertices are at integer positions $0 = u_0 < u_1 < u_2 < \ldots$ such that $u_{n+1} - u_n \leq m$, for $m \geq 1$. Besides, let us assume $\mathbb{P}(R_n < m) \in (0,1)$ for all n.*

(I) *If $\sum_{n=0}^{\infty}[\mathbb{P}(R_n < tm)]^t < \infty$ for some $t \geq 1$ then $\mathbb{P}[V] > 0$.*

(II) *If for some random variable R, which distribution is $\mathbf{P}$, the following conditions hold*

- $\mathbf{P}(R \geq k) - \mathbb{P}(R_n \geq k) \leq b_k$ *for all $k \geq 0$ and all $n \geq 0$,*
- $\lim_{n\to\infty} n[\mathbf{P}(R \geq n) - b_n] > m$,
- $\lim_{n\to\infty} b_n = 0$.

Then $\mathbb{P}[V] > 0$.

(III) $\mathbb{P}(V) \geq \prod_{j=0}^{\infty} \left[1 - \prod_{i=0}^{j} \mathbb{P}(R_{j-i} < (i+1)m) \right].$

2.2. **Reverse Firework Process.** Consider $\{u_i\}_{i\in\mathbb{N}}$ a set of vertices of $\mathbb{N}$ such that $0 = u_0 < u_1 < u_2 < \cdots$ and a set of independent random

variables $\{R_i\}_{i\in\mathbb{N}}$ assuming values in $\mathbb{N}$ which joint distribution is $\mathbb{P}$. The *Reverse Firework Process* can be defined as follows. At time 0, only the origin is activated. At time 1, explosions of size R_i towards the origin, come from all vertices of $\{u_i\}_{i\in\mathbb{N}}$. All vertices $u_i \leq R_i$ are activated. As a rule, at discrete times t the set of vertices u_j which can find a vertex activated at time $t-1$ within a distance R_j to its left, are activated. Let us call this set A_t. If for some t, A_t is empty the process stops. If the process never stops we say it survives. We call the process *homogeneous* if all R_i have the same distribution and $u_i = i$ for all i, otherwise we call it *heterogeneous*. We focus to the cases $\mathbb{P}(R_i < 1) \in (0,1)$ for all i. Unless stated differently, we assume $u_i = i$ for all i.

Let S be the event *"the reverse process survives"*.

2.2.1. *The homogeneous case.*

Theorem 2.8. *Consider the Reverse Homogeneous Firework Process. We have that*

    (I) If $\mathbb{E}(R) = \infty$ then $\mathbb{P}(S) = 1$.

    (II) If $\mathbb{E}(R) < \infty$ then $\mathbb{P}(S) = 0$.

Remark 2.9. For a random variable R, having a power law distribution as in (2.3), we have that

    • if $1 < \alpha \leq 2$ then $\mathbb{E}[R] = \infty$,

- if $\alpha > 2$ then $\mathbb{E}[R] < \infty$.

In conclusion, if R has a power law distribution as in (2.3), with $\alpha = 2$, then $\mathbb{P}[V] = 0$ for the *Homogeneous Firework Process* by Remark 2.5 and $\mathbb{P}[S] = 1$ for the *Reverse Homogeneous Firework Process*.

2.2.2. *The heterogeneous case.*

Theorem 2.10. *Consider the Reverse Heterogeneous Firework Process. It holds that*

(I) $\sum_{k=1}^{\infty} \mathbb{P}(R_{n+k} \geq k) = \infty$ *for all n if and only if $\mathbb{P}(S) = 1$.*

(II) *If* $\sum_{n=1}^{\infty} \prod_{k=1}^{\infty} \mathbb{P}(R_{n+k} < k) < \infty$ *then* $\mathbb{P}(S) > 0$.

Remark 2.11. Let $\rho = \sum_{n=1}^{\infty} \prod_{k=1}^{\infty} \mathbb{P}(R_{n+k} < k)$. Observe now that Theorem 2.10 gives no additional information for Theorem 2.8, as in the homogeneous case ρ equals either 0 ($\mathbb{E}[R] = \infty$) or ∞ ($\mathbb{E}[R] < \infty$).

Remark 2.12. By a coupling argument and Theorem 2.8 one can see that if there is a random variable R, which distribution is $\mathbf{P}$, with $\mathbb{E}[R] < \infty$ ($\mathbb{E}[R] = \infty$), such that $\mathbb{P}(R_n \geq k) \leq \mathbf{P}(R \geq k)$ ($\mathbb{P}(R_n \geq k) \geq \mathbf{P}(R \geq k)$) for all k then $\mathbb{P}(S) = 0$ ($\mathbb{P}(S) = 1$).

3. Proofs

Next we present some basic facts, starting from the Raabes test (Fort [7, p. 32] or Bonar and Khoury [3, p. 48]).

Raabes Test. *For $a_n > 0$, let us define*

$$L = \lim_{n \to \infty} n \left(\frac{a_n}{a_{n+1}} - 1 \right).$$

Then

- *If $L > 1$ then $\sum_{n=1}^{\infty} a_n < \infty$.*

- *If $L < 1$ then $\sum_{n=1}^{\infty} a_n = \infty$.*

- *If $L = 1$ and $n\left(a_n/a_{n+1} - 1 \right) \leq 1$, for n large enough,*

  then $\sum_{n=1}^{\infty} a_n = \infty$.

The following result (Bremaud [4, p. 422]) is useful for what comes

next

Lemma 3.1. *Let $\{a_n\}_{n \geq 1}$ be a sequence of real numbers in $(0, 1)$. Then,*

$$\prod_{i=0}^{\infty}(1 - a_k) = 0 \Leftrightarrow \sum_{i=0}^{\infty} a_k = \infty. \tag{3.1}$$

Remark 3.2. Consider that the actionable vertices are at integer positions $0 = u_0 < u_1 < u_2 < \ldots$ such that $u_{n+1} - u_n \leq m$, for $m \geq 1$. From the definition of V_n one can see that

- $V_{k+1} \supset V_k \bigcap \left\{ \bigcup_{i=0}^{k}(R_{k-i} \geq (i+1)m) \right\}$,

- V_k e $\bigcup_{i=0}^{k}(R_{k-i} \geq (i+1)m)$ are increasing events,

- $\mathbb{P}(V_n) > 0$ for all n.

From FKG inequality (Grimmett [8, p.34]) we can assure that

$$\mathbb{P}(V_{k+1}) \geq \mathbb{P}(V_k \cap \{\cup_{i=0}^{k}(R_{k-i} \geq (i+1)m)\}) \tag{3.2}$$

$$\geq \mathbb{P}(V_k)\mathbb{P}(\{\cup_{i=0}^{k}(R_{k-i} \geq (i+1)m)\})$$

$$= \mathbb{P}(V_k)\Big[1 - \prod_{i=0}^{k}\mathbb{P}(R_{k-i} < (i+1)m)\Big]$$

and then

$$\mathbb{P}(V_n) \geq \prod_{j=0}^{n-1}\Big[1 - \prod_{i=0}^{j}\mathbb{P}(R_{j-i} < (i+1)m)\Big].$$

Therefore

$$\mathbb{P}(V) \geq \prod_{j=0}^{\infty}\Big[1 - \prod_{i=0}^{j}\mathbb{P}(R_{j-i} < (i+1)m)\Big]. \tag{3.3}$$

Inequality (3.2) becomes an equality if $u_i = mi$ for all $i \in \mathbb{N}$ and some $m \in \mathbb{N}$. From the latter set of displays and (3.1) follows next proposition.

Proposition 3.3. *Consider a Heterogeneous Firework Process which actionable vertices are at integer positions $0 = u_0 < u_1 < u_2 < \cdots$ such that $u_{n+1} - u_n \leq m$. Let $a_n = \prod_{i=0}^{n}\mathbb{P}(R_{n-i} < (i+1)m)$ and assume $\mathbb{P}(R_i < m) \in (0,1)$.*

$$\textit{If } \sum_{n=0}^{\infty} a_n < \infty \textit{ then } \mathbb{P}[V] > 0. \tag{3.4}$$

3.1. **Firework Process.**

Proof of Theorem 2.1. Assume $\sum_{n=0}^{\infty} a_n < \infty$. From Proposition 3.3, with $m = 1$, we have that $\mathbb{P}[V] > 0$.

Assume now $\sum_{n=0}^{\infty} a_n = \infty$. First consider the event

$$C = \{\exists n \text{ such that } \forall u_i > n \ \exists x \text{ such that } x < u_i \le x + R_x\}.$$

In words that means that from some position on, all vertex belong to the radius of influence of some other vertices. Those later vertices not necessarily have been activated.

Next, consider the following event

$$B(u_n) = \{u_n > x + R_x, \text{ for all } x < u_n\}$$

In words, the vertex u_n does not belong to the radius of influence of any vertex to its left.

Assuming all independent random variables having the same distribution as R and that $u_i = i$ $(B_n = B(u_n))$,

$$\mathbb{P}(B_n) = \mathbb{P}\left(\bigcap_{i=1}^{n}[R_{n-i} < i]\right) = \prod_{i=1}^{n}\mathbb{P}(R < i) = a_{n-1}.$$

Conditional independence of the B_is as stated next, for $i > j$

$$
\begin{aligned}
\mathbb{P}(B_i \cap B_j) &= \mathbb{P}\left(\bigcap_{k=1}^{i-j}[R_{i-k} < k] \cap \bigcap_{k=1}^{j}[R_{j-k} < k]\right) \\
&= \prod_{k=1}^{i-j}\mathbb{P}(R < k)\prod_{k=1}^{i}\mathbb{P}(R < k) \\
&= \mathbb{P}(B_{i-j})\mathbb{P}(B_j)
\end{aligned}
$$

makes the B_is satisfy the definition of a renewal event in [6, p.308]. So, from the fact that $\sum_{n=1}^{\infty} \mathbb{P}(B_n) = \infty$, one can rely on Theorem 2 of Section XIII.3 of [6, p.312] to see that

$$\mathbb{P}(B_n \text{ infinitely often }) = 1.$$

From this we conclude that $\mathbb{P}[V] = 0$, as

$$V^c \supset C^c \supset \{B_n \text{ infinitely often }\}.$$

Inequality (2.1) follows from (3.3) and inequality (2.2) follows from the fact that

$$V^C \supseteq \bigcup_{k=0}^{\infty} \left[R_0 = k, \bigcap_{j=1}^{k} [R_j \leq k - j] \right].$$

$\square$

Proof of Corollary 2.2. Observe that, as $a_n = \prod_{i=0}^{n} \mathbb{P}(R < i + 1)$

$$\frac{a_n}{a_{n+1}} - 1 = \frac{\mathbb{P}(R \geq n + 2)}{\mathbb{P}(R < n + 2)}.$$

Therefore

$$\lim_{n \to \infty} n \left(\frac{a_n}{a_{n+1}} - 1 \right) = \lim_{n \to \infty} n \mathbb{P}(R \geq n). \tag{3.5}$$

From (3.5), Raabes Test and Theorem 2.1, follow (I), (II) and (III).

$\square$

Proof of Corollary 2.4. Observe that

$$\frac{1}{(\alpha - 1)(n + 1)^{\alpha - 1}} = \int_{n+1}^{\infty} \frac{1}{x^{\alpha}} dx \leq \sum_{j=n+1}^{\infty} \frac{1}{j^{\alpha}}$$

$$\leq \int_{n+1}^{\infty} \frac{1}{(x - 1)^{\alpha}} dx = \frac{1}{(\alpha - 1)n^{\alpha - 1}}.$$

Then

$$\frac{Z_{\alpha}}{(\alpha - 1)} \frac{1}{(n + 1)^{\alpha - 1}} \leq \mathbb{P}(R \geq n) \leq \frac{Z_{\alpha}}{(\alpha - 1)} \frac{1}{n^{\alpha - 1}}.$$

Consequently

$$\lim_{n\to\infty} n\mathbb{P}(R \geq n) = \begin{cases} +\infty & \text{if } \alpha < 2, \\ 6/\pi^2 & \text{if } \alpha = 2, \\ 0 & \text{if } \alpha > 2. \end{cases}$$

From Corollary 2.2, the conclusion follows. $\qquad\square$

Proof of Theorem 2.7

Let

$$a_n = \prod_{j=0}^{n} \mathbb{P}(R_{n-j} < (j+1)m).$$

Proof of (I). As

$$\sum_{n=t}^{\infty} [\mathbb{P}(R_n < tm)]^t < \infty$$

implies that

$$\sum_{n=t}^{\infty} \left[\max_{j\in\{0,\dots,t-1\}} \{\mathbb{P}(R_{n-j} < tm)\} \right]^t < \infty,$$

and as

$$a_n \leq \prod_{j=0}^{t-1} \mathbb{P}(R_{n-j} < tm) \leq \left[\max_{j\in\{0,\dots,t-1\}} \{\mathbb{P}(R_{n-j} < tm)\} \right]^t,$$

the series which terms are a_n converges. So we can use (3.4) in order

to get the result.

Proof of (II). Let

$$r_n = \prod_{j=0}^{n} [\mathbf{P}(R < (j+1)m) + b_{(j+1)m}].$$

As

$$n\left(\frac{r_n}{r_{n+1}} - 1\right) = \frac{n[\mathbf{P}(R \geq (n+2)m) - b_{(n+2)m}]}{\mathbf{P}(R < (n+2)m) + b_{(n+2)m}},$$

from the hypothesis

$$\lim_{n \to \infty} n\left(\frac{r_n}{r_{n+1}} - 1\right) > 1.$$

But $a_n \leq r_n$, therefore the series which terms are a_n is convergent and so we can use Proposition 3.3 to get the desired result.

Proof of (III). It follows from (3.3). □

3.2. **Reverse Firework Process.** First consider the following variation of the *Homogeneous Firework Process*. Instead of having just the origin activated at time zero, we consider that all vertices to its left are also activated at time zero. The set of independent random variables which defines the radius of influence of all vertices is $\{F_i\}_{i \in \mathbb{Z}}$, all having the same distribution as R, the random variable which defines the *Reverse Homogeneous Firework Process*.

For this variation of the *Homogeneous Firework Process* let us define the following events

- $\mathcal{V}_n = $ *the vertex n is hit by an explosion,*
- $\mathcal{V} = $ *the process survives.*

By analogy *"to survive"* in this variation means to hit infinitely many vertices of $\mathbb{N}$. It follows that

$$\mathcal{V} = \bigcap_{n=0}^{\infty} \bigcup_{j=0}^{\infty} [F_{n-j} \geq j+1]. \tag{3.6}$$

Proposition 3.4. *If* $\mathbb{E}(R) < \infty$, *then* $\mathbb{P}(\mathcal{V}) = 0$.

Proof of Proposition 3.4 Let us define the following events

$$\mathcal{A}_n = \bigcup_{i=-\infty}^{n-1} \{F_i \geq 2n - i\}$$

and

$$\mathcal{B}_n = \bigcup_{i=n}^{2n-1} \{F_i \geq 2n - i\}.$$

Observe that

$$\mathcal{V}_{2n} \subseteq \mathcal{V}_n \cap [\mathcal{A}_n \cup \mathcal{B}_n].$$

Therefore

$$\mathbb{P}(\mathcal{V}_{2n}) \leq \mathbb{P}(\mathcal{A}_n) + \mathbb{P}(\mathcal{B}_n)\mathbb{P}(\mathcal{V}_n).$$

Now

$$
\begin{aligned}
\mathbb{P}(\mathcal{A}_n) &\leq \sum_{i=-\infty}^{n-1} \mathbb{P}(F_i \geq 2n - i) \\
&= \sum_{i=n+1}^{\infty} \mathbb{P}(F_{2n-i} \geq i) \\
&= \sum_{i=n+1}^{\infty} \mathbb{P}(R \geq i) \longrightarrow 0,
\end{aligned}
$$

and

$$\mathbb{P}(\mathcal{B}_n) = \mathbb{P}\left(\bigcup_{i=n}^{2n-1} \{F_i \geq 2n - i\}\right)$$

$$= 1 - \prod_{i=n}^{2n-1} \mathbb{P}(F_i < 2n - i)$$

$$\leq 1 - \prod_{i=1}^{\infty} \mathbb{P}(R < i).$$

Then, (3.1) and $\mathbb{E}(R) < \infty$ guarantee the existence of $\lambda \in (0,1)$ such that

$$\mathbb{P}(\mathcal{B}_n) \leq \lambda$$

for all n. So, as for the homogeneous case $\mathbb{P}(\mathcal{A}_n) \geq \mathbb{P}(\mathcal{A}_{n+1})$,

$$\lim_{n \to \infty} \mathbb{P}(\mathcal{V}_n) = 0$$

and this implies that $\mathbb{P}(\mathcal{V}) = 0$ as $\mathcal{V}_{n+1} \subset \mathcal{V}_n$. $\qquad\square$

Proof of Theorem 2.8

Let $\{R_i\}_{i \in \mathbb{N}}$ independent random variables distributed as R. Observe that

$$S = \bigcap_{n=0}^{\infty} \bigcup_{j=1}^{\infty} [R_{n+j} \geq j]. \tag{3.7}$$

By using FKG inequality (Grimmett [8, p.34]) and the fact that intersections of increasing events is an increasing event, we have that

$$\mathbb{P}\left(\bigcap_{n=0}^{n_0} \bigcup_{j=1}^{\infty} [R_{n+j} \geq j]\right) \geq \prod_{n=0}^{n_0} \mathbb{P}\left(\bigcup_{j=1}^{\infty} [R_{n+j} \geq j]\right)$$

for all n_0. Taking the limit $n_0 \to \infty$, by the continuity of probability

$$\mathbb{P}\left(\bigcap_{n=0}^{\infty}\bigcup_{j=1}^{\infty}[R_{n+j}\geq j]\right)\geq\prod_{n=0}^{\infty}\mathbb{P}\left(\bigcup_{j=1}^{\infty}[R_{n+j}\geq j]\right).$$

Therefore

$$\mathbb{P}(S)\geq\prod_{n=0}^{\infty}\left[1-\prod_{j=1}^{\infty}[1-\mathbb{P}(R_{n+j}\geq j)]\right]. \tag{3.8}$$

Proof of (I). From the hypothesis

$$\sum_{j=1}^{\infty}\mathbb{P}(R\geq j)=\infty. \tag{3.9}$$

Now, (3.1) and (3.9) implies that

$$\prod_{j=1}^{\infty}[1-\mathbb{P}(R\geq j)]=0,$$

and $\mathbb{P}(S)=1$ follows by (3.8).

Proof of (II). By Proposition 3.4, (3.6) and the fact that R_i and F_i have the same distribution

$$\mathbb{P}\left(\bigcap_{n=0}^{\infty}\bigcup_{j=0}^{\infty}[R_{n-j}\geq j+1]\right)=0. \tag{3.10}$$

By the other hand, as R_i are all distributed as R

$$\mathbb{P}\left(\bigcap_{n=0}^{\infty}\bigcup_{j=0}^{\infty}[R_{n-j}\geq j+1]\right)=\mathbb{P}\left(\bigcap_{n=0}^{\infty}\bigcup_{j=0}^{\infty}[R_{n+j+1}\geq j+1]\right),$$

and therefore, by (3.7) and (3.10), $\mathbb{P}(S)=0$. $\qquad\square$

Proof of Theorem 2.10

Proof of (I). Assuming that $\sum_{k=1}^{\infty} \mathbb{P}(R_{n+k} \geq k) = \infty$ for all n and considering (3.1), one can see that $\prod_{k=1}^{\infty}[1 - \mathbb{P}(R_{n+k} \geq k)] = 0$ for all n. Therefore, by (3.8), $\mathbb{P}(S) = 1$. By the other side, as $\mathbb{P}(S) \leq 1 - \prod_{k=1}^{\infty} \mathbb{P}(R_{n+k} < k)$ for all n, if $\mathbb{P}(S) = 1$ we have that $\prod_{k=1}^{\infty}[1 - \mathbb{P}(R_{n+k} \geq k)] = 0$ for all n. Now, from (3.1), $\sum_{k=1}^{\infty} \mathbb{P}(R_{n+k} \geq k) = \infty$ for all n.

Proof of (II). From $\sum_{n=1}^{\infty} \prod_{k=1}^{\infty} \mathbb{P}(R_{n+k} < k) < \infty$, follows that, by the use of (3.1), $\prod_{n=0}^{\infty}[1 - \prod_{k=1}^{\infty}[1 - \mathbb{P}(R_{n+k} \geq k)]] > 0$. Then, by (3.8) we have that $\mathbb{P}(S) > 0$. $\qquad\square$

4. Final Remarks and Examples

We consider two discrete propagation phenomena modeling in their homogeneous and heterogeneous versions. While the *Firework Process* models a phenomena where there is at all times a finite number of individuals trying to spread an information for an infinite group of individuals, the *Reverse Firework Process* models a phenomena where there is always an infinite number of individuals willing and working towards to heard about that information from a finite quantity of informed individuals. Our results show that the four versions are qualitatively different.

Considering the *Homogeneous Firework Process*, Remark 2.3 shows that the information will no be spread for an infinite number of individuals if $\mathbb{E}[R]$ is finite. To have a *radius of influence* with infinite

expectation is also no guarantee for the information to reach an infinite number of individuals, as Remark 2.5 shows. Besides, the probability of not reaching an infinite amount of individuals is at least $\mathbb{P}[R = 0]$. Conversely, in the *Heterogeneous Firework Process*, to have an infinite expectation guarantees almost surely that the information will spread out among an infinite amount of individuals, as Theorem 2.8 points out. Furthermore in the case where the *radius of influence* has a power law distribution as in (2.3), the process works in opposite direction as Remark 2.9 shows for $\alpha = 2$. The processes agree for R whose expectation is finite. Next we present some final examples pointing to some extreme cases.

Let $\{b_n\}_{n \in \mathbb{N}}$ be a non-increasing sequence convergent to 0 and such that $b_0 < 1$.

Example 4.1. It is possible to have in the *Heterogeneous Firework Process* the expectation of the radius of influence infinite for all vertices together and the process dies out almost surely.

(i) $\mathbb{P}(R_n = 0) = 1 - b_n$ and $\mathbb{P}(R_n = k) = b_{n+k-1} - b_{n+k}$ for $k \geq 1$.

(ii) $\sum_{n=0}^{\infty} b_n = \infty$.

(iii) $\lim_{n \to \infty} n b_n = 0$.

Observe that $\mathbb{E}(R_n) = \infty$ for all n from *(ii)*. Besides $\mathbb{P}[V] = 0$ from *(iii)*, because

$$\mathbb{P}(V_n) \leq \sum_{k=0}^{n-1} \mathbb{P}(R_k \geq n - k) = \sum_{k=0}^{n-1} b_{n-1} = (n-1)b_n. \qquad (4.1)$$

Example 4.2. It is possible to have in the *Heterogeneous Firework Process* the expectation of the radius of influence finite for all vertices and the process survives with positive probability. Assume that $\sum_{n=0}^{\infty} b_n < \infty$, while

- $\mathbb{P}(R_n = 0) = b_n$
- $\mathbb{P}(R_n = 1) = 1 - b_n$

Then $\mathbb{E}(R_n) < 1$ for all n and $\mathbb{P}(V) > 0$ by part (I) of Theorem 2.7 with $m = t = 1$.

Example 4.3. Next we present an example where $\mathbb{P}[S] = 1$ for a *Reverse Heterogeneous Firework Process* while $\mathbb{P}[V] = 0$ for a *Heterogeneous Firework Process*. For this aim consider

(i) $\mathbb{P}(R_n = 0) = 1 - b_n$ and $\mathbb{P}(R_n = n) = b_n$.

(ii) $\sum_{n=0}^{\infty} b_n = \infty$.

(iii) $\lim_{n \to \infty} n b_n = 0$.

Observe that even though $\lim_{n \to \infty} \mathbb{E}[R_n] = 0$ and $\lim_{n \to \infty} \mathbb{P}(R_n = 0) = 1$, from Theorem 2.10 and *(ii)* it is true for the *Reverse Heterogeneous Firework Process* that $\mathbb{P}(S) = 1$. In the opposite direction, by (4.1) and *(iii)* one have that $\mathbb{P}[V] = 0$ for the *Heterogeneous Firework Process*.

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