Finite dimensional reduction and convergence to equilibrium for incompressible Smectic-A liquid crystal flows

Antonio SEGATTI * and Hao WU†

Abstract

We consider a hydrodynamic system that models the Smectic-A liquid crystal flow. The model consists of the Navier-Stokes equation for the fluid velocity coupled with a fourth-order equation for the layer variable φ, endowed with periodic boundary conditions. We analyze the long-time behavior of the solutions within the theory of infinite-dimensional dissipative dynamical systems. We first prove that in $2D$, the problem possesses a global attractor $\mathcal{A}$ in certain phase space. Then we establish the existence of an exponential attractor $\mathcal{M}$ which entails that the global attractor $\mathcal{A}$ has finite fractal dimension. Moreover, we show that each trajectory converges to a single equilibrium by means of a suitable Lojasiewicz–Simon inequality. Corresponding results in $3D$ are also discussed.

Keywords: Smectic-A liquid crystal flow, Navier–Stokes equations, global attractor, exponential attractor, convergence to equilibrium.
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1 Introduction

Smectic liquid crystal is in a liquid crystalline phase, which possesses not only some degree of orientational order like the nematic liquid crystal, but also some degree of positional order (layer structure). The local orientation of the liquid crystal molecules is usually denoted by a director field $\mathbf{d}$. In the nematic state, molecules tend to align themselves along a preferred direction with no positional order of centers of mass. In the smectic phase, molecules organize themselves into layers that are nearly incompressible and of near constant width [6]. The layers are characterized by the iso-surfaces of a scalar function φ. A key property that distinguishes the smectic-A liquid crystals is that, the molecules tend to align themselves along the direction perpendicular to the layers. The study on the continuum theory for the smectic-A phase has a long history, see for instance, [4, 5, 16, 27]. A general nonlinear continuum theory for smectic-A liquid crystals applicable to situations with large deformations and non-trivial flows was established by E in [7]. In [7], the following hydrodynamic system was proposed

$$\rho_t + \mathbf{v} \cdot \nabla \rho = 0, \tag{1.1}$$

*Dipartimento di Matematica 'F.Casorati', Università di Pavia, Via Ferrata 1, Pavia 27100, Italy, *antonio.segatti@unipv.it*.

†Shanghai Key Laboratory for Contemporary Applied Mathematics and School of Mathematical Sciences, Fudan University, Han Dan Road No. 220, Shanghai 200433, P.R. China, *haowufd@yahoo.com*. Corresponding author.

$$\rho \mathbf{v}_t + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \nabla \cdot (\sigma^e + \sigma^d), \tag{1.2}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{1.3}$$

$$\varphi_t + \mathbf{v} \cdot \nabla \varphi = \lambda[\nabla \cdot (\xi \nabla \varphi) - K\Delta^2 \varphi], \tag{1.4}$$

where

$$\sigma^d = \mu_1(\mathbf{d}^T D(\mathbf{v})\mathbf{d})\mathbf{d} \otimes \mathbf{d} + \mu_4 D(\mathbf{v}) + \mu_5(D(\mathbf{v})\mathbf{d} \otimes \mathbf{d} + \mathbf{d} \otimes D(\mathbf{v})\mathbf{d}),$$
$$\sigma^e = -\xi \mathbf{d} \otimes \mathbf{d} + K\nabla(\nabla \cdot \mathbf{d}) \otimes \mathbf{d} - K(\nabla \cdot \mathbf{d})\nabla\nabla\varphi.$$

In the above system, ρ is the density of the material, $\mathbf{v}$ is the flow velocity and φ denotes the layer variable. In the Smectic-A phase, molecule orientational direction lies normal to the layer that $\mathbf{d} = \nabla\varphi$. The scalar function p represents the pressure of the fluid, σ^d is the viscous (dissipative) stress tensor and σ^e is the elastic stress tensor (Ericksen tensor). As usual, $D(\mathbf{v})$ indicates the symmetric velocity gradient, $D(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + \nabla^\top \mathbf{v})$. Due to the incompressibility of the fluid, there holds $\nabla \cdot D(\mathbf{v}) = \frac{1}{2}\Delta \mathbf{v}$. $\mu_1 \geq 0$, $\mu_4 > 0$ and $\mu_5 \geq 0$ are dissipative coefficients in the stress tensor. The constant $K > 0$ arises in the free energy (cf. [7]) and $\lambda > 0$ is elastic relaxation time.

System (1.1)–(1.4) can be viewed as the analog for the Smectic-A liquid crystal of the Ericksen–Leslie system [6, 9, 19] for the nematic liquid crystal flow. Equation (1.1) represents the conservation of mass, equation (1.2) is the conservation of linear momentum, (1.3) implies the incompressibility of the fluid and equation (1.4) is the angular momentum equation. ξ is the Lagrange multiplier corresponding to the constraint associated with the incompressibility of the layers such that $|\nabla\varphi| = 1$. In order to relax this constraint, an often used approach is to introduce the Ginzburg-Landau penalization function $f(\mathbf{d}) = \frac{1}{\epsilon^2}(|\mathbf{d}|^2 - 1)\mathbf{d}$ $(0 < \epsilon \leq 1)$ with the associated potential function $F(\mathbf{d}) = \frac{1}{4\epsilon^2}(|\mathbf{d}|^2 - 1)^2$ such that $f(\mathbf{d}) = \frac{\delta F}{\delta \mathbf{d}}$ (cf. [3, 22]). Replacing the original Lagrange multiplier term $\xi\mathbf{d}$ in σ^e as well as in (1.4) by $f(\mathbf{d})$, we arrive at the evolution system that will be considered in the present paper:

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} - \frac{\mu_4}{2}\Delta \mathbf{v} + \nabla p = \nabla \cdot (\tilde{\sigma}^d + \tilde{\sigma}^e), \tag{1.5}$$

$$\nabla \cdot v = 0, \tag{1.6}$$

$$\varphi_t + \mathbf{v} \cdot \nabla \varphi = \lambda(-K\Delta^2\varphi + \nabla \cdot f(\mathbf{d})), \tag{1.7}$$

where

$$\tilde{\sigma}^d = \mu_1(\mathbf{d}^T D(\mathbf{v})\mathbf{d})\mathbf{d} \otimes \mathbf{d} + \mu_5(D(\mathbf{v})\mathbf{d} \otimes \mathbf{d} + \mathbf{d} \otimes D(\mathbf{v})\mathbf{d}),$$
$$\tilde{\sigma}^e = -f(\mathbf{d}) \otimes \mathbf{d} + K\nabla(\nabla \cdot \mathbf{d}) \otimes \mathbf{d} - K(\nabla \cdot \mathbf{d})\nabla\mathbf{d}.$$

The first well-posedness result of the hydrodynamic system for Smectic-A liquid crystal flow mentioned above was obtained in [22]. The author considered an approximate system like (1.5)–(1.7) but with variable density (thus one also has a mass transport equation for ρ like (1.1)) in an open bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. The system is subject to no-slip boundary condition for $\mathbf{v}$ and time-independent Dirichlet–Neumann boundary conditions for φ. The author derived the energy dissipative relation of the system and proved the existence of global weak solutions in both $2D$ and $3D$ by using a semi-Galerkin procedure. Moreover, he described the global regularity of weak solutions (for large enough μ_4 if $n = 3$) and provided a preliminary analysis on the stability of the system. Quite recently, system (1.5)–(1.7) with constant density and subject to no-slip boundary condition for $\mathbf{v}$ but time-dependent Dirichlet–Neumann boundary data for φ was studied in [3]. The authors proved the existence of weak solutions that are bounded up to infinity time for the initial-boundary problem with arbitrary initial data. The existence of time-periodic weak solutions is also obtained. Assuming the viscosity μ_4 is sufficiently large, the author studied the global in time regularity of the solution and proved the existence and uniqueness of regular solutions for both the initial-valued problem and the time-periodic problem.

In our present paper, we consider the problem in the n-dimensional torus ($n = 2, 3$) $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$, namely, system (1.5)–(1.7) is subjected to periodic boundary conditions. One of the possible reason for this choice is as follows. Contrary to the system for nematic liquid crystal flow (cf. e.g., [20]), now the equation (1.7) for φ is of fourth order type and thus lacks of the maximum principle. In particular, we lose the control of $\|\mathbf{d}\|_{\mathbf{L}^\infty}$. We note that, the bound of $\|\mathbf{d}\|_{\mathbf{L}^\infty}$ plays an important role in the subsequent analysis in order to prove the regularity of solutions to system (1.5)–(1.7) (cf. Lemma 3.6 and Lemma 5.1). Higher-order estimates of solutions can be obtained from some higher-order differential inequalities in the sprit of [20]. However, without the estimate of $\|\mathbf{d}\|_{\mathbf{L}^\infty}$, we are not able to control certain higher-order nonlinear terms to derive the required higher-order differential inequalities. It seems that this is also necessary in order to complete the calculations in [22]. This difficulty can be bypassed if one additionally assume that the viscosity μ_4 is sufficiently large (cf. Lemma 5.2, see also [3]). In the periodic boundary case, the key observation is that we can first obtain a uniform estimate on $\|\mathbf{d}\|_{\mathbf{H}^2}$, which by the embedding $\mathbf{H}^2 \hookrightarrow \mathbf{L}^\infty$ yields the bound of $\|\mathbf{d}\|_{\mathbf{L}^\infty}$. The proof relies on integration by parts, thus if we take the boundary conditions as in [3, 22], we are not able to get rid of certain extra boundary terms.

The main propose the present paper is to be a first step towards the mathematical study of the long-time behavior of global solutions to the periodic boundary problem of system (1.5)–(1.7). In the $2D$ case, we are interested in the study of finite dimensional global attractors. We recall that a global attractor is the smallest compact attracting set of the phase space which is fully invariant for the dynamics and attracts all the bounded subsets of the phase space for large times. Thus, it is certainly a major step in the understanding of the long time dynamics of the given evolutional system. In particular, when the global attractor is proved to have finite fractal or Hausdorff dimension, then, although the phase space is infinite dimensional, the dynamics of the system becomes finite dimensional for large times and can be described with a finite numbers of parameters. This is the so called finite dimensional reduction. We refer to [31] for a detailed description. We will prove the finite dimensionality of the global attractor by showing the existence of an exponential attractor, which is a semi-invariant, compact set attracting exponentially fast the bounded subsets of the phase space. Moreover, it has finite fractal dimension and contains the global attractor. We refer to [8] and to [28] for a detailed introduction of this concept and for discussion on its importance. This approach has the advantage that, contrary to the volume contraction method (see [31]), it does not need any differentiability property of the semigroup. As a second step, we will study the long-time behavior of single trajectories, i.e., the convergence to single equilibrium. This is a nontrivial problem because the structure of the set of equilibria can be quite complicated and, moreover, may form a continuum. In particular, under our current periodic boundary conditions, one may expect that the dimension of the set of equilibria is at least n. This is because a shift in each variable should give another steady state. Moreover, we note that for our system, every constant vector $\mathbf{d}_0$ with unit-length ($|\mathbf{d}_0| = 1$) serves as an absolute minimizer of the functional E in (4.7). We shall apply the Łojasiewicz–Simon approach (cf. L. Simon [30]) to prove the convergence and obtain estimates on the convergence rate (see [2, 12, 14, 15, 29, 32, 33] and the references therein for applications to various evolution equations). In $3D$ case, some partial results can be obtained. Since the $\mathbf{L}^\infty$-estimate of $\mathbf{d}$ is still available, we can show the local existence of strong solutions for arbitrary initial data by higher-order energy estimates. Assuming the viscosity μ_4 is sufficiently large, we also obtain the global existence of strong solution. Finally, we show that the global weak/strong solutions will converge to single equilibrium as in the $2D$ case. In particular, we prove the well-posedness and long-time behavior of global strong solutions when the initial data is close to a local minimizer of the energy E using the Łojasiewicz–Simon inequality, which improves the results in the literature that only the case near an absolute minimizer is considered (cf. [22], see also [20, 32] for the nematic liquid crystal flow).

The remaining part of the paper is organized as follows. Section 2 is devoted to some preliminaries and the main results of the paper. In Section 3, we prove that in the $2D$ case, the semigroup generated by our model on a suitable phase space possesses the global attractor $\mathcal{A}$ and an exponential attractor $\mathcal{M}$. This allows us to infer that $\mathcal{A}$ has finite fractal dimension. In Section 4, in the $2D$ case, we demonstrate that each trajectory converges to a single equilibrium and also find a convergence rate estimate. Finally, in Section 5, we discuss the results in $3D$ case.

2 Preliminaries and Main Results

We denote the Lebesgue spaces with $L^p(\mathbb{T}^n)$ (or simply L^p), $p \in [1, \infty]$, and their norms with $\|\cdot\|_{L^p}$. When $p = 2$, we simply denote the L^2-norm by $\|\cdot\|$ and its inner product by $(\cdot, \cdot)$. With H^s, $s \in \mathbb{R}$ we indicate the Sobolev spaces $H^s(\mathbb{T}^n)$ endowed with norm $\|\cdot\|_{H^s}$. To simplify the notations, we will denote the vector spaces $(L^p)^n$, $(H^s)^n$, $(L^p)^{n\times n}$, $(H^s)^{n\times n}$... by $\mathbf{L}^p$ and $\mathbf{H}^s$, respectively, and their norms are denoted in the same way as above. For any norm space X, we denote its subspace by $\dot{X}$ such that $\dot{X} = \{w \in X : \int_{\mathbb{T}^n} w dx = 0\}$. As customary, we introduce the following standard functional spaces for the Navier–Stokes equation

$$H := \{\mathbf{v} \in \mathbf{L}^2(\mathbb{T}^n),\ \nabla \cdot \mathbf{v} = 0\}, \quad V := \{\mathbf{v} \in \mathbf{H}^1(\mathbb{T}^n),\ \nabla \cdot \mathbf{v} = 0\}, \quad V' := \text{the dual of } V.$$

$\langle \cdot, \cdot \rangle$ denotes the duality product between V' and V. The shorthand notation D_{ij} will be used for the entries of the matrix D. We indicate with the same symbol C different constants. Special dependence will be indicated if it is necessary. Analogously, $\mathcal{D} : \mathbb{R}^+ \to \mathbb{R}^+$ denotes a generic monotone function. Throughout the paper, the Einstein summation convention will be used.

We introduce the notions of weak/strong solutions to problem (1.5)–(1.7):

Definition 2.1. *(1) $(\mathbf{v}, \varphi)$ is a weak solution to problem (1.5)–(1.7) in $[0, T) \times \mathbb{T}^n$ ($T \in (0, +\infty)$), if $\mathbf{v} \in L^\infty(0, T; H) \cap L^2(0, T; V)$, $\varphi \in L^\infty(0, T; H^2) \cap L^2(0, T; H^4)$ and verifying*

$$\langle \partial_t \mathbf{v}, \mathbf{w} \rangle + ((\mathbf{v} \cdot \nabla)\mathbf{v}, \mathbf{w}) + \frac{\mu_4}{2}(\nabla \mathbf{v}, \nabla \mathbf{w}) = (\tilde{\sigma}^d + \tilde{\sigma}^e, \nabla \mathbf{w}), \quad \forall \mathbf{w} \in V,$$
$$\varphi_t + \mathbf{v} \cdot \nabla \varphi = \lambda(-K\Delta^2 \varphi + \nabla \cdot f(\nabla \varphi)), \quad a.e.\ in\ [0, T] \times \mathbb{T}^n$$

$$\mathbf{v}(0) = \mathbf{v}_0, \quad \varphi(0) = \varphi_0, \quad in\ \mathbb{T}^n.$$

(2) A weak solution $(\mathbf{v}, \varphi)$ *to problem* (1.5)–(1.7) *is a strong solution, if for* $T > 0$, $\mathbf{v} \in L^\infty(0,T;V) \cap L^2(0,T;\mathbf{H}^2)$, $\varphi \in L^\infty(0,T;H^4) \cap L^2(0,T;H^6)$, *and the system* (1.5)–(1.7) *is satisfied point-wisely in* $[0,T) \times \mathbb{T}^n$.

The calculation in [22] (with different boundary conditions but the proof is the same) implies that system (1.5)–(1.7) has a dissipative nature, in particular, the following *basic energy law* holds

Proposition 2.1. *Let* $(\mathbf{v}, \varphi)$ *be a smooth solution to the system* (1.5)–(1.7)*. Define the total energy*

$$\mathcal{E}(t) = \frac{1}{2}\|\mathbf{v}(t)\|^2 + \frac{K}{2}\|\Delta\varphi(t)\|^2 + \int_{\mathbb{T}^n} F(\mathbf{d})(t)\, dx. \tag{2.1}$$

Then following identity holds:

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &= -\int_{\mathbb{T}^n} \left(\mu_1(\mathbf{d}^\top D(\mathbf{v})\mathbf{d})^2 + \frac{\mu_4}{2}|\nabla \mathbf{v}|^2 + 2\mu_5|D(\mathbf{v})\mathbf{d}|^2\right) dx \\ &\quad -\lambda \left\|-K\Delta^2\varphi + \nabla\cdot f(\mathbf{d})\right\|^2. \end{aligned} \tag{2.2}$$

We can prove the existence of weak solutions to (1.5)–(1.7) by applying the semi-Galerkin approximation scheme as in [22] (cf. also [3, 20, 21]). The proof is similar to [3, 22] and we omit the details here.

Theorem 2.1. [Existence of weak solution] *Suppose* $n = 2, 3$*. For any* $(\mathbf{v}_0, \varphi_0) \in H \times H^2$, *system* (1.5)–(1.7) *admits at least one weak solution.*

A weak/strong uniqueness result was obtained in [3] for system (1.5)–(1.7) with different boundary conditions (see [22] for a statement for the system with variable density). A similar argument yields the same conclusion for our case:

Theorem 2.2. [Weak/strong uniqueness] *If* $(\mathbf{v}_1, \varphi_1)$ *and* $(\mathbf{v}_2, \varphi_2)$ *are respectively a weak and a strong solution of* (1.5)–(1.7) *in* $[0,T]$, *then* $(\mathbf{v}_1, \varphi_1) \equiv (\mathbf{v}_2, \varphi_2)$ *almost everywhere in* $[0,T] \times \mathbb{T}^n$.

Here are the main results of the paper:

Theorem 2.3. *Suppose* $n = 2$.

(1) Any weak solution to system (1.5)–(1.7) *becomes strong for strictly positive times such that for any* $t > 0$,

$$\|(\mathbf{v}, \varphi)(t)\|_{V \times H^4} + \int_t^{t+1} \|\Delta \mathbf{v}(s)\|^2 + \|\varphi(s)\|_{H^6}^2 ds \le \mathcal{D}(\|(\mathbf{v}_0, \varphi_0)\|_{H \times H^2}, t), \tag{2.3}$$

$\mathcal{D}$ *is a positive function depending on* $\|\mathbf{v}_0\|$, $\|\varphi_0\|_{H^2}$, t *and coefficients of the system. In particular,* $\lim_{t \to 0^+} \mathcal{D}(t) = +\infty$.

(2) For any $(\mathbf{v}_0, \varphi_0) \in V \times H^4$, *system* (1.5)–(1.7) *admits a unique strong solution.*

Theorem 2.4. *Suppose* $n = 2$*. Denote the phase space* $\mathcal{H} \times H^2_c$, *where* $\mathcal{H} = \{\mathbf{v} \in H : \int_{\mathbb{T}^2} \mathbf{v} dx = \mathbf{h}\}$ *and* $H^2_c = \{\varphi \in H^2 : \int_{\mathbb{T}^2} \varphi dx = c\}$, *with* $\mathbf{h}$ *being any given constant vector in* $\mathbb{R}^2$ *and* c *is an arbitrary constant.*

(1) System (1.5)–(1.7) *processes a global attractor* $\mathcal{A}$ *with finite fractal dimension in* $\mathcal{H} \times H^2_c$*. Moreover,* $\mathcal{A}$ *is bounded in* $V \times H^4$ *and it is generated by all the complete trajectories.*

(2) System (1.5)–(1.7) *possesses an exponential attractor* $\mathcal{M}$ *in* $\mathcal{H} \times H^2_c$*, which is bounded in* $V \times H^4$*.*

Theorem 2.5. *Suppose* $n = 2$*. For any* $\mathbf{v}_0 \in \dot{H}$*,* $\varphi_0 \in H^2$*, the global weak solutions to problem* (1.5)–(1.7) *has the following property:*

$$\lim_{t\to+\infty}(\|\mathbf{v}(t)\|_{\mathbf{H}^1} + \|\varphi(t) - \varphi_\infty\|_{H^4}) = 0, \tag{2.4}$$

where $\varphi_\infty \in H^4$ *is a solution to the following periodic elliptic problem:*

$$-K\Delta^2\varphi_\infty + \nabla\cdot f(\nabla\varphi_\infty) = 0, \quad x \in \mathbb{T}^2, \text{with} \int_{\mathbb{T}^2}\varphi_\infty dx = \int_{\mathbb{T}^2}\varphi_0 dx, \tag{2.5}$$

Moreover, there exists a positive constant C *depending on* $\mathbf{v}_0, \varphi_0, \varphi_\infty, K, \lambda, \mu' s$ *such that*

$$\|\mathbf{v}(t)\|_{\mathbf{H}^1} + \|\varphi(t) - \varphi_\infty\|_{H^4} \le C(1+t)^{-\frac{\theta}{(1-2\theta)}}, \quad \forall\, t \ge 1. \tag{2.6}$$

$\theta \in (0, \frac{1}{2})$ *is usually called Łojasiewicz exponent and it depends on* φ_∞*.*

For any $\mathbf{v}_0 \in \dot{V}$*,* $\varphi_0 \in H^4$*, the global strong solution to problem* (1.5)–(1.7) *has the same property* (2.4) *and* (2.6) *holds for* $t \ge 0$*.*

Theorem 2.6. *Suppose* $n = 3$*.*

(1) For any $(\mathbf{v}_0, \varphi_0) \in V \times H^4$*, problem* (1.5)–(1.7) *admits a unique local strong solution.*

(2) For any $(\mathbf{v}_0, \varphi_0) \in V \times H^4$*, if* $\mu_4 \ge \underline{\mu}_4(\mathbf{v}_0, \varphi_0)$ *is sufficiently large (cf.* (5.11)*), problem* (1.5)–(1.7) *admits a unique global strong solution.*

(3) Let $(\mathbf{v}, \varphi)$ *be the weak solution to problem* (1.5)–(1.7) *on* $[0, +\infty)$*. Then there is some* $T^* > 0$ *such that* $\mathbf{v} \in L^\infty(T^*, \infty; V) \cap L^2_{loc}(T^*, \infty; \mathbf{H}^2)$*,* $\varphi \in L^\infty(T^*, \infty; H^4) \cap L^2_{loc}(T^*, \infty; H^6)$*.*

(4) Let $\varphi^* \in H^2$ *be a local/absolute minimizer of* $E(\varphi)$ *(cf.* (4.7)*). For any* $\mathbf{v}_0 \in V$*,* $\varphi_0 \in H^4$ *satisfying* $\|\mathbf{v}_0\|_{\mathbf{H}^1} \le 1$*,* $\|\varphi_0 - \varphi^*\|_{H^4} \le 1$*, there are constants* $\sigma_1, \sigma_2 \in (0, 1]$ *which depend on* φ^* *and coefficients of the system such that if* $\|\mathbf{v}_0\| \le \sigma_1$ *and* $\|\varphi_0 - \varphi^*\|_{H^2} \le \sigma_2$*, then problem* (1.5)–(1.7) *admits a unique global strong solution.*

(5) If we further assume that $\int_{\mathbb{T}^3}\mathbf{v}_0 dx = 0$*, then the global weak/strong solution to* (1.5)–(1.7) *enjoys the same long-time behavior as in Theorem 2.5, with* $t \ge 1$ *in* (2.6) *being replaced by* $t \ge T^*$ *for the weak solution.*

Remark 2.1. *Due to the periodic boundary conditions, we can easily see that the mean value of* $\mathbf{v}$ *and* φ *are conserved in the evolution:*

$$\int_{\mathbb{T}^n}\mathbf{v}(t)dx = \int_{\mathbb{T}^n}\mathbf{v}_0 dx, \quad \int_{\mathbb{T}^n}\varphi(t)dx = \int_{\mathbb{T}^n}\varphi_0 dx, \quad \forall\, t \ge 0.$$

For the sake of simplicity, by replacing $\mathbf{v}$ *(respectively* φ*) with* $\mathbf{v}_0 - \int_{\mathbb{T}^n}\mathbf{v}_0 dx$ *(respectively with* $\varphi_0 - \int_{\mathbb{T}^n}\varphi_0 dx$*), we shall always assume that* $\int_{\mathbb{T}^n}\mathbf{v}_0 dx \equiv 0$ *and* $\int_{\mathbb{T}^n}\varphi_0 dx \equiv 0$ *in the subsequent proof. Since system* (1.5)–(1.7) *is invariant under a shift of* φ *by any constant, the transformation on* φ *will not influence all our results. However, when we shift the velocity* $\mathbf{v}$ *to make it has a zero mean, there will be one extra lower-order term in the equations* (1.5) *and* (1.7) *respectively. This difference will not influence most results we obtain except the convergence of global solutions to equilibria (Theorem 2.5 and point (5) in Theorem 2.6). If the mean value of* $\mathbf{v}$ *is not zero, we cannot apply the Poincaré inequality to obtain the decay of* $\|\mathbf{v}\|_{\mathbf{H}^1}$ *from the convergence of* $\|\nabla\mathbf{v}\|$*.*

Remark 2.2. *If we simply set* $\mathbf{v} = 0$*, system* (1.5)–(1.7) *is reduce to the single equation* $\varphi_t = \lambda(-K\Delta^2\varphi + \nabla \cdot f(\nabla\varphi))$*, which has been used to model epitaxial growth of thin films with slope selection in* $2D$*, where* φ *denotes a scaled height function of a thin film (cf. [17, 23]). Existence and uniqueness of the weak solutions as well as some preliminary results on long-time behavior of the solutions as time goes to infinity (like sequent convergence) was obtained in [18].*

3 Global Attractor and Exponential Attractors in $2D$

In this section we study the long time behavior of the system (1.5)–(1.7) in terms of global and exponential attractors. As suggested by Remark 2.1, we work in the phase spaces

$$\Phi := \dot{H} \times \dot{H}^2, \quad \Phi_1 := \dot{V} \times \dot{H}^4$$

with the norms $\|(\mathbf{v}, \varphi)\|_\Phi^2 := \|\mathbf{v}\|_H^2 + \|\varphi\|_{H^2}^2$, $\|(\mathbf{v}, \varphi)\|_{\Phi_1}^2 := \|\mathbf{v}\|_V^2 + \|\varphi\|_{H^4}^2$, respectively. It is obvious that Φ_1 is compactly embedded into Φ.

Recall the definition of the global attractor (cf. [31])

Definition 3.1. *Suppose* $\mathcal{X}$ *is a complete metric space. Given a semigroup* $S(t) : \mathcal{X} \mapsto \mathcal{X}$*, a subset* $\mathcal{A} \subset \mathcal{X}$ *is the global attractor if (i) The set* $\mathcal{A}$ *is compact in* $\mathcal{X}$*; (ii) It is strictly invariant:* $S(t)\mathcal{A} = \mathcal{A}$*,* $t \geq 0$*; (iii) For every bounded set* $B \subset \mathcal{X}$ *and for every neighborhood* $\mathcal{O} = \mathcal{O}(\mathcal{A})$ *of* $\mathcal{A}$ *in* $\mathcal{M}$*, there exists a time* $T = T(\mathcal{O})$ *such that* $S(t)B \subset \mathcal{O}(\mathcal{A})$ *for all* $t \geq T$*.*

As far as our system is concerned, we define $S(t) : \Phi \mapsto \Phi$ to be the map $(\mathbf{v}_0, \varphi_0) \mapsto (\mathbf{v}(t), \varphi(t))$. Unfortunately, Theorem 2.1 does not guarantee that $S(t)$ is well defined on the phase space Φ, since we are not able to prove a uniqueness result for weak solutions. We will refer to $S(t)$ as a *solution operator*, being aware of the fact that, in principle, $S(t)(\mathbf{v}_0, \varphi_0)$ could be multi-valued due to the possible non-uniqueness. In the cases in which uniqueness holds, with a little abuse of notation, we will still indicate with $S(t)$ the corresponding *semigroup*. As a consequence of the possible non-uniqueness, as it will be further explained later, we will not directly construct the global attractor on the phase space Φ but rather on the "lifted" phase space of ℓ-trajectories.

3.1 Dissipativity

The following lower-order uniform estimate follows from the basic energy law:

Lemma 3.1. *Suppose* $n = 2, 3$*. For* $\mathbf{v}_0 \in \dot{H}$*,* $\varphi_0 \in \dot{H}^2$*, the weak solution to* (1.5)–(1.7) *has the following uniform estimates*

$$\|\mathbf{v}(t)\| + \|\varphi(t)\|_{H^2} \leq C, \quad t \geq 0, \tag{3.1}$$

where $C > 0$ *is a constant depending on* $\|\mathbf{v}_0\|, \|\varphi_0\|_{H^2}, K$*. Moreover,*

$$\int_0^{+\infty} \left(\|\nabla \mathbf{v}(t)\|^2 + \left\| -K\Delta^2\varphi(t) + \nabla \cdot f(\mathbf{d}(t)) \right\|^2 \right) dt \leq \max\left\{ \frac{2}{\mu_4}, \frac{1}{\lambda} \right\} \mathcal{E}(0). \tag{3.2}$$

Next, we prove some dissipative estimates for the weak solutions to (1.5)–(1.7).

Lemma 3.2. *Suppose $n = 2$. For $\mathbf{v}_0 \in \dot{H}$, $\varphi_0 \in \dot{H}^2$, any weak solution of* (1.5)–(1.7) *verifies*

$$\|(\mathbf{v}(t), \varphi(t))\|_\Phi^2 \le \mathcal{D}(\|(\mathbf{v}_0, \varphi_0)\|_\Phi) e^{-\alpha t} + C, \tag{3.3}$$

where the positive constants C and α are independent on the solution and depend only on the coefficients of the system.

Proof. Multiplying (1.7) with φ and integrating over $\mathbb{T}^2$, we get

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 + K\lambda\|\Delta\varphi\|^2 + \frac{\lambda}{\epsilon^2}\int_{\mathbb{T}^2} |\nabla\varphi|^4 dx = -\int_{\mathbb{T}^2} (\mathbf{v}\cdot\nabla)\varphi\varphi dx + \frac{\lambda}{\epsilon^2}\int_{\mathbb{T}^2} |\nabla\varphi|^2 dx. \tag{3.4}$$

The righthand side of (3.4) can be estimated as follows

$$\begin{aligned} \frac{\lambda}{\epsilon^2}\int_{\mathbb{T}^2} |\nabla\varphi|^2 dx &\le \frac{\lambda}{4\epsilon^2}\int_{\mathbb{T}^2} |\nabla\varphi|^4 dx + \frac{\lambda}{\epsilon^2}|Q|, \\ -\int_{\mathbb{T}^2} (\mathbf{v}\cdot\nabla)\varphi\varphi dx \le \|\mathbf{v}\|\|\nabla\varphi\|_{L^4}\|\varphi\|_{L^4} &\le \frac{1}{2\delta_1}\|\mathbf{v}\|^2 + \frac{\delta_1}{2}\|\nabla\varphi\|_{L^4}^2\|\varphi\|_{L^4}^2, \end{aligned}$$

$\delta_1 > 0$ is a small constant to be determined later. For $\mathbf{v} \in \dot{V}$, we infer from the Poincaré inequality that $\|\mathbf{v}\| \le C_P\|\nabla\mathbf{v}\|$, where the constant $C_P > 0$ depends only on $\mathbb{T}^2$. For $\varphi \in \dot{H}^2$, we infer from the Sobolev embedding theorem, Poincaré inequality and Hölder inequality that

$$|\mathbb{T}^2|^{-\frac{1}{4}}\|\varphi\| \le \|\varphi\|_{L^4} \le C_1\|\nabla\varphi\| \le C_1|\mathbb{T}^2|^{\frac{1}{4}}\|\nabla\varphi\|_{\mathbf{L}^4},$$

where C_1 is constant depending only on $\mathbb{T}^2$. As a result,

$$\begin{aligned} &\frac{\delta_1}{2}\|\nabla\varphi\|_{\mathbf{L}^4}^2\|\varphi\|_{L^4}^2 \le \frac{\delta_1}{2}C_1^2|\mathbb{T}^2|^{\frac{1}{2}}\int_{\mathbb{T}^2} |\nabla\varphi|^4 dx, \\ &\|\varphi\|^2 \le C_1^2|\mathbb{T}^2|\|\nabla\varphi\|_{\mathbf{L}^4}^2 \le \frac{\lambda}{4\epsilon^2}\int_{\mathbb{T}^2} |\nabla\varphi|^4 dx + \frac{\epsilon^2 C_1^4|\mathbb{T}^2|^2}{\lambda}. \end{aligned}$$

Hence, we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 + K\lambda\|\Delta\varphi\|^2 + \left(\frac{\lambda}{2\epsilon^2} - \frac{\delta_1}{2}C_1^2|\mathbb{T}^2|^{\frac{1}{2}}\right)\int_{\mathbb{T}^2} |\nabla\varphi|^4 dx + \|\varphi\|^2 \\ \le\ &\frac{C_P}{2\delta_1}\|\nabla\mathbf{v}\|^2 + \frac{\lambda}{\epsilon^2}|\mathbb{T}^2| + \frac{\epsilon^2 C_1^4|\mathbb{T}^2|^2}{\lambda}. \end{aligned} \tag{3.5}$$

Multiplying (3.5) by $\delta_2 > 0$ and adding it to the basic energy law (2.2), we obtain

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|\mathbf{v}\|^2 + \frac{K}{2}\|\Delta\varphi\|^2 + \int_{\mathbb{T}^2} F(\mathbf{d})\, dx + \frac{\delta_2}{2}\|\varphi\|^2\right] + \int_{\mathbb{T}^2} \left[\mu_1(D_{kp}d_kd_p)^2 + 2\mu_5|D\mathbf{d}|^2\right] dx \\ &+\lambda\left\|-K\Delta^2\varphi + \nabla\cdot f(\nabla\varphi)\right\|^2 + \left(\frac{\mu_4}{2} - \frac{\delta_2 C_P}{2\delta_1}\right)\|\nabla\mathbf{v}\|^2 + \delta_2 K\lambda\|\Delta\varphi\|^2 \\ &+\delta_2\left(\frac{\lambda}{2\epsilon^2} - \frac{\delta_1}{2}C_1^2|\mathbb{T}^2|^{\frac{1}{2}}\right)\int_{\mathbb{T}^2} |\nabla\varphi|^4 dx + \delta_2\|\varphi\|^2 \\ \le\ &\delta_2\left(\frac{\lambda}{\epsilon^2}|\mathbb{T}^2| + \frac{\epsilon^2 C_1^4|\mathbb{T}^2|^2}{\lambda}\right). \end{aligned} \tag{3.6}$$

Take δ_1, δ_2 that satisfying

$$\delta_1 = \frac{\lambda}{2\epsilon^2 C_1^2|\mathbb{T}^2|^{\frac{1}{2}}}, \quad \delta_2 = \frac{\mu_4\delta_1}{2C_P} = \frac{\lambda\mu_4}{4\epsilon^2 C_P C_1^2|\mathbb{T}^2|^{\frac{1}{2}}}.$$

We deduce from (3.6) that

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\|\mathbf{v}\|^2+\frac{K}{2}\|\Delta\varphi\|^2+\int_{\mathbb{T}^2}F(\mathbf{d})\,dx+\frac{\delta_2}{2}\|\varphi\|^2\right]+\frac{\mu_4}{4}\|\nabla\mathbf{v}\|^2\\
&+\delta_2K\lambda\|\Delta\varphi\|^2+\frac{\delta_2\lambda}{4\epsilon^2}\int_{\mathbb{T}^2}|\nabla\varphi|^4dx\le\delta_2\left(\frac{\lambda}{\epsilon^2}|\mathbb{T}^2|+\frac{\epsilon^2C_1^4|\mathbb{T}^2|^2}{\lambda}\right). \qquad (3.7)
\end{aligned}
$$

Define $\Psi(t):=\mathcal{E}(t)+\frac{\delta_2}{2}\|\varphi\|^2$. It is easy to see that

$$
C_3\left(\|\mathbf{v}\|^2+\|\Delta\varphi\|^2+\|\varphi\|^2+\int_{\mathbb{T}^2}|\nabla\varphi|^4dx+1\right)\ge\Psi(t)\ge C_2(\|\mathbf{v}\|^2+\|\varphi\|_{H^2}^2),
$$

where C_2, C_3 are positive constants depending on $\mathbb{T}^2, K, \epsilon, \lambda, \mu_4$ but not on the solution. Thus, we can conclude that there exist two positive constants C_4, C_5 depending only on $\mathbb{T}, K, \epsilon, \lambda, \mu_4$ such that

$$
\frac{d}{dt}\Psi(t)+C_4\Psi(t)\le C_5.
$$

As a result,

$$
\|\mathbf{v}(t)\|^2+\|\varphi(t)\|_{H^2}^2\le\frac{1}{C_2}\Psi(t)\le\frac{1}{C_2}e^{-C_4t}\Psi(0)+\frac{C_5}{C_2C_4},\quad\forall\, t\ge0.
$$

The proof is complete. □

3.2 Higher-order estimates

Next, we show that the weak solutions turn out to be regular for strictly positive times. This, will imply the compactness of the solution operator $S(t)$. The following lemma plays an important role in the subsequent proof. It is worthwhile noting that, since the coupling in equation (1.7) is weak, this result is valid both for $n=2,3$.

Lemma 3.3. *Suppose $n=2,3$. We have*

$$
\frac{d}{dt}\|\nabla\Delta\varphi\|^2+\lambda K\left\|\nabla\Delta^2\varphi\right\|^2\le C(\|\nabla\mathbf{v}\|^2+\|\nabla\Delta\varphi\|^2)\|\nabla\Delta\varphi\|^2+C\|\nabla\mathbf{v}\|^2+C, \qquad (3.8)
$$

where C is a positive constant depending on $\|\mathbf{v}_0\|$, $\|\varphi_0\|_{H^2}$ and coefficients of the system.

Proof. We just work in the $3D$ case and it is easy to verify that the same result holds in $2D$. Multiplying (1.7) by $\Delta^3\varphi$, integrating over $\mathbb{T}^3$, due to the periodic boundary condition, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\Delta\varphi\|^2+\lambda K\left\|\nabla\Delta^2\varphi\right\|^2\\
=&\ -\int_{\mathbb{T}^3}\nabla\Delta^2\varphi\cdot\nabla(\mathbf{v}\cdot\nabla\varphi)dx-\lambda\int_{\mathbb{T}^3}\nabla\Delta^2\varphi\cdot\nabla\left[\nabla\cdot f(\mathbf{d})\right]dx. \qquad (3.9)
\end{aligned}
$$

By the uniform estimates (3.3), the Agmon inequality and Gagliardo–Nirenberg inequality in $3D$, we get

$$
\begin{aligned}
\|\nabla\varphi\|_{L^\infty}&\le C\|\varphi\|_{H^3}^{\frac12}\|\varphi\|_{H^2}^{\frac12}\le C(\|\nabla\Delta\varphi\|^{\frac12}+1),\\
\|\nabla\nabla\varphi\|_{L^3}&\le C(\|\nabla\Delta\varphi\|^{\frac12}\|\Delta\varphi\|^{\frac12}+\|\Delta\varphi\|)\le C(\|\nabla\Delta\varphi\|^{\frac12}+1).
\end{aligned}
$$

Now we estimate the right-hand side of (3.9) term by term.

$$\begin{aligned}\left|\int_{\mathbb{T}^2}\nabla\Delta^2\varphi\cdot\nabla(\mathbf{v}\cdot\nabla\varphi)dx\right| &\leq C\|\nabla\Delta^2\varphi\|(\|\nabla\mathbf{v}\|\|\nabla\varphi\|_{\mathbf{L}^\infty}+\|\mathbf{v}\|_{\mathbf{L}^6}\|\nabla\nabla\varphi\|_{\mathbf{L}^3})\\ &\leq C\|\nabla\Delta^2\varphi\|\|\nabla\mathbf{v}\|(\|\nabla\Delta\varphi\|^{\frac{1}{2}}+1)\\ &\leq \varepsilon\|\nabla\Delta^2\varphi\|^2+\|\nabla\mathbf{v}\|^2\|\nabla\Delta\varphi\|^2+C\|\nabla\mathbf{v}\|^2.\end{aligned}$$

$$\begin{aligned}&\lambda\left|\int_{\mathbb{T}^2}\nabla\Delta^2\varphi\cdot\nabla\left[\nabla\cdot f(\mathbf{d})\right]dx\right| = -\frac{\lambda}{\epsilon^2}\int_{\mathbb{T}^2}\nabla\Delta^2\varphi\cdot\nabla\left[(3|\nabla\varphi|^2-1)\Delta\varphi\right]dx\\ \leq\ & C\|\nabla\Delta^2\varphi\|\|\nabla\Delta\varphi\|+C\|\nabla\Delta^2\varphi\|\|\nabla\varphi\|^2_{\mathbf{L}^\infty}\|\nabla\Delta\varphi\|\\ &+C\|\nabla\Delta^2\varphi\|\|\nabla\varphi\|_{\mathbf{L}^\infty}\|\nabla\nabla\varphi\|_{\mathbf{L}^3}\|\Delta\varphi\|_{L^6}\\ \leq\ & \varepsilon\|\nabla\Delta^2\varphi\|^2+C\|\nabla\Delta\varphi\|^2+C\|\nabla\Delta\varphi\|^2\|\nabla\varphi\|^4_{\mathbf{L}^\infty}+C\|\nabla\nabla\varphi\|^2_{\mathbf{L}^3}\|\Delta\varphi\|^2_{L^6}\|\nabla\varphi\|^2_{\mathbf{L}^\infty}\\ \leq\ & \varepsilon\|\nabla\Delta^2\varphi\|^2+C\|\nabla\Delta\varphi\|^4+C.\end{aligned}$$

Taking $\varepsilon=\frac{\lambda K}{4}$, we infer from the above estiamtes that (3.8) holds. The proof is complete. □

Denote

$$\mathcal{Q}=-K\Delta^2\varphi+\nabla\cdot f(\mathbf{d}).$$

By the definition of $\mathcal{Q}$ and the Sobolev embedding theorem, we can easily derive the the following estimates.

Lemma 3.4. *Suppose $n=2,3$. We have $\|\nabla\Delta\varphi\|\leq C\|\mathcal{Q}\|^{\frac{1}{2}}+C$, $\|\Delta^2\varphi\|\leq\frac{2}{K}\|\mathcal{Q}\|+C$, $\|\nabla\Delta^2\varphi\|\leq\frac{2}{K}\|\nabla\mathcal{Q}\|+C$, where C is a constant depending on $\|\varphi\|_{H^2},K,\epsilon$. Moreover, $\|\Delta^3\varphi\|\leq\frac{2}{K}\|\Delta\mathcal{Q}\|+C$, where C is a constant depending on $\|\varphi\|_{H^3},K,\epsilon$.*

Next, we prove the following higher-order estimate for φ:

Lemma 3.5. *Suppose $n=2,3$. For any $\mathbf{v}_0\in\dot{H}$, $\varphi_0\in\dot{H}^2$, the weak solution to (1.5)–(1.7) satisfies*

$$\|\varphi(t)\|_{H^3}\leq\frac{1+t}{t}\mathcal{D}(\|(\mathbf{v}_0,\varphi_0)\|_\Phi),\quad\forall t>0. \tag{3.10}$$

Moreover, if we assume in addition that $\varphi_0\in H^3$, $\|\varphi(t)\|_{H^3}$ can be bounded by a constant depending on $\|\mathbf{v}_0\|$ and $\|\varphi_0\|_{H^3}$ uniformly in time.

Proof. We infer from Lemma 3.1 and Lemma 3.4 that for any $r>0$ and $t\geq 0$,

$$\sup_{t\geq0}\int_t^{t+r}\|\nabla\Delta\varphi(\tau)\|^2d\tau\leq\sup_{t\geq0}C\int_t^{t+r}\|\mathcal{Q}(\tau)\|d\tau+Cr\leq C\int_0^\infty\|\mathcal{Q}(\tau)\|^2+Cr\leq C(1+r), \tag{3.11}$$

$$\sup_{t\geq0}\int_t^{t+r}\|\nabla\mathbf{v}(\tau)\|^2d\tau\leq\int_0^{+\infty}\|\nabla\mathbf{v}(\tau)\|^2d\tau\leq C. \tag{3.12}$$

It follows from (3.8) and the uniform Gronwall lemma [31, Lemma III.1.1] that

$$\|\nabla\Delta\varphi(t+r)\|^2\leq C\left(1+\frac{1}{r}\right),\quad\forall t\geq0, \tag{3.13}$$

which together with (3.3) yields (3.10).

If we assume that $\varphi_0 \in H^3$, then by (3.8), (3.11), (3.12) and the standard Gronwall inequality, we have

$$\begin{aligned}
& \|\nabla\Delta\varphi(t)\|^2 \\
\leq\ & \|\nabla\Delta\varphi_0\|^2 \exp\left(C\int_0^t (\|\nabla \mathbf{v}(\tau)\|^2 + \|\nabla\Delta\varphi(\tau)\|^2 + 1)d\tau\right) \\
& + C\int_0^t (\|\nabla \mathbf{v}(s)\|^2 + 1)\exp\left(-C\int_0^s (\|\nabla \mathbf{v}(\tau)\|^2 + \|\nabla\Delta\varphi(\tau)\|^2 + 1)d\tau\right) ds \\
\leq\ & \|\nabla\Delta\varphi_0\|^2 \exp\left(C\int_0^1 (\|\nabla \mathbf{v}(\tau)\|^2 + \|\nabla\Delta\varphi(\tau)\|^2 + 1)d\tau\right) + C\int_0^1 (\|\nabla \mathbf{v}(s)\|^2 + 1)ds \\
\leq\ & C, \quad \forall\, t \in [0,1],
\end{aligned}$$

where C is a constant depending on $\|\mathbf{v}_0\|$, $\|\varphi_0\|_{H^3}$. Taking $r = 1$ in (3.13) and using (3.3), we obtain the uniform estimate on $\|\varphi(t)\|_{H^3}$ for all $t \geq 0$. The proof is complete. □

By the Sobolov embedding theorem, we easily deduce the follow result

Corollary 3.1. *Suppose $n = 2, 3$. For any $\mathbf{v}_0 \in \dot{H}$, $\varphi_0 \in \dot{H}^2$, we have*

$$\|\nabla\varphi(t)\|_{\mathbf{L}^\infty} \leq \frac{1+t}{t}\mathcal{D}(\|(\mathbf{v}_0, \varphi_0)\|_\Phi), \quad \forall t > 0. \tag{3.14}$$

Moreover, if we assume in addition that $\varphi_0 \in H^3$, $\|\nabla\varphi(t)\|_{\mathbf{L}^\infty}$ can be bounded by a constant depending on $\|\mathbf{v}_0\|$ and $\|\varphi_0\|_{H^3}$ uniformly in time.

Using Corollary 3.1, we are able to derive the higher-order energy inequality in $2D$.

Lemma 3.6. *Suppose $n = 2$. Let*

$$\mathbf{A}(t) = \|\nabla \mathbf{v}(t)\|^2 + \alpha\|\mathcal{Q}(t)\|^2,$$

where $\alpha > 0$ is a small constant to be chosen later (cf. (3.24) below). We have

$$\frac{d}{dt}\mathbf{A}(t) + \frac{\mu_4}{4}\|\Delta \mathbf{v}\|^2 + \frac{\alpha\lambda K}{2}\|\Delta\mathcal{Q}\|^2 \leq C(\mathbf{A}^2(t) + \mathbf{A}(t)), \quad \forall\, t \geq t_1 > 0, \tag{3.15}$$

where $t_1 > 0$ is arbitrary and C is a constant depending on $\|\mathbf{v}_0\|$, $\|\varphi_0\|_{H^2}$ and t_1. Moreover, if we assume that $\varphi_0 \in H^3$, (3.15) holds for $t \geq 0$ with C being dependent of $\|\mathbf{v}_0\|$, $\|\varphi_0\|_{H^3}$.

Proof. Recall the computation in [3, pp. 1475] that $\nabla\cdot\tilde{\sigma}^e = -(\nabla\cdot f(\mathbf{d}))\mathbf{d} - \nabla F(\mathbf{d}) + K\Delta^2\varphi\nabla\varphi - K\nabla\left(\frac{|\nabla\varphi|^2}{2}\right)$. We note that (1.5) can be written in the following form

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} - \frac{\mu_4}{2}\Delta\mathbf{v} + \nabla P = \nabla\cdot\tilde{\sigma}^d + (K\Delta^2\varphi - \nabla\cdot f(\mathbf{d}))\mathbf{d}, \tag{3.16}$$

where $P = p + \nabla\left(\frac{K|\nabla\varphi|^2}{2} + F(\mathbf{d})\right)$. Using (3.16), we have

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{v}\|^2 = -\int_{\mathbb{T}^2} \mathbf{v}_t\cdot\Delta\mathbf{v}dx \\
=\ & \int_{\mathbb{T}^2} (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\Delta\mathbf{v}dx - \frac{\mu_4}{2}\|\Delta\mathbf{v}\|^2 - \mu_1\int_{\mathbb{T}^2}[\nabla\cdot((\mathbf{d}^\top D(\mathbf{v})\mathbf{d})\mathbf{d}\otimes\mathbf{d})]\cdot\Delta\mathbf{v}dx \\
& -\mu_5\int_{\mathbb{T}^2}[\nabla\cdot((D(\mathbf{v})\mathbf{d}\otimes\mathbf{d} + \mathbf{d}\otimes D(\mathbf{v})\mathbf{d}))]\cdot\Delta\mathbf{v}dx
\end{aligned}$$

$$-\int_{\mathbb{T}^2}[(K\Delta^2\varphi-\nabla\cdot f(\mathbf{d}))\mathbf{d}]\cdot\Delta\mathbf{v}dx. \tag{3.17}$$

Using the periodic boundary condition and integration by parts, the right-hand side of (3.17) can be manipulated as follows

$$\begin{aligned}
&-\mu_1\int_{\mathbb{T}^2}\nabla\cdot[(\mathbf{d}^\top D(\mathbf{v})\mathbf{d})\mathbf{d}\otimes\mathbf{d}]\cdot\Delta\mathbf{v}dx=-\mu_1\int_{\mathbb{T}^2}\nabla_j(d_kD_{kp}d_pd_id_j)\nabla_l\nabla_lv_idx\\
=&-\mu_1\int_{\mathbb{T}^2}\nabla_l(d_kD_{kp}d_pd_id_j)\nabla_l\nabla_jv_idx=-\mu_1\int_{\mathbb{T}^2}\nabla_l(d_kD_{kp}d_pd_id_j)\nabla_lD_{ij}dx\\
=&-\mu_1\int_{\mathbb{T}^2}(d_kd_p\nabla_lD_{kp})^2dx-2\mu_1\int_{\mathbb{T}^2}D_{kp}\nabla_ld_kd_pd_id_j\nabla_lD_{ij}dx\\
&-2\mu_1\int_{\mathbb{T}^2}D_{kp}d_kd_pd_i\nabla_ld_j\nabla_lD_{ij}dx:=-\mu_1\int_{\mathbb{T}^2}(d_id_j\nabla_lD_{ij})^2dx+I_1+I_2.
\end{aligned}$$

$$\begin{aligned}
&-\mu_5\int_{\mathbb{T}^2}\nabla\cdot[(D(\mathbf{v})\mathbf{d}\otimes\mathbf{d}+\mathbf{d}\otimes D(\mathbf{v})\mathbf{d})]\cdot\Delta\mathbf{v}dx\\
=&-\mu_5\int_{\mathbb{T}^2}\nabla_j(D_{ik}d_kd_j)\nabla_l\nabla_lv_idx-\mu_5\int_{\mathbb{T}^2}\nabla_i(d_jD_{ik}d_k)\nabla_l\nabla_lv_jdx\\
=&-\mu_5\int_{\mathbb{T}^2}\nabla_l(D_{ik}d_kd_j)\nabla_l\nabla_jv_idx-\mu_5\int_{\mathbb{T}^2}\nabla_l(d_jD_{ik}d_k)\nabla_l\nabla_iv_jdx\\
=&-2\mu_5\int_{\mathbb{T}^2}\nabla_l(D_{ik}d_kd_j)\nabla_lD_{ij}dx\\
=&-2\mu_5\int_{\mathbb{T}^2}(\nabla_lD_{ik}d_k)^2dx-2\mu_5\int_{\mathbb{T}^2}\nabla_ld_jd_kD_{ik}\nabla_lD_{ij}dx-2\mu_5\int_{\mathbb{T}^2}d_j\nabla_ld_kD_{ik}\nabla_lD_{ij}dx\\
:=&-2\mu_5\int_{\mathbb{T}^2}(\nabla_lD_{ik}d_k)^2dx+I_3+I_4.
\end{aligned}$$

$$-\int_{\mathbb{T}^2}[(K\Delta^2\varphi-\nabla\cdot f(\mathbf{d}))\mathbf{d}]\cdot\Delta\mathbf{v}dx:=I_5.$$

Summing up, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{v}\|^2+\frac{\mu_4}{2}\|\Delta\mathbf{v}\|^2+\mu_1\int_{\mathbb{T}^2}(d_id_j\nabla_lD_{ij})^2dx+2\mu_5\int_{\mathbb{T}^2}(\nabla_lD_{ik}d_k)^2dx\\
=&\int_{\mathbb{T}^2}(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\Delta\mathbf{v}dx+\sum_{k=1}^5I_k.
\end{aligned}\tag{3.18}$$

By Lemma 3.5 and Corollary 3.1, for any $t_1>0$, we have obtained the uniform estimate:

$$\|\varphi(t)\|_{H^3}+\|\nabla\varphi(t)\|_{\mathbf{L}^\infty}\le M,\quad\forall t\ge t_1>0. \tag{3.19}$$

We now apply the Gagliardo–Nirenberg inequality, Lemma 3.4 and (3.19) to estimate the right-hand side of (3.18).

$$\int_{\mathbb{T}^2}(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\Delta\mathbf{v}dx\le\|\mathbf{v}\|_{\mathbf{L}^4}\|\nabla\mathbf{v}\|_{\mathbf{L}^4}\|\Delta\mathbf{v}\|\le\frac{\mu_4}{8}\|\Delta\mathbf{v}\|^2+C\|\nabla\mathbf{v}\|^4,$$

Since

$$\|\nabla\mathbf{v}\|^2\|\nabla\mathbf{d}\|^2_{\mathbf{L}^\infty}\|\mathbf{d}\|^2_{\mathbf{L}^\infty}\le C\|\nabla\mathbf{v}\|^2(\|\Delta^2\varphi\|+1)$$

$$\leq \; C\|\nabla \mathbf{v}\|^2(\|\mathcal{Q}\|+1) \leq C\|\mathcal{Q}\|^2 + C\|\nabla \mathbf{v}\|^4 + C\|\nabla \mathbf{v}\|^2,$$

we have

$$\begin{aligned} I_1 &\leq \frac{\mu_1}{4}\int_{\mathbb{T}^2}(d_id_j\nabla_l D_{ij})^2dx + C\|\nabla \mathbf{v}\|^2\|\nabla \mathbf{d}\|^2_{\mathbf{L}^\infty}\|\mathbf{d}\|^2_{\mathbf{L}^\infty} \\ &\leq \frac{\mu_1}{4}\int_{\mathbb{T}^2}(d_id_j\nabla_l D_{ij})^2dx + C\|\mathcal{Q}\|^2 + C\|\nabla \mathbf{v}\|^4 + C\|\nabla \mathbf{v}\|^2. \end{aligned}$$

For I_2, after integrating by parts, we have

$$\begin{aligned} I_2 &= -2\mu_1\int_{\mathbb{T}^2} D_{kp}d_kd_pd_i\nabla_l d_j\nabla_l D_{ij}dx \\ &= 2\mu_1\int_{\mathbb{T}^2}\nabla_l D_{kp}d_kd_pd_i\nabla_l d_jD_{ij}dx + 4\mu_1\int_{\mathbb{T}^2}D_{kp}\nabla_l d_kd_pd_i\nabla_l d_jD_{ij}dx \\ &\quad +2\mu_1\int_{\mathbb{T}^2}D_{kp}d_kd_p\nabla_l d_i\nabla_l d_jD_{ij}dx + 2\mu_1\int_{\mathbb{T}^2}D_{kp}d_kd_pd_i\nabla_l\nabla_l d_jD_{ij}dx \\ &:= I_{2a}+I_{2b}+I_{2c}+I_{2d}, \end{aligned}$$

where

$$\begin{aligned} I_{2a} &\leq \frac{\mu_1}{4}\int_{\mathbb{T}^2}(d_id_j\nabla_l D_{ij})^2dx + C\|\nabla \mathbf{v}\|^2\|\nabla \mathbf{d}\|^2_{\mathbf{L}^\infty}\|\mathbf{d}\|^2_{\mathbf{L}^\infty} \\ &\leq \frac{\mu_1}{4}\int_{\mathbb{T}^2}(d_id_j\nabla_l D_{ij})^2dx + C\|\mathcal{Q}\|^2 + C\|\nabla \mathbf{v}\|^4 + C\|\nabla \mathbf{v}\|^2, \\ I_{2b}+I_{2c} &\leq C\|\nabla \mathbf{v}\|^2\|\nabla \mathbf{d}\|^2_{\mathbf{L}^\infty}\|\mathbf{d}\|^2_{\mathbf{L}^\infty} \leq C\|\mathcal{Q}\|^2 + C\|\nabla \mathbf{v}\|^4 + C\|\nabla \mathbf{v}\|^2, \\ I_{2d} &\leq C\|\mathbf{d}\|^3_{\mathbf{L}^\infty}\|\Delta \mathbf{d}\|\|\nabla \mathbf{v}\|^2_{\mathbf{L}^4} \leq C(\|\Delta^2\varphi\|^{\frac{1}{2}}+1)\|\Delta \mathbf{v}\|\|\nabla \mathbf{v}\| \\ &\leq \frac{\mu_4}{8}\|\Delta \mathbf{v}\|^2 + C\|\mathcal{Q}\|^2 + C\|\nabla \mathbf{v}\|^4 + C\|\nabla \mathbf{v}\|^2. \end{aligned}$$

As a consequence,

$$I_2 \leq \frac{\mu_1}{4}\int_{\mathbb{T}^2}(d_id_j\nabla_l D_{ij})^2dx + \frac{\mu_4}{8}\|\Delta \mathbf{v}\|^2 + C\|\mathcal{Q}\|^2 + C\|\nabla \mathbf{v}\|^4 + C\|\nabla \mathbf{v}\|^2.$$

Next,

$$\begin{aligned} I_3+I_4 &\leq \|\Delta \mathbf{v}\|\|\nabla \mathbf{v}\|\|\mathbf{d}\|_{\mathbf{L}^\infty}\|\nabla \mathbf{d}\|_{\mathbf{L}^\infty} \leq \varepsilon\|\Delta \mathbf{v}\|^2 + C\|\nabla \mathbf{v}\|^2\|\nabla \mathbf{d}\|^2_{\mathbf{L}^\infty}\|\mathbf{d}\|^2_{\mathbf{L}^\infty} \\ &\leq \frac{\mu_4}{8}\|\Delta \mathbf{v}\|^2 + C\|\mathcal{Q}\|^2 + C\|\nabla \mathbf{v}\|^4 + C\|\nabla \mathbf{v}\|^2. \end{aligned}$$

$$I_5 = \int_{\mathbb{T}^2}\mathcal{Q}\mathbf{d}\cdot\Delta \mathbf{v}dx \leq \|\Delta \mathbf{v}\|\|\mathcal{Q}\|\|\nabla\varphi\|_{\mathbf{L}^\infty} \leq \frac{\mu_4}{8}\|\Delta \mathbf{v}\|^2 + C\|\mathcal{Q}\|^2.$$

It follows from (3.18) and the above estimates that

$$\begin{aligned} &\frac{d}{dt}\|\nabla \mathbf{v}\|^2 + \frac{\mu_4}{2}\|\Delta \mathbf{v}\|^2 + \mu_1\int_{\mathbb{T}^2}(d_id_j\nabla_l D_{ij})^2dx + 4\mu_5\int_{\mathbb{T}^2}(\nabla_l D_{ik}d_k)^2dx \\ \leq\; & C\|\mathcal{Q}\|^2 + C\|\nabla \mathbf{v}\|^4 + C\|\nabla \mathbf{v}\|^2, \quad t \geq t_1. \end{aligned} \tag{3.20}$$

On the other hand, by equation (1.7) and integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{Q}(t)\|^2 = -K\int_{\mathbb{T}^2}\mathcal{Q}\Delta^2\varphi_t dx + \int_{\mathbb{T}^2}\mathcal{Q}(\nabla\cdot f(\mathbf{d}))_t dx$$

$$
\begin{aligned}
&= -\lambda K\|\Delta \mathcal{Q}\|^2 + K\int_{\mathbb{T}^2} \Delta \mathcal{Q}\cdot \Delta(\mathbf{v}\cdot\nabla\varphi)dx \\
&\quad +\frac{1}{\epsilon^2}\int_{\mathbb{T}^2}\nabla\mathcal{Q}\cdot[(|\nabla\varphi|^2-1)\nabla(\mathbf{v}\cdot\nabla\varphi)]dx - \frac{\lambda}{\epsilon^2}\int_{\mathbb{T}^2}\nabla\mathcal{Q}\cdot[(|\nabla\varphi|^2-1)\nabla\mathcal{Q}]dx \\
&\quad +\frac{2}{\epsilon^2}\int_{\mathbb{T}^2}\nabla\mathcal{Q}\cdot[(\nabla\varphi\cdot\nabla(\mathbf{v}\cdot\nabla\varphi))\nabla\varphi]dx - \frac{2\lambda}{\epsilon^2}\int_{\mathbb{T}^2}\nabla\mathcal{Q}\cdot[(\nabla\varphi\cdot\nabla\mathcal{Q})\nabla\varphi]dx \\
&:= -\lambda K\|\Delta\mathcal{Q}\|^2 + \sum_{k=1}^{5} J_k. \qquad (3.21)
\end{aligned}
$$

The terms $J_1, ..., J_5$ on the right hand side of (3.21) can be estimated as follows.

$$
\begin{aligned}
J_1 &= K\int_{\mathbb{T}^2}\Delta\mathcal{Q}\Delta\mathbf{v}\cdot\nabla\varphi dx + 2K\int_{\mathbb{T}^2}\Delta\mathcal{Q}\nabla_k v_i\nabla_k\nabla_i\varphi dx + K\int_{\mathbb{T}^2}\Delta\mathcal{Q}\,\mathbf{v}\cdot\nabla\Delta\varphi dx \\
&:= J_{1a} + J_{1b} + J_{1c},
\end{aligned}
$$

where by the uniform estimate (3.19), Lemma 3.4 and the Sobolev embedding theorem, we get

$$
\begin{aligned}
J_{1a} &\le K\|\nabla\varphi\|_{\mathbf{L}^\infty}\|\Delta\mathbf{v}\|\|\Delta\mathcal{Q}\| \le \frac{\mu_4}{16\alpha}\|\Delta\mathbf{v}\|^2 + \frac{4\alpha K^2M^2}{\mu_4}\|\Delta\mathcal{Q}\|^2, \\
J_{1b} &\le C\|\Delta\mathcal{Q}\|\|\nabla\mathbf{v}\|_{\mathbf{L}^4}\|\varphi\|_{W^{2,4}} \le C\|\Delta\mathcal{Q}\|\|\nabla\mathbf{v}\|^{\frac12}\|\Delta\mathbf{v}\|^{\frac12} \\
&\le \frac{\lambda K}{8}\|\Delta\mathcal{Q}\|^2 + \frac{\mu_4}{16\alpha}\|\Delta\mathbf{v}\|^2 + C\|\nabla\mathbf{v}\|^2, \\
J_{1c} &\le C\|\Delta\mathcal{Q}\|\|\mathbf{v}\|_{\mathbf{L}^4}\|\varphi\|_{W^{3,4}} \le C\|\Delta\mathcal{Q}\|\|\nabla\mathbf{v}\|(\|\nabla\Delta\varphi\|_{\mathbf{L}^4}+1) \\
&\le \frac{\lambda K}{8}\|\Delta\mathcal{Q}\|^2 + C\|\nabla\mathbf{v}\|^2(\|\mathcal{Q}\|+1) \\
&\le \frac{\lambda K}{8}\|\Delta\mathcal{Q}\|^2 + C\|\mathcal{Q}\|^2 + C\|\nabla\mathbf{v}\|^4 + C\|\nabla\mathbf{v}\|^2.
\end{aligned}
$$

Next,

$$
\begin{aligned}
J_2 + J_4 &\le C\|\nabla\mathcal{Q}\|(\|\nabla\varphi\|^2_{\mathbf{L}^\infty}+1)(\|\nabla\mathbf{v}\|\|\nabla\varphi\|_{\mathbf{L}^\infty} + \|\mathbf{v}\|_{\mathbf{L}^4}\|\varphi\|_{W^{2,4}}) \\
&\le C(\|\Delta\mathcal{Q}\|^{\frac12}\|\mathcal{Q}\|^{\frac12} + \|\mathcal{Q}\|)\|\nabla\mathbf{v}\| \\
&\le \frac{\lambda K}{8}\|\Delta\mathcal{Q}\|^2 + C\|\mathcal{Q}\|^2 + C\|\nabla\mathbf{v}\|^2, \\
J_3 + J_5 &\le C\|\nabla\mathcal{Q}\|^2(\|\nabla\varphi\|^2_{\mathbf{L}^\infty}+1) \le \frac{\lambda K}{8}\|\Delta\mathcal{Q}\|^2 + C\|\mathcal{Q}\|^2.
\end{aligned}
$$

Inserting the above estimates into (3.21), we obtain that

$$
\frac{d}{dt}\|\mathcal{Q}(t)\|^2 + \left(\lambda K - \frac{8\alpha K^2M^2}{\mu_4}\right)\|\Delta\mathcal{Q}\|^2 - \frac{\mu_4}{4\alpha}\|\Delta\mathbf{v}\|^2 \le C\|\mathcal{Q}\|^2 + C\|\nabla\mathbf{v}\|^4 + C\|\nabla\mathbf{v}\|^2. \quad (3.22)
$$

Multiplying (3.22) by α and adding it to (3.20), we have

$$
\begin{aligned}
&\frac{d}{dt}\mathbf{A}(t) + \frac{\mu_4}{4}\|\Delta\mathbf{v}\|^2 + \alpha\left(\lambda K - \frac{8\alpha K^2M^2}{\mu_4}\right)\|\Delta\mathcal{Q}\|^2 \\
\le\ & C(1+\alpha)(\|\mathcal{Q}\|^2 + \|\nabla\mathbf{v}\|^4 + \|\nabla v\|^2), \quad \forall\ t \ge t_1. \qquad (3.23)
\end{aligned}
$$

Taking

$$
\alpha = \frac{\lambda\mu_4}{16KM^2}, \qquad (3.24)
$$

we conclude from (3.23) that (3.15) holds. The lemma is proved. □

Lemma 3.7. *Suppose $n = 2$. For any $\mathbf{v}_0 \in \dot{H}$, $\varphi_0 \in H^2$, the weak solution to* (1.5)–(1.7) *satisfies*

$$\|(\mathbf{v},\varphi)(t)\|_{\Phi_1} \le C(t_2), \quad \forall\, t \ge t_2 > 0, \tag{3.25}$$

where $t_2 > 0$ is arbitrary and $C(t_2)$ is a positive constant depending on $\|\mathbf{v}_0\|$, $\|\varphi_0\|_{H^2}$, t_2 and coefficients of the system. In particular, $\lim_{t_2 \to 0^+} C(t_2) = +\infty$. If we assume in addition that $\mathbf{v}_0 \in V$ and $\varphi_0 \in H^4$, $\|(\mathbf{v},\varphi)(t)\|_{\Phi_1}$ can be bounded by a constant depending on $\|(\mathbf{v}_0,\varphi_0)\|_{\Phi_1}$ uniformly in time.

Proof. By Corollary 3.1, Lemma 3.1 and the definition of α (cf. (3.24)), we infer that for arbitrary $t_1 > 0$,

$$\int_{t_1}^{+\infty} \mathbf{A}(t)dt < +\infty. \tag{3.26}$$

Since (3.15) holds for $t \ge t_1$, we can apply the uniform Gronwall lemma [31, Lemma III.1.1] to get the following uniform estimate: for any $r > 0$,

$$\mathbf{A}(t+r) \le C(t_1)\left(1 + \frac{1}{r}\right), \quad \forall t \ge t_1,$$

where $C(t_1)$ is a positive constant depending on $\|\mathbf{v}_0\|$, $\|\varphi_0\|_{H^2}$, t_1. Since t_1 and r are arbitrary positive constants, we can prove the uniform estimate (3.25) for any $t_2 > 0$.

If the initial data is more regular, namely, $\mathbf{v}_0 \in V$ and $\varphi_0 \in H^4$, by Lemma 3.5 we can easily show that $\|(\mathbf{v},\varphi)(t)\|_{\Phi_1}$ can be uniformly bounded by a constant depending on $\|(\mathbf{v}_0,\varphi_0)\|_{\Phi_1}$. The proof is complete. □

Corollary 3.2. *Suppose $n = 2$. For any $\mathbf{v}_0 \in \dot{H}$, $\varphi_0 \in H^2$, there exists $t^* > 0$ depending on $\|\mathbf{v}_0\|$, $\|\varphi_0\|_{H^2}$, such that for all $t \ge t^*$, the weak solution to* (1.5)–(1.7) *satisfies*

$$\|(\mathbf{v},\varphi)(t)\|_{\Phi_1} \le M, \quad \forall\, t \ge t^*, \tag{3.27}$$

where M is independent of $\mathbf{v}_0, \varphi_0$.

Proof. It follows from Lemma 3.2 that there exists t_3 depending on $\|\mathbf{v}_0\|$, $\|\varphi_0\|_{H^2}$, such that for all $t \ge t_3$, the weak solution to (1.5)–(1.7) satisfies

$$\|(\mathbf{v},\varphi)(t)\|_{\Phi} \le M_1, \quad \forall\, t \ge t_3, \tag{3.28}$$

where M_1 is independent of $\mathbf{v}_0, \varphi_0$. Now, Lemma 3.1 and Lemma 3.4 imply that for $t \ge t_3$,

$$\sup_{t \ge t_3} \int_t^{t+1} (\|\nabla \mathbf{v}(\tau)\|^2 + \|\nabla\Delta\varphi(\tau)\|^2)d\tau \le C \int_{t_3}^{\infty} (\|\mathcal{Q}(\tau)\|^2 + \|\nabla \mathbf{v}(\tau)\|^2) + C \le C,$$

with C depending on M_1. Then (3.8) and the uniform Gronwall lemma yield that $\|\varphi(t+1)\|_{H^3} \le C$, for $t \ge t_3$. As a consequence, $\|\nabla\varphi(t)\|_{\mathbf{L}^\infty} \le M_2$ for all $t \ge t_3 + 1$. For $t \ge t_3 + 1$, we fix α in (3.24) with $\alpha = \frac{\lambda\mu_4}{16KM_2^2}$. Then (3.15) holds with C only depending on M_1, M_2. Applying the uniform Gronwall inequality once more, we have $\mathbf{A}(t) \le M_3$, for $t \ge t_3 + 2$, where M_3 depends on M_1, M_2. Finally, taking $t^* = t_3 + 2$, we conclude the proof. □

Proof of Theorem 2.1. First, thanks to Lemma 3.7, we see that

$$\mathbf{A}(t) \le C(t), \quad \forall t > 0, \tag{3.29}$$

where $C(t)$ depends on $\|(\mathbf{v}_0, \varphi_0)\|_\Phi$ and $C(t) \nearrow +\infty$ for $t \searrow 0^+$ but remains uniformly bounded for $t \nearrow +\infty$. Integrating (3.15) over $[t, t+1]$, recalling Lemma 3.1, we have

$$\int_t^{t+1} \big(\|\Delta \mathbf{v}(s)\|^2 + \|\varphi(s)\|_{H^6}^2\big)\, ds \le C(t), \quad \forall t > 0. \tag{3.30}$$

As a consequence of (3.29) and (3.30), we immediately have

$$\int_t^{t+1} \big(\|\mathbf{v}(s)\cdot\nabla\mathbf{v}(s)\|^2 + \|\mathbf{v}(s)\cdot\nabla\varphi(s)\|_{H^2}^2\big)\, ds \le C(t), \quad \forall t > 0.$$

On the other hand, it is easy to check that (3.29) and (3.30) gives an analogous $L^2(t, t+1; \mathbf{L}^2(\mathbb{T}^2))$ estimate of $\nabla\cdot(\tilde{\sigma}^d + \tilde{\sigma}^e)$ for any $t > 0$. Hence, by direct comparison with (1.5)–(1.7), we can see that

$$\int_t^{t+1} \big(\|\partial_t \mathbf{v}(s)\|^2 + \|\partial_t\varphi(s)\|_{H^2}^2\big)\, ds \le C(t), \quad \forall t > 0.$$

The $2D$ smoothing property is thus proved.

Finally, similar to [20], we know that if the initial data are regular, the existence of a weak solution together with high-order estimates implies a strong solution, and by Theorem 2.2, the strong solution is actually unique.

3.3 The global attractor and exponential attractors

Lemma 3.2 and Corollary 3.2 entail that there exists a compact absorbing set in Φ. If we had uniqueness for the weak solutions, this would be sufficient to prove the existence of the global attractor by using the classical theory on dynamical systems (see, e.g., [31]). We can overcome this difficulty essentially relying on the regularization of weak solutions to strong solutions for strictly positive times proved in Theorem 2.3. This implies that, for strictly positive times, we have enough regularity to ensure uniqueness by Theorem 2.2. As a consequence, we have the following weaker form of uniqueness, to which we refer as *unique continuation*:

Proposition 3.1. *Suppose $n = 2$. For any two weak solutions $(\mathbf{v}_1, \varphi_1)$ and $(\mathbf{v}_2, \varphi_2)$ such that $(\mathbf{v}_1(T), \varphi_1(T)) = (\mathbf{v}_2(T), \varphi_2(T))$ at some $T > 0$, then it holds $(\mathbf{v}_1, \varphi_1) \equiv (\mathbf{v}_2, \varphi_2)$ for any $t \ge T$.*

A possible way to construct the global attractor is to apply the theory of ℓ-trajectories introduced by Málek and Nečas in [24] and later developed by Málek and Pražák in [25] (For other possible approaches, the reader is referred to, e.g., Ball [1] or to Remark 3.1 in this paper). Besides, we can also use the ℓ-trajectory method to study the existence of an exponential attractor.

For the sake of convenience, we recall some highlight points of the ℓ-trajectory method here. Roughly speaking, the ℓ-trajectory method consists in lifting the dynamics from the physical phase space to a space of trajectories with an arbitrary but fixed length $\ell > 0$. More precisely, for our current problem, by ℓ-trajectory we mean any solution to (1.5)-(1.7) defined on the time interval $[0,\ell]$. Then, we endow the space of ℓ-trajectories denoted by $\mathscr{X}_\ell$ with the topology of $L^2(0,\ell;\Phi)$. Note that weak solutions to (1.5)–(1.7) lie (at least) in

$$C_w([0,\ell];\Phi) := \{(\mathbf{v},\varphi) \in L^\infty(0,\ell;\Phi) : \langle(\mathbf{v},\varphi),(\mathbf{u},\psi)\rangle_{\Phi,\Phi'} \in C([0,\ell]),\ \forall\ (\mathbf{u},\psi) \in \Phi'\},$$

which makes it reasonable to talk about the point values of trajectories.

The *unique continuation* property implies that from an end point of an ℓ-trajectory there starts at most one solution. Combined with the existence theorem, this implies that if $(\mathbf{u},\phi) \in \mathscr{X}_\ell$ and $T > \ell$, then there exists a unique $(\mathbf{v},\varphi)$ which is a solution to (1.5)–(1.7) on $[0,T]$ such that $(\mathbf{u},\phi) = (\mathbf{v},\varphi)|_{[0,\ell]}$. Then we can define the semigroup $\mathscr{S}(t)$ on $\mathscr{X}_\ell$:

$$(\mathscr{S}(t)(\mathbf{u},\phi))(\tau) := (\mathbf{v}(t+\tau),\varphi(t+\tau)), \quad \tau \in [0,\ell]. \tag{3.31}$$

From now on, without loss of generality, we will fix $\ell = 1$. Corollary 3.2 implies that there exists $R > 0$ such that

$$B_1 = \{(\mathbf{v},\varphi) \in \Phi_1 : \|(\mathbf{v},\varphi)\|_{\Phi_1} \le R\} \subset \Phi_1 \subset\subset \Phi$$

is a compact, absorbing set for the solution map $S(t)$. Theorem 2.2 entails that the solution operator $S(t)$ confined on $\mathcal{B}_1$ is indeed a semigroup. Let

$$\mathcal{B}_1 := \overline{\bigcup_{t\in[0,T_0]} S(t)B_1}^{\Phi_1} \tag{3.32}$$

where $T_0 > 0$ is a time such that $S(t)B_1 \subset B_1$ for all $t \ge T_0$ and the closure is taken with respect to the weak topology of Φ_1. Then B_1 is a compact, absorbing and positive invariant set for $S(t)$. Define

$$\mathscr{B}_1^1 = \{(\mathbf{u},\phi) \in \mathscr{X}_\ell : (\mathbf{u},\phi)(0) \in \mathcal{B}_1\}. \tag{3.33}$$

Note that $\mathscr{B}_1^1$ is indeed closed with respect to the topology of $L^2(0,1;\Phi)$. Using Corollary 3.2 and Proposition 3.1, one can verify that all the assumptions in [25, Theorem 2.1] are satisfied and as a result, the dynamical system $(\mathscr{S}(t),\mathscr{X}_\ell)$ possesses the global attractor $\mathbb{A}$. Next, we introduce the following map evaluation map

$$e : L^2(0,1;\Phi) \mapsto \Phi \quad \text{defined by } e((\mathbf{u},\phi)) = (\mathbf{u}(1),\phi(1)). \tag{3.34}$$

Define $\mathbf{B} = e(\mathscr{B}_1^1)$. We see that $\mathbf{B} \subset \Phi_1$, thus $S(t) : \Phi \to \Phi$ is a semigroup on $\mathbf{B}$ and $\mathbf{B}$ is positively invariant. If we can show that the map e is Lipschitz continuous on $\mathscr{B}_1^1$ (which is indeed true, see (3.52) below), then we can project the global attractor $\mathbb{A}$ back to the physical space Φ obtaining the usual global attractor $\mathcal{A} = e(\mathbb{A})$ for the dynamic system $(S(t),\mathbf{B})$. Since $\mathbf{B}$ is actually absorbing, $\mathcal{A}$ is also a global attractor in the phase space Φ.

Remark 3.1. *If one is interested only in the existence of the global attractor, one can reason as follows, without invoking the ℓ-trajectory theory. First of all, combining Theorem 2.1, 2.2 and 2.3, we have that the restriction of the solution operator, named $\tilde{S}(t)$, to the bounded sets of Φ_1 is a semigroup. Moreover, Corollary 3.2 give the dissipativity of $\tilde{S}(t)$ with respect to the Φ_1 metric. As a consequence, the standard theory of dynamical systems gives the existence of the global attractor $\mathcal{A}$ attracting the bounded sets of Φ_1 but with respect to the Φ-topology. Finally, the smoothing property implies that $\mathcal{A}$ is indeed the attractor for the weak solutions, since it attracts also the Φ-bounded sets.*

Our next step is to prove the finite dimensionality (in terms of fractal dimension) of the global attractor $\mathcal{A}$ constructed above and the existence of an exponential attractor. As anticipated in the introduction, the finite dimensionality of the global attractor will be deduced as a consequence of the existence of a finer attracting set, the exponential attractor. We recall the following (cf. [8])

Definition 3.1. *A compact subset $\mathcal{M}$ of the phase space Φ is called an exponential attractor for the semigroup $S(t)$ if the following conditions are satisfied: (E1) The set $\mathcal{M}$ is positively invariant, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for all $t \geq 0$; (E2) The fractal dimension (see, e.g., [26, 31]) of $\mathcal{M}$ in Φ is finite; (E3) The set $\mathcal{M}$ attracts exponentially fast the image of the bounded subsets of the phase space Φ. Namely, there exist $C, \beta > 0$ such that*

$$\mathrm{dist}_\Phi(S(t)B, \mathcal{M}) \leq C\, e^{-\beta t}, \quad \forall \text{ bounded set } B \subset \Phi, \quad \forall t \geq 0. \tag{3.35}$$

Note that, by construction, the exponential attractor, when it exists, always contains the global attractor. Thus, property $(E2)$ gives that the global attractor has finite fractal dimension too. Besides its importance in proving the finite dimensionality of the global attractor, the existence of an exponential attractor is of interest in itself. In fact, it may resolve some of the major drawbacks of the global attractor, namely its arbitrary slow attraction, which makes the global attractor very sensitive to perturbation and to numerical approximation, and the difficulty in estimating its rate of convergence. We refer the readers to the recent survey [28] for more details and additional references.

To prove the existence of an exponential attractor $\mathcal{M}$, we first use the following existence theorem proposed in [10], which gives an efficient strategy to obtain the existence of an exponential attractor for the discrete semigroup generated by the iterations of a proper map $\mathbb{S}$. Then in a second step, we construct the desired exponential attractor for the semigroup with continuous time.

Lemma 3.8. (cf. [10]) *Let $\mathscr{H}$ and $\mathscr{H}_1$ be two Banach spaces such that $\mathscr{H}_1$ is compactly embedded into $\mathscr{H}$. Suppose $\mathbb{B}_1$ is a bounded closed subset of $\mathscr{H}$. Let us give a map $\mathbb{S} : \mathbb{B}_1 \to \mathbb{B}_1$ such that*

$$\|\mathbb{S}b_1 - \mathbb{S}b_2\|_{\mathscr{H}_1} \leq L\|b_1 - b_2\|_{\mathscr{H}}, \quad \forall\, b_1, b_2 \in \mathbb{B}_1, \tag{3.36}$$

where the constant L is independent of b_1 and b_2. Then, the discrete semigroup $\{\mathbb{S}(n),\, n \in \mathbb{N}\}$ generated on $\mathbb{B}_1$ by the iterations of the map $\mathbb{S}$ possesses an exponential attractor, i.e., there exists a compact set $\mathcal{M}_d \subset \mathbb{B}_1$ such that (E1) $\mathcal{M}_d$ is positively invariant: $\mathbb{S}\mathcal{M}_d \subset \mathcal{M}_d$; (E2) The fractal dimension of $\mathcal{M}_d$ in $\mathscr{H}$ is finite: $\dim_f(\mathcal{M}_d, \mathscr{H}) \leq M < +\infty$; (E3) $\mathcal{M}_d$ attracts exponentially the images of $\mathbb{B}_1$ under the iterations of the map $\mathbb{S}$: $\mathrm{dist}_{\mathscr{H}}(\mathbb{S}(n)\mathbb{B}_1, \mathcal{M}_d) \leq Ce^{-\kappa n}$. Moreover, the positive constants M, C and κ can be expressed explicitly in terms of the squeezing constant L, *the size of the set $\mathbb{B}_1$ and the entropy of the compact embedding $\mathscr{H}_1 \subset\subset \mathscr{H}$.*

In order to apply Lemma 3.8, one has to properly define the map $\mathbb{S}$, together with the spaces $\mathscr{H}$, $\mathscr{H}_1$ and $\mathbb{B}_1$. A typical choice for dissipative problems like (1.5)–(1.7), would be (recall (3.32))

$$\mathbb{S} := S(1), \quad \mathscr{H} := \Phi, \quad \mathscr{H}_1 := \Phi_1, \quad \mathbb{B}_1 := \mathcal{B}_1.$$

Unfortunately, a closer inspection to system (1.5)–(1.7) reveals that the above choice is not completely satisfactory in the sense that proving a point-wise (in time) estimate for the difference of two solutions in the norm of Φ_1 appears to be difficult due to the highly nonlinear character of the problem. We overcome this difficulty by using the method of ℓ-trajectories to construct proper spaces $\mathscr{H}$ and $\mathscr{H}_1$ and then verify the assumptions of Lemma 3.8.

As we did for the construction of the global attractor, we still set $\ell = 1$. Let us define $\mathbb{B}_1 := \mathscr{B}_1^1$ (recall (3.33)) and

$$\mathscr{H} := L^2(0,1;\Phi), \quad \mathscr{H}_1 := L^2(0,1;\Phi_1) \cap (W^{1,1}(0,1;V') \times H^1(0,1;L^2)). \tag{3.37}$$

It follows from the Aubin–Lions compactness lemma that the embedding $\mathscr{H}_1 \subset \mathscr{H}$ is compact. We will apply Lemma 3.8 to the map $\mathbb{S} = \mathscr{S}(1)$ (see (3.31)) acting on the set $\mathbb{B}_1$. To this end, we only need to check the smoothing property (3.36). All the results in this subsection holds only in the two dimensional case. Moreover, we do not need any particular restriction on the values of the structural constants in the equations. Nevertheless, it would be quite interesting and important to find an explicit (and possibly sharp) dependence of the fractal dimension of the attractor with respect to the coefficients in the equations.

Lemma 3.9. *Suppose $n = 2$. Let $(\mathbf{v}_1, \varphi_1)$ and $(\mathbf{v}_2, \varphi_2)$ be two solutions to problem* (1.5)–(1.7) *with initial conditions in $\mathcal{B}_1$. Denote $\bar{\mathbf{v}} := \mathbf{v}_1 - \mathbf{v}_2$ and $\bar{\varphi} := \varphi_1 - \varphi_2$. Then, the following estimate holds*

$$\|\partial_t \bar{\mathbf{v}}\|_{L^1(1,2;V')} + \|\partial_t \bar{\varphi}\|_{L^2(1,2;L^2)} + \|(\bar{\mathbf{v}}, \bar{\varphi})\|_{L^2(1,2;\Phi_1)} \le C\|(\bar{\mathbf{v}}, \bar{\varphi})\|_{L^2(0,1;\Phi)}. \tag{3.38}$$

Proof. We know from Lemma 3.7 that for $i = 1, 2$

$$\|(\mathbf{v}_i, \varphi_i)(t)\|_{\Phi_1} \le C, \quad \forall\, t \ge 0. \tag{3.39}$$

Then we test the equation for $\bar{\mathbf{v}}$ with $\bar{\mathbf{v}}$ and the equation for $\bar{\varphi}$ with $\Delta^2\bar{\varphi}$, respectively. We obtain

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}(\|\bar{\mathbf{v}}\|^2 + \|\Delta\bar{\varphi}\|^2) + \frac{\mu_4}{2}\|\Delta\bar{\mathbf{v}}\|^2 + \lambda K\|\Delta^2\bar{\varphi}\|^2 \\
= & -\int_{\mathbb{T}^2} (\bar{\mathbf{v}}\cdot\nabla)\mathbf{v}_2\cdot\bar{\mathbf{v}}\,dx - \int_{\mathbb{T}^2} (\tilde{\sigma}_1^d - \tilde{\sigma}_2^d)\cdot\nabla\bar{\mathbf{v}}\,dx - \int_{\mathbb{T}^2} (\tilde{\sigma}_1^e - \tilde{\sigma}_2^e)\cdot\nabla\bar{\mathbf{v}}\,dx \\
& - \int_{\mathbb{T}^2} (\bar{\mathbf{v}}\nabla\varphi_1 + \mathbf{v}_2\nabla\bar{\varphi})\Delta^2\bar{\varphi}\,dx + \lambda\int_{\mathbb{T}^2} \nabla\cdot(f(\mathbf{d}_1) - f(\mathbf{d}_2))\Delta^2\bar{\varphi}\,dx := \sum_{i=1}^5 K_i.
\end{aligned} \tag{3.40}$$

Using estimate (3.39), it is not difficult to see that

$$K_1 \le \|\bar{\mathbf{v}}\|_{\mathbf{L}^4}^2\|\nabla\mathbf{v}_2\| \le \varepsilon\|\nabla\bar{\mathbf{v}}\|^2 + C(\varepsilon)\|\bar{\mathbf{v}}\|^2, \tag{3.41}$$

$$\begin{aligned}
K_4 &\le \|\bar{\mathbf{v}}\|\|\nabla\varphi_1\|_{\mathbf{L}^\infty}\|\Delta^2\bar{\varphi}\| + \|\mathbf{v}_2\|_{\mathbf{L}^4}\|\nabla\bar{\varphi}\|_{\mathbf{L}^4}\|\Delta^2\bar{\varphi}\| \\
&\le \varepsilon\|\Delta^2\bar{\varphi}\|^2 + C(\varepsilon)(\|\bar{\mathbf{v}}\|^2 + \|\bar{\varphi}\|_{H^2}^2),
\end{aligned} \tag{3.42}$$

$$\begin{aligned}
K_5 &\le \|\nabla\cdot(f(\mathbf{d}_1) - f(\mathbf{d}_2))\|\|\Delta^2\bar{\varphi}\| \\
&\le C[(\|\varphi_1\|_{L^\infty}^2 + 1)\|\Delta\bar{\varphi}\| + \|\Delta\varphi_2\|_{L^6}\|\nabla\varphi_1 + \nabla\varphi_2\|_{\mathbf{L}^6}\|\nabla\bar{\varphi}\|_{\mathbf{L}^6}]\|\Delta^2\bar{\varphi}\| \\
&\le \varepsilon\|\Delta^2\bar{\varphi}\|^2 + C(\varepsilon)\|\bar{\varphi}\|_{H^2}^2.
\end{aligned} \tag{3.43}$$

To estimate K_2, we need to control $\|\tilde{\sigma}_1^d - \tilde{\sigma}_2^d\|$.

$$\tilde{\sigma}_1^d - \tilde{\sigma}_2^d = \mu_1[(\mathbf{d}_1^\top D(\mathbf{v}_1)\mathbf{d}_1)\mathbf{d}_1\otimes\mathbf{d}_1 - (\mathbf{d}_2^\top D(\mathbf{v}_2)\mathbf{d}_2)\mathbf{d}_2\otimes\mathbf{d}_2]$$

$$
\begin{aligned}
&+\mu_5(D(\mathbf{v}_1)\mathbf{d}_1\otimes\mathbf{d}_1 - D(\mathbf{v}_2)\mathbf{d}_2\otimes\mathbf{d}_2) + \mu_5(\mathbf{d}_1\otimes D(\mathbf{v}_1)\mathbf{d}_1 - \mathbf{d}_2\otimes D(\mathbf{v}_2)\mathbf{d}_2)\\
:=\ & J_1 + J_2 + J_3.
\end{aligned}
$$

We give a detailed L^2-estimate only for the terms J_1 in the above decomposition, since for the other two (lower-order) terms J_2, J_3, the argument is essentially the same and actually simpler. We have

$$
\begin{aligned}
J_1 &:= (\mathbf{d}_1^\top D(\mathbf{v}_1)\mathbf{d}_1)\mathbf{d}_1\otimes\mathbf{d}_1 - (\mathbf{d}_2^\top D(\mathbf{v}_2)\mathbf{d}_2)\mathbf{d}_2\otimes\mathbf{d}_2 = \mathbf{d}_1^\top D(\bar{\mathbf{v}})\mathbf{d}_1\mathbf{d}_1\otimes\mathbf{d}_1\\
&\quad +(\mathbf{d}_1^\top - \mathbf{d}_2^\top)D(\mathbf{v}_2)\mathbf{d}_1\mathbf{d}_1\otimes\mathbf{d}_1 + \mathbf{d}_2^\top D(\mathbf{v}_2)\Big(\mathbf{d}_1\mathbf{d}_1\otimes\mathbf{d}_1 - \mathbf{d}_2\mathbf{d}_2\otimes\mathbf{d}_2\Big)\\
&= J_{1a} + J_{1b} + J_{1c}. \qquad (3.44)
\end{aligned}
$$

The term J_{1a} multiplied with $\nabla\bar{\mathbf{v}}$ produces a nonnegative (hence negligible in the estimate) term since

$$
\begin{aligned}
J_{1a} : \nabla\bar{\mathbf{v}} &= J_{1a} : D(\bar{\mathbf{v}}) = (\mathbf{d}_1^\top D(\bar{\mathbf{v}})\mathbf{d}_1\mathbf{d}_1\otimes\mathbf{d}_1) : D(\bar{\mathbf{v}})\\
&= \mathbf{d}_1^\top(D(\bar{\mathbf{v}}))\mathbf{d}_1(\mathbf{d}_1\otimes\mathbf{d}_1) : D(\bar{\mathbf{v}})) = |\mathbf{d}_1^\top D(\bar{\mathbf{v}})\mathbf{d}_1|^2 \ge 0.
\end{aligned}
$$

Then using Sobolev embedding theorem and (3.39), we have

$$
\begin{aligned}
\|J_{1b}\| &\le \|\mathbf{d}_1^\top - \mathbf{d}_2^\top\|_{\mathbf{L}^\infty}\|D(\mathbf{v}_2)\|\|\mathbf{d}_1(\mathbf{d}_1\otimes\mathbf{d}_1)\|_{\mathbf{L}^\infty} \le C\|\bar\varphi\|_{H^3} \le C\|\bar\varphi\|_{H^4}^{\frac12}\|\bar\varphi\|_{H^2}^{\frac12}, \qquad (3.45)\\
\|J_{1c}\| &\le \|\mathbf{d}_2\|_{\mathbf{L}^\infty}\|D(\mathbf{v}_2)\|(\|\bar{\mathbf{d}}\mathbf{d}_1\otimes\mathbf{d}_1\|_{\mathbf{L}^\infty} + \|\mathbf{d}_2\bar{\mathbf{d}}\otimes\mathbf{d}_1\|_{\mathbf{L}^\infty} + \|\mathbf{d}_1\mathbf{d}_2\otimes\bar{\mathbf{d}}\|_{\mathbf{L}^\infty})\\
&\le C\|\bar\varphi\|_{H^3} \le C\|\bar\varphi\|_{H^4}^{\frac12}\|\bar\varphi\|_{H^2}^{\frac12}. \qquad (3.46)
\end{aligned}
$$

As a consequence,

$$
K_2 \le \|\nabla\bar{\mathbf{v}}\|\|\tilde\sigma_1^d - \tilde\sigma_2^d\| \le \varepsilon\|\nabla\bar{\mathbf{v}}\|^2 + \varepsilon\|\bar\varphi\|_{H^4}^2 + C\|\bar\varphi\|_{H^2}^2.
$$

Concerning K_3, we have

$$
\begin{aligned}
\tilde\sigma_1^e - \tilde\sigma_2^e &= K[\nabla(\nabla\cdot\mathbf{d}_1)\otimes\mathbf{d}_1 - \nabla(\nabla\cdot\mathbf{d}_2)\otimes\mathbf{d}_2] - K[(\nabla\cdot\mathbf{d}_1)\nabla\mathbf{d}_1 - (\nabla\cdot\mathbf{d}_2)\mathbf{d}_2]\\
&\quad -[f(\mathbf{d}_1)\otimes\mathbf{d}_1 - f(\mathbf{d}_2)\otimes\mathbf{d}_2] := J_4 + J_5 + J_6.
\end{aligned}
$$

By the Sobolev embedding and (3.39), we obtain

$$
\begin{aligned}
J_4 &= \|\nabla(\nabla\cdot\mathbf{d}_1)\otimes\mathbf{d}_1 - \nabla(\nabla\cdot\mathbf{d}_2)\otimes\mathbf{d}_2\| \le \|\bar{\mathbf{d}}\|_{\mathbf{H}^2}\|\mathbf{d}_1\|_{\mathbf{L}^\infty} + \|\mathbf{d}_2\|_{\mathbf{H}^2}\|\bar{\mathbf{d}}\|_{\mathbf{L}^\infty}\\
&\le C\|\bar\varphi\|_{H^3} \le C\|\bar\varphi\|_{H^4}^{\frac12}\|\bar\varphi\|_{H^2}^{\frac12},\\
J_5 &= \|(\nabla\cdot\mathbf{d}_1)\otimes\nabla\mathbf{d}_1 - (\nabla\cdot\mathbf{d}_2)\otimes\nabla\mathbf{d}_2\| \le \|\bar{\mathbf{d}}\|_{\mathbf{H}^1}\|\nabla\mathbf{d}_1\|_{\mathbf{L}^\infty} + \|\mathbf{d}_2\|_{\mathbf{H}^1}\|\nabla\bar{\mathbf{d}}\|_{\mathbf{L}^\infty}\\
&\le C\|\bar\varphi\|_{H^2} + C\|\bar\varphi\|_{H^3} \le C\|\bar\varphi\|_{H^4}^{\frac12}\|\bar\varphi\|_{H^2}^{\frac12},\\
J_6 &\le C\|\bar\varphi\|_{H^2},
\end{aligned}
$$

which imply that

$$
K_3 \le \|\nabla\bar{\mathbf{v}}\|\|\bar\varphi\|_{H^4}^{\frac12}\|\bar\varphi\|_{H^2}^{\frac12} \le \varepsilon\|\nabla\bar{\mathbf{v}}\|^2 + \varepsilon\|\bar\varphi\|_{H^4}^2 + C\|\bar\varphi\|_{H^2}^2.
$$

Now we test the equation for $\bar\varphi$ by $\bar\varphi + \bar\varphi_t$. Similar computations give

$$
\frac{d}{dt}(\|\bar\varphi\|^2 + \|\Delta\bar\varphi\|^2) + \|\bar\varphi_t\|^2 + \|\bar\varphi\|_{H^2}^2 \le C(\|\bar{\mathbf{v}}\|^2 + \|\bar\varphi\|_{H^2}^2). \qquad (3.47)
$$

Summing (3.40) with (3.47), choosing ε sufficiently small, we obtain

$$\frac{d}{dt}(\|\bar{\mathbf{v}}\|^2 + 2\|\Delta\bar{\varphi}\|^2 + \|\bar{\varphi}\|^2) + \|\nabla\bar{\mathbf{v}}\|^2 + \|\partial_t\bar{\varphi}\|^2 + \|\bar{\varphi}\|_{H^4}^2 \le C(\|\bar{\mathbf{v}}\|^2 + \|\bar{\varphi}\|_{H^2}^2). \tag{3.48}$$

By the Gronwall Lemma, for any $0 \le y - t \le 2$,

$$\|\bar{\mathbf{v}}(y)\|^2 + \|\bar{\varphi}(y)\|_{H^2}^2 \le Ce^{2C}\left(\|\bar{\mathbf{v}}(t)\|^2 + \|\bar{\varphi}(t)\|_{H^2}^2\right). \tag{3.49}$$

Taking $t \in [0,1]$ and integrating (3.48) over $[t,2]$, we infer from (3.49) that

$$\begin{aligned} &\|(\bar{\mathbf{v}},\bar{\varphi})(2)\|_{\Phi}^2 + \int_t^2 (\|(\bar{\mathbf{v}},\bar{\varphi})(r)\|_{\Phi_1}^2 + \|\partial_t\bar{\varphi}(r)\|^2)dr \\ \le\ & C\left(\|(\bar{\mathbf{v}},\bar{\varphi})(t)\|_{\Phi}^2 + \int_t^2 \|(\bar{\mathbf{v}},\bar{\varphi})(r)\|_{\Phi}^2 dr\right) \le C\|(\bar{\mathbf{v}},\bar{\varphi})(t)\|_{\Phi}^2. \end{aligned} \tag{3.50}$$

Integrating (3.50) with respect to t over $[0,1]$, we finally obtain

$$\begin{aligned} \|(\bar{\mathbf{v}},\bar{\varphi})\|_{L^2(1,2;\Phi_1)}^2 + \|\partial_t\bar{\varphi}\|_{L^2(1,2;L^2)}^2 &\le \int_0^1\int_t^2 \|(\bar{\mathbf{v}},\bar{\varphi})(r)\|_{\Phi_1}^2 + \|\partial_t\bar{\varphi}(r)\|^2)drdt \\ &\le C\|(\bar{\mathbf{v}},\bar{\varphi})\|_{L^2(0,1;\Phi)}^2. \end{aligned} \tag{3.51}$$

It remains to estimate $\|\partial_t\bar{\mathbf{v}}\|_{L^1(1,2;V')}$. We use a duality argument. First, we recall that, for $u \in L^1(1,2;V')$,

$$\|u\|_{L^1(1,2;V')} = \sup_{\varphi}\left|\int_1^2 \langle u,\varphi\rangle dr\right|,$$

where the sup is taken over the function $\phi \in L^\infty(1,2;V)$ such that $\|\phi\|_{L^\infty(1,2;V)} = 1$ and the duality pairing is between V' and V. Consequently, thanks to (3.39) and (3.51), there holds

$$\begin{aligned} \int_1^2 \|\partial_t\bar{\mathbf{v}}(r)\|_{V'}dr &\le \int_1^2 \|\nabla\bar{\mathbf{v}}(r)\|dr + \int_1^2 (\|\bar{\mathbf{v}}(r)\|\|\nabla\mathbf{v}_1(r)\| + \|\nabla\bar{\mathbf{v}}(r)\|\|\mathbf{v}_2(r)\|)dr \\ &\quad + \int_1^2 (\|\tilde{\sigma}_1^d(r) - \tilde{\sigma}_2^d(r)\| + \|\tilde{\sigma}_1^e(r) - \tilde{\sigma}_2^e(r)\|)dr \\ &\le C\left(\int_0^1 \|(\bar{\mathbf{v}},\bar{\varphi})(r)\|_{\Phi}^2 dr\right)^{\frac{1}{2}}. \end{aligned}$$

The proof is complete. □

Thanks to Lemma 3.9, we have verified that the map $\mathbb{S} := \mathscr{S}(1)$ satisfies all of the assumptions of the abstract result Lemma 3.8. Therefore, the discrete semigroup $\{\mathscr{S}(n),\ n \in \mathbb{N}\}$ possesses an exponential attractor $\mathbb{M}_d$ in the trajectory space $\mathbb{B}_1$ endowed with the topology of $\mathscr{H} = L^2(0,1;\Phi)$.

Multiplying (3.48) with t and using the Gronwall lemma, we easily obtain that

$$\|(\bar{\mathbf{v}},\bar{\varphi})(1)\|_{\Phi}^2 \le C\int_0^1 \|(\bar{\mathbf{v}},\bar{\varphi})(r)\|_{\Phi}^2 dr, \tag{3.52}$$

which means that the map e (cf. (3.34) for the definition) is Lipschitz continuous on $\mathbb{B}_1$. This yields that projecting $\mathbb{M}_d$ back to the phase space Φ via

$$\mathcal{M}_d := e(\mathbb{M}_d) = \mathbb{M}_d\big|_{t=1}, \tag{3.53}$$

the resulting $\mathcal{M}_d$ is indeed the exponential attractor for the discrete semigroup $\{S(n),\, n \in \mathbb{N}\}$ acting on $\mathbf{B} = e(\mathbb{B}_1)$ (endowed with the topology of Φ).

We note that for all the trajectories $(\mathbf{v}, \varphi)$ starting from $\mathcal{B}_1$, there holds

$$\int_0^1 \|\partial_t \mathbf{v}(s)\|^2 + \|\partial_t \varphi(s)\|_{H^2}^2 ds \le C.$$

This, together with (3.49) imply that the map $(t, (\mathbf{v}_0, \varphi_0)) \mapsto S(t)((\mathbf{v}_0, \varphi_0))$ is Lipschitz continuous on $[0, 1] \times \mathcal{B}_1$ with respect to the $\mathbb{R} \times \Phi$ metric. Thus, the desired exponential attractor $\mathcal{M}$ with continuous time (and on the whole phase space since $\mathcal{B}_1$ is absorbing) is given by the standard expression (see [11] and [25] for further information)

$$\mathcal{M} := \bigcup_{t \in [0,1]} S(t)\mathcal{M}_d.$$

The proof of Theorem 2.4 is complete.

4 Convergence to Equilibrium in $2D$

Theorem 2.1 indicates that the total energy $\mathcal{E}(t)$ (cf. (2.1)) is decreasing with respect to time, consequently, it serves as a global Lyapunov functional for system (1.5)–(1.7). The ω-limit set of $(\mathbf{v}_0, \varphi_0) \in \dot{H} \times \dot{H}^2$ is defined as follows:

$$\begin{aligned} \omega(\mathbf{v}_0, \varphi_0) \;=\; & \{(\mathbf{v}_\infty, \varphi_\infty) \in \dot{H} \times \dot{H}^2 :\ \exists\ \{t_i\}_{i=1}^\infty \nearrow +\infty, \text{ such that} \\ & (\mathbf{v}(t_i), \varphi(t_i)) \to (\mathbf{v}_\infty, \varphi_\infty) \text{ in } \dot{H} \times H^2\}. \end{aligned} \tag{4.1}$$

It follows from Lemma 3.7 and the well-known result on dynamical systems (cf. [31, Lemma I.1.1]) that

Lemma 4.1. $\omega(\mathbf{v}_0, \varphi_0)$ *is a non-empty bounded connected subset in* $\dot{V} \times \dot{H}^4$. *Furthermore, (i) it is invariant under the nonlinear semigroup* $S(t)$. *(ii)* $\mathcal{E}$ *is constant on* $\omega(\mathbf{v}_0, \varphi_0)$. *(iii)* $\omega(\mathbf{v}_0, \varphi_0)$ *consists of steady states of system* (1.5)–(1.7).

We note that the energy inequality obtained in Lemma 3.6 not only yields uniform higher-order energy estimates of weak solutions (cf. (3.25)), but also indicates that the asymptotic limit points of weak solutions to problem (1.5)–(1.7) actually have a special form.

Lemma 4.2. *For* $\mathbf{v}_0 \in \dot{H}$, $\varphi_0 \in \dot{H}^2$, *the weak solutions to* (1.5)–(1.7) *have the following property*

$$\lim_{t \to +\infty} (\|\mathbf{v}(t)\|_{\mathbf{H}^1} + \|\mathcal{Q}(t)\|) = 0. \tag{4.2}$$

Proof. We recall that for any $t_1 > 0$, (3.26) holds. Using Lemma 3.6 and [35, Lemma 6.2.1], we conclude that $\lim_{t\to+\infty} \mathbf{A}(t) = 0$, which together with Corollary 3.1 and the Poincaré inequality leads to our conclusion. □

Lemma 4.2 implies that for any initial data $\mathbf{v}_0 \in \dot{H}$, $\varphi_0 \in \dot{H}^2$, their corresponding asymptotic limit points $(\mathbf{v}_\infty, \varphi_\infty)$ satisfy the following stationary problem (using the form (3.16)):

$$-\nabla P_\infty = \mathbf{v}_\infty = 0, \tag{4.3}$$

$$-K\Delta^2 \varphi_\infty + \nabla \cdot f(\nabla \varphi_\infty) = 0, \tag{4.4}$$

$$\int_{\mathbb{T}^2} \varphi_\infty dx = 0. \tag{4.5}$$

Lemma 4.3. *When $n = 2, 3$, for any $\varphi_0 \in L^1$, problem*

$$-K\Delta^2\varphi + \nabla \cdot f(\nabla\varphi) = 0, \quad \int_{\mathbb{T}^n} \varphi dx = \int_{\mathbb{T}^n} \varphi_0 dx \tag{4.6}$$

admits at least one weak solution ϕ, which is in fact smooth. If ϵ is properly large, then the weak solution is unique.

Proof. It is easy to verify that the energy $E(\varphi) = \frac{K}{2}\|\Delta\varphi\|^2 + \int_{\mathbb{T}^2} F(\nabla\varphi)dx$ admits at least one minimizer ϕ in $H^2 \cap \{\varphi \in L^1, \int_{\mathbb{T}^n} \varphi dx = \int_{\mathbb{T}^n} \varphi_0 dx\}$, which is a weak solution to problem (4.6). Moreover, for any weak solution ϕ to (4.6), we have

$$K\|\Delta\phi\|^2 + \frac{1}{\epsilon^2}\int_{\mathbb{T}^n} |\nabla\phi|^4 dx = \frac{1}{\epsilon^2}\|\nabla\phi\|^2 \le \frac{1}{2\epsilon^2}\int_{\mathbb{T}^n} |\nabla\phi|^4 dx + \frac{1}{2\epsilon^2}|\mathbb{T}^n|,$$

which together with the Poincaré inequality $\|\phi\| \le C_P(\|\nabla\phi\| + |\int_{\mathbb{T}^n} \phi dx|)$ implies that $\|\phi\|_{H^2}$ can be bounded by a constant depending on $|\int_{\mathbb{T}^n} \varphi_0 dx|$, ϵ, C_P and $|\mathbb{T}^n|$. A bootstrap argument yields that ϕ is actually smooth and for $m \in \mathbb{N}$, $\|\phi\|_{H^m}$ can be bounded by a constant depending on $|\int_{\mathbb{T}^n} \varphi_0 dx|$, ϵ, C_P and $|\mathbb{T}^n|$. Finally, let ϕ_1 and ϕ_2 be two solutions of (4.6), using the fact that $\int_{\mathbb{T}^n}(f(\nabla\phi_1) - f(\nabla\phi_2)) \cdot \nabla(\phi_1 - \phi_2)dx \ge 0$, then we infer from the Poincaré inequality that $K\|\Delta(\phi_1 - \phi_2)\|^2 \le \frac{1}{\epsilon^2}\|\nabla(\phi_1 - \phi_2)\|^2 \le \frac{C_P^2}{\epsilon^2}\|\Delta(\phi_1 - \phi_2)\|^2$. As a result, if $\epsilon > C_P K^{-\frac{1}{2}}$, then $\phi_1 = \phi_2$. (We refer to [18] for a similar problem but with different boundary conditions) ☐

4.1 Convergence to equilibrium

Lemma 4.2 yields the convergence of velocity field $\mathbf{v}$. In what follows, we study the convergence for φ. First, $\epsilon > C_P K^{-\frac{1}{2}}$, we infer from Lemma 4.3 that $\omega(\mathbf{v}_0, \varphi_0)$ consists of a single point $(0, \varphi_\infty)$ where φ_∞ is the unique solution to (4.4)–(4.5). However, if $\epsilon \le C_P K^{-\frac{1}{2}}$, we lose the uniqueness of steady states. Alternatively, we shall use the Łojasiewicz–Simon approach. Denote $A = -\Delta$ with $D(A) = \{\phi \in H^2, \int_{\mathbb{T}^n} \phi dx = 0\}$. Then A is self-adjoint and positive definite. Let H_A be the dual space of $H^1_* = \{\phi \in H^1, \int_{\mathbb{T}^n} \phi dx = 0\}$. Then the norm on H_A is given by $\|\phi\|_A^2 = \int_{\mathbb{T}^n} \phi A^{-1}\phi dx = \|A^{-\frac{1}{2}}\phi\|^2$.

We introduce the following Łojasiewicz–Simon type inequality:

Lemma 4.4. *Suppose $n = 2, 3$. Let ψ be the critical point of energy*

$$E(\varphi) = \frac{K}{2}\|\Delta\varphi\|^2 + \int_{\mathbb{T}^n} F(\nabla\varphi)dx. \tag{4.7}$$

Then, there exist constants $\beta > 0$, $\theta \in (0, \frac{1}{2})$ depending on ψ such that for any $\varphi \in H^3$ with $\|\varphi - \psi\|_{H^2} < \beta$ and $\int_{\mathbb{T}^n} \varphi dx = \int_{\mathbb{T}^n} \psi dx$, there holds

$$\| - K\Delta^2\varphi + \nabla \cdot f(\nabla\varphi)\|_A \ge |E(\varphi) - E(\psi)|^{1-\theta}. \tag{4.8}$$

Proof. Slightly modifying the arguments in [13,29], we can easily prove that there exist constants $\beta_1 > 0$, $\theta \in (0, \frac{1}{2})$ depending on ψ such that for any $\varphi \in H^3$ with $\|\varphi - \psi\|_{H^3} < \beta_1$ and $\int_{\mathbb{T}^n} \varphi dx = \int_{\mathbb{T}^n} \psi dx$, (4.8) holds. Next, we slightly relax the smallness condition and show that (4.8) still holds if one only requires that φ falls into a certain H^2-neighborhood of ψ. For any $\varphi \in H^3$ satisfying $\int_{\mathbb{T}^n} \varphi dx = \int_{\mathbb{T}^n} \psi dx$, using the regularity theory for elliptic problem, we can see that

$$\|\varphi - \psi\|_{H^3} \le M\|\Delta^2(\varphi - \psi)\|_A, \tag{4.9}$$

where M is a constant independent of φ. On the other hand, if $\|\varphi - \psi\|_{H^2} \le 1$ (which implies that $\|\varphi\|_{H^2} \le \|\psi\|_{H^2} + 1$), then by Sobolev embedding theorem, we get

$$\|\nabla \cdot f(\nabla\varphi) - \nabla \cdot f(\nabla\psi)\|_A \le C_1 \|\varphi - \psi\|_{H^2},$$
$$|E(\varphi) - E(\psi)|^{1-\theta} \le C_2 \|\varphi - \psi\|_{H^2}^{1-\theta},$$

where C_1, C_2 depend on $\|\psi\|_{H^2}$ and $\|\varphi\|_{H^2}$ (by our assumption, the later one can be bounded by using only $\|\psi\|_{H^2}$). As a consequence, there exists a (sufficiently small) $\beta \in (0, 1]$ independent of φ, such that if $\|\varphi - \psi\|_{H^2} < \beta$, then

$$\|\nabla \cdot f(\nabla\varphi) - \nabla \cdot f(\nabla\psi)\|_A + |E(\varphi) - E(\psi)|^{1-\theta} < \frac{\beta_1 K}{2M}. \tag{4.10}$$

Now for any $\varphi \in H^3$ satisfying $\int_{\mathbb{T}^n} \varphi dx = \int_{\mathbb{T}^n} \psi dx$ and $\|\varphi - \psi\|_{H^2} < \beta$, there are only two possibilities: (i) If $\|\varphi - \psi\|_{H^3} < \beta_1$, then (4.8) holds. (ii) If $\|\varphi - \psi\|_{H^3} \ge \beta_1$, noticing that ψ satisfies (4.5), we deduce from (4.9) and (4.10) that

$$\begin{aligned}
\| - K\Delta^2\varphi + \nabla \cdot f(\nabla\varphi)\|_A &= \| - K\Delta^2(\varphi - \psi) + \nabla \cdot f(\nabla\varphi) - \nabla \cdot f(\nabla\psi)\|_A \\
&\ge K\|\Delta^2(\varphi - \psi)\|_A - \|\nabla \cdot f(\nabla\varphi) - \nabla \cdot f(\nabla\psi)\|_A \\
&\ge \frac{K}{M}\|\varphi - \psi\|_{H^3} - \|\nabla \cdot f(\nabla\varphi) - \nabla \cdot f(\nabla\psi)\|_A \\
&> \frac{\beta_1 K}{2M} > |E(\varphi) - E(\psi)|^{1-\theta}.
\end{aligned}$$

The proof is complete. □

For any initial data $(\mathbf{v}_0, \varphi_0) \in \dot{H} \times \dot{H}^2$, it follows from Lemma 3.7 that $\|\varphi\|_{H^4}$ is uniformly bounded for $t \ge t_2 > 0$. Therefore, there is an increasing unbounded sequence $\{t_i\}_{i\in\mathbb{N}}$ and a function $\varphi_\infty \in H^4$ satisfying (4.4)–(4.5) such that

$$\lim_{t_i\to+\infty} \|\varphi(t_i) - \varphi_\infty\|_{H^3} = 0, \quad \lim_{t_i\to+\infty} \mathcal{E}(t_i) = E(\varphi_\infty). \tag{4.11}$$

By (1.7), we have

$$\|\varphi_t\| \le \|\mathbf{v} \cdot \nabla\varphi\| + \|\mathcal{Q}\| \le \|\nabla\mathbf{v}\|\|\varphi\|_{H^2} + \|\mathcal{Q}\|. \tag{4.12}$$

We first exclude the trivial case, i.e., that there exists a $t_0 > 0$ such that $\mathcal{E}(t_0) = E(\varphi_\infty)$. In this case, for all $t \ge t_0$, we deduce from (2.2) that $\|\nabla\mathbf{v}(t)\| = \|\mathcal{Q}(t)\| = 0$. It follows from (4.12) that for $t \ge t_0$, $\|\varphi_t\| = 0$. Namely, φ is independent of time for all $t \ge t_0$. Due to (4.11), we conclude that $\varphi(t) \equiv \varphi_\infty$ for $t \ge t_0$. In this case, there is nothing else to prove.

Therefore, without loss of generality, for all $t > 0$, we suppose that $\mathcal{E}(t) > E(\varphi_\infty)$. For arbitrary $t > 0$, we know that $\varphi \in L^2(t, t+1; H^4) \cap H^1(t, t+1; L^2)$ which implies that $\varphi \in C([t, t+1], H^2)$. Due to this continuity, by a standard contradiction argument (see [15]), we can prove that there is a (sufficiently large) $t_0 > 0$ such that for all $t \ge t_0$, $\|\varphi(t) - \varphi_\infty\|_{H^2} < \beta$. Namely, for all $t \ge t_0$, $\varphi(t)$ satisfies the conditions in Lemma 4.4. Apply Lemma 4.4, (2.2) and the Poincaré inequality, we obtain

$$\begin{aligned}
&-\frac{d}{dt}(\mathcal{E}(t) - E(\varphi_\infty))^\theta = -\theta(\mathcal{E}(t) - E(\varphi_\infty))^{\theta-1}\frac{d}{dt}\mathcal{E}(t) \ge \theta \frac{\frac{\mu_4}{2}\|\nabla\mathbf{v}\|^2 + \lambda\|\mathcal{Q}\|^2}{\|\mathbf{v}\|^{2(1-\theta)} + \|\mathcal{Q}\|_A} \\
\ge\ & C(\|\nabla\mathbf{v}\| + \|\mathcal{Q}\|), \quad \forall\, t \ge t_0,
\end{aligned} \tag{4.13}$$

which implies that $\int_{t_0}^{+\infty}(\|\nabla \mathbf{v}(\tau)\| + \|\mathcal{Q}(\tau)\|)d\tau < +\infty$, and by (4.12), $\int_{t_0}^{+\infty} \|\varphi_t(\tau)\|d\tau < +\infty$. This easily yields the convergence of $\varphi(t)$ in L^2 as $t \to +\infty$. Since φ is compact in H^3, we infer from (4.11) that $\lim_{t\to+\infty} \|\varphi(t) - \varphi_\infty\|_{H^3} = 0$. By the Sobolev embedding theorem, we have

$$K\|\Delta^2\varphi(t)-\Delta^2\varphi_\infty\| \le \|\mathcal{Q}(t)\|+\|\nabla\cdot f(\nabla\varphi(t))-\nabla\cdot f(\nabla\varphi_\infty)\| \le \|\mathcal{Q}(t)\|+C\|\varphi(t)-\varphi_\infty\|_{H^2}, \quad (4.14)$$

where C depends on $\|\varphi(t)\|_{H^3}$ and $\|\varphi_\infty\|_{H^3}$. As a consequence, we can conclude from (4.2) and the H^3-convergence of φ that

$$\lim_{t\to+\infty} \|\varphi(t) - \varphi_\infty\|_{H^4} = 0. \quad (4.15)$$

4.2 Convergence rate

It remains to prove the convergence rate. This can be done in two steps: the first consists in obtaining, via the Łojasiewicz–Simon inequality (cf. e.g., [14]), the convergence rate for the lower order terms. In the second step, we will use the energy method to obtain the convergence rate for the higher order terms. From Lemma 4.4 and (4.13), we have

$$\frac{d}{dt}(\mathcal{E}(t) - E(\varphi_\infty)) + C(\mathcal{E}(t) - E(\varphi_\infty))^{2(1-\theta)} \le 0, \quad \forall\, t \ge t_0, \quad (4.16)$$

and as a consequence,

$$\mathcal{E}(t) - E(\varphi_\infty) \le C(1+t)^{-\frac{1}{1-2\theta}}, \quad \forall\, t \ge t_0.$$

Integrating (4.13) on $(t, +\infty)$, where $t \ge t_0$, it follows from (4.12) that

$$\int_t^{+\infty} \|\varphi_t(\tau)\|d\tau \le C\int_t^{+\infty} (\|\nabla \mathbf{v}(\tau)\| + \|\mathcal{Q}(\tau)\|)d\tau \le C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad (4.17)$$

which implies

$$\|\varphi(t) - \varphi_\infty\| \le C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad t \ge t_0. \quad (4.18)$$

It follows from the basic energy law (2.2) and (4.5) that

$$\frac{d}{dt}y(t) + \frac{\mu_4}{2}\|\nabla \mathbf{v}\|^2 + \lambda\|\mathcal{Q}\|^2 \le 0, \quad (4.19)$$

where

$$y(t) = \frac{1}{2}\|\mathbf{v}(t)\|^2 + \frac{K}{2}\|\Delta\varphi(t) - \Delta\varphi_\infty\|^2 + \int_{\mathbb{T}^2}[F(\nabla\varphi(t)) - F(\nabla\varphi_\infty) + \nabla\cdot f(\nabla\varphi_\infty)(\varphi(t) - \varphi_\infty)]dx.$$

A direct calculation yields

$$\begin{aligned} &\int_{\mathbb{T}^2}[F(\nabla\varphi) - F(\nabla\varphi_\infty) + \nabla\cdot f(\nabla\varphi_\infty)(\varphi - \varphi_\infty)]dx \\ = &\ \frac{1}{4\epsilon^2}\int_{\mathbb{T}^2}(\mathbf{d} - \mathbf{d}_\infty)\cdot\left[(|\mathbf{d}|^2 + |\mathbf{d}_\infty|^2 + \mathbf{d}\cdot\mathbf{d}_\infty)(\mathbf{d} - \mathbf{d}_\infty)\right]dx \\ &+\frac{1}{4\epsilon^2}\int_{\mathbb{T}^2}(\mathbf{d} - \mathbf{d}_\infty)\cdot\left[|\mathbf{d}_\infty|^2(\mathbf{d} - \mathbf{d}_\infty) + \mathbf{d}_\infty(\mathbf{d} + \mathbf{d}_\infty)\cdot(\mathbf{d} - \mathbf{d}_\infty)\right]dx - \frac{1}{2\epsilon^2}\|\mathbf{d} - \mathbf{d}_\infty\|^2. \end{aligned}$$

Thus, we have

$$\left|\int_{\mathbb{T}^2}[F(\nabla\varphi) - F(\nabla\varphi_\infty) + \nabla\cdot f(\nabla\varphi_\infty)(\varphi - \varphi_\infty)]dx\right| \le C\|\nabla\varphi - \nabla\varphi_\infty\|^2,$$

which together with the Poincaré inequality implies

$$y(t) \geq \frac{1}{2}\|\mathbf{v}\|^2 + \frac{K}{4}\|\Delta\varphi - \Delta\varphi_\infty\|^2 - C\|\varphi - \varphi_\infty\|^2. \tag{4.20}$$

On the other hand, it follows from (3.43) and (4.14) that

$$\begin{aligned} K\|\Delta^2\varphi - \Delta^2\varphi_\infty\| &\leq \|\mathcal{Q}\| + C\|\varphi - \varphi_\infty\|_{H^2} \leq \|\mathcal{Q}\| + C\|\Delta^2\varphi - \Delta^2\varphi_\infty\|^{\frac{1}{2}}\|\varphi - \varphi_\infty\|^{\frac{1}{2}} \\ &\leq \|\mathcal{Q}\| + \frac{K}{2}\|\Delta^2\varphi - \Delta^2\varphi_\infty\| + C\|\varphi - \varphi_\infty\|, \end{aligned} \tag{4.21}$$

which yields

$$\begin{aligned} y(t) &\leq \frac{1}{2}\|\mathbf{v}\|^2 + \frac{K}{2}\|\Delta\varphi - \Delta\varphi_\infty\|^2 + C\|\nabla\varphi - \nabla\varphi_\infty\|^2 \\ &\leq C\|\nabla\mathbf{v}\|^2 + C\|\Delta^2\varphi - \Delta^2\varphi_\infty\|\|\varphi - \varphi_\infty\| + C\|\Delta^2\varphi - \Delta^2\varphi_\infty\|^{\frac{1}{2}}\|\varphi - \varphi_\infty\|^{\frac{3}{2}} \\ &\leq C\|\nabla\mathbf{v}\|^2 + C\|\mathcal{Q}\|^2 + C\|\varphi - \varphi_\infty\|^2. \end{aligned} \tag{4.22}$$

For $t \geq t_0 > 0$, Lemma 3.7 implies that $\mathbf{A}(t) \leq C$ that combined with (3.15) yields

$$\frac{d}{dt}\mathbf{A}(t) \leq C\mathbf{A}(t). \tag{4.23}$$

It follows from (4.18), (4.19) and (4.20)–(4.23) that there exist constants $M_1, M_2 > 0$ such that

$$\frac{d}{dt}[y(t) + M_1\mathbf{A}(t)] + M_2[y(t) + M_1\mathbf{A}(t)] \leq C\|\varphi(t) - \varphi_\infty\|^2 \leq C(1+t)^{-\frac{2\theta}{1-2\theta}}, \quad \forall\, t \geq t_0. \tag{4.24}$$

By a similar argument as in [33], we conclude from (4.24) that

$$\begin{aligned} y(t) + M_1\mathbf{A}(t) &\leq [y(t_0) + M_1\mathbf{A}(t_0)]e^{\gamma(t_0 - t)} + Ce^{-M_2 t}\int_{t_0}^{t} e^{M_2\tau}(1+\tau)^{-\frac{2\theta}{1-2\theta}}\,d\tau \\ &\leq C(1+t)^{-\frac{2\theta}{1-2\theta}}, \quad \forall\, t \geq t_0. \end{aligned} \tag{4.25}$$

Finally, from (4.18), (4.21) and (4.25) we obtain the required estimate

$$\|\mathbf{v}(t)\|_{\mathbf{H}^1} + \|\varphi(t) - \varphi_\infty\|_{H^4} \leq C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad \forall t \geq t_0. \tag{4.26}$$

5 Results in $3D$

Lemma 5.1. *Suppose $n = 3$. We have*

$$\frac{d}{dt}\mathbf{A}(t) + \frac{\mu_4}{4}\|\Delta\mathbf{v}\|^2 + \frac{\alpha\lambda K}{2}\|\Delta\mathcal{Q}\|^2 \leq C_*(\mathbf{A}^3(t) + \mathbf{A}(t)), \quad \forall\, t \geq t_1 > 0, \tag{5.1}$$

where $t_1 > 0$ is arbitrary and $\alpha > 0$, $C_* > 0$ are constants depending on $\|\mathbf{v}_0\|$, $\|\varphi_0\|_{H^2}$ and t_1. Moreover, if we assume that $\varphi_0 \in H^3$, (5.1) holds for $t \geq 0$ with C_* being dependent of $\|\mathbf{v}_0\|$, $\|\varphi_0\|_{H^3}$.

Proof. Using Lemma 3.5 and Corollary 3.1, we modify the calculations in Lemma 3.6 by using the $3D$ version of embedding theorems. It is not difficult to see that we are still able to choose $\alpha > 0$ sufficiently small in $\mathbf{A}(t)$ such that our conclusion holds true. □

The existence of local strong solution for arbitrary viscosity $\mu_4 > 0$ is a direct consequence of Lemma 5.1.

Theorem 5.1. *Suppose $n = 3$. For any $(\mathbf{v}_0, \phi_0) \in V \times H^4$, problem (1.5)–(1.7) admits a unique local strong solution.*

Proof. Since $\varphi_0 \in H^4$, we have uniform estimates for $\|\varphi\|_{H^3}$ and $\|\nabla\varphi\|_{\mathbf{L}^\infty}$. (5.1) is valid for $t \geq 0$. Considering the ODE problem:

$$\frac{d}{dt}Y(t) = C_*[(Y(t))^3 + Y(t)], \quad Y(0) = \mathbf{A}(0), \tag{5.2}$$

we denote by $I = [0, T_{max})$ the interval of existence of the maximal solution $Y(t)$. We thus have $\lim_{t\to T_{max}^-} Y(t) = +\infty$. It easily follows that for any $t \in I$, $0 \leq \mathbf{A}(t) \leq Y(t)$. Consequently, $\mathbf{A}(t)$ exists on I. This and Theorem 2.2 imply the local existence of a unique strong solution of problem (1.5)–(1.7). □

Proposition 5.1. *Suppose $n = 3$, $\mathbf{v}_0 \in \dot{V}$, $\varphi_0 \in H^4$. For any $R \in (0, \infty)$, whenever $\|\nabla \mathbf{v}_0\|^2 + \|\mathcal{Q}(0)\|^2 \leq R$, there is a small constant $\varepsilon_0 \in (0, 1)$ depending only on R and coefficients of the system such that either (i) Problem (1.5)–(1.7) has a unique global strong solution $(\mathbf{v}, \varphi)$, or (ii) there is a $T_1 \in (0, +\infty)$ such that $\mathcal{E}(T_1) < \mathcal{E}(0) - \varepsilon_0$.*

Proof. The proof follows from the argument in [20] for simplified nematic liquid crystal model. A statement was also given for the Smectic-A system with variable density in [22] without proof. For the convenience of the readers, we sketch the proof here. We suppose that $(\mathbf{v}, \varphi)$ is a weak solution with initial data $(\mathbf{v}_0, \varphi_0)$ such that $\|\nabla \mathbf{v}_0\|^2 + \|\mathcal{Q}(0)\|^2 \leq R$. Then C_* in (5.1) is determined by R. Moreover, due to Corollary 3.1 we have uniform estimate on $\|\nabla\varphi(t)\|_{\mathbf{L}^\infty}$ for all $t \geq 0$ which only depends on R. We fix through R the constant α in the definition of $\mathbf{A}(t)$. Consider the ODE problem (5.2) with $Y(0) = \max\{1, \alpha\}R \geq \mathbf{A}(0)$. Let $Y(t)$ denote the unique maximal solution defined on $[0, T_{max})$. The time T_{max} is determined by $Y(0)$ and C_* in such a way that it is increasing when $Y(0)$ is decreasing. Now we take

$$t_0 = \frac{1}{2}T_{max}(Y(0), C_*), \quad \varepsilon_0 = \frac{Rt_0}{2}\min\left\{\frac{\mu_4}{2}, \lambda\right\}.$$

If (ii) is not true, we have $\mathcal{E}(t) \geq \mathcal{E}(0) - \varepsilon_0$ for all $t \geq 0$. From the basic energy law (2.2), we infer that

$$\int_{\frac{t_0}{2}}^{t_0} \mathbf{A}(t)dt \leq \int_0^\infty \mathbf{A}(t)dt \leq \kappa\varepsilon_0, \quad \text{with } \kappa = \max\{1, \alpha\}\max\{2\mu_4^{-1}, \lambda^{-1}\}.$$

Hence, there exists a $t_* \in [\frac{t_0}{2}, t_0]$ such that $\mathbf{A}(t^*) \leq \frac{2\kappa\varepsilon_0}{t_0} \leq Y(0)$. Restarting the flow (5.2) from t_*, we infer from the above argument that $\mathbf{A}(t)$ remains bounded at least on $[0, \frac{3t_0}{2}] \subset [0, t_* + t_0]$ with the same bound as that on $[0, t_0]$. As a consequence, an iteration argument shows that $\mathbf{A}(t)$ is bounded for all $t \geq 0$. The proof is complete. □

As an immediate consequences of the above result, we can prove (1) eventual regularization of weak solutions and (2) the well-posedness of strong solutions near the *absolute* minimizers of energy E (cf. (4.7)) (cf. [20, 22]).

Corollary 5.1. *Suppose* $n = 3$.

(1) Let $(\mathbf{v}, \varphi)$ *be the weak solution to problem* (1.5)–(1.7) *on* $[0, +\infty)$. *Then there is some* $T^* > 0$ *such that* $\mathbf{v} \in L^\infty(T^*, \infty; V) \cap L^2_{loc}(T^*, \infty; \mathbf{H}^2)$, $\varphi \in L^\infty(T^*, \infty; H^4) \cap L^2_{loc}(T^*, \infty; H^6)$.

(2) Let $\varphi^* \in H^2$ *be an absolute minimizer of* $E(\varphi)$ *in the sense that* $E(\varphi^*) \le E(\varphi)$ *for all* $\varphi \in H^2$. *For any* $\mathbf{v}_0 \in \dot{V}$, $\varphi_0 \in H^4$ *satisfying* $\|\mathbf{v}_0\|_{\mathbf{H}^1} \le 1$ *and* $\|\varphi_0 - \varphi^*\|_{H^4} \le 1$, *there is a constant* σ *which may depend on coefficients of the system and* φ^* *such that if* $\|\mathbf{v}_0\| \le \sigma$ *and* $\|\varphi_0 - \varphi^*\|_{H^2} \le \sigma$, *then problem* (1.5)–(1.7) *admits a unique global strong solution.*

Next, we improve the second part of Corollary 5.1 by proving the well-posedness of strong solutions close to *local* minimizers of the energy E:

Theorem 5.2. *Suppose* $n = 3$. *Let* $\varphi^* \in H^2$ *be a local minimizer of* $E(\varphi)$ *in the sense that* $E(\varphi^*) \le E(\varphi)$ *for all* $\varphi \in H^2$ *satisfying* $\|\varphi - \varphi^*\|_{H^2} < \delta$. *For any* $\mathbf{v}_0 \in \dot{V}$, $\varphi_0 \in H^4$ *satisfying* $\|\mathbf{v}_0\|_{\mathbf{H}^1} \le 1$ *and* $\|\varphi_0 - \varphi^*\|_{H^4} \le 1$, *there exist constants* $\sigma_1, \sigma_2 \in (0, 1]$ *which may depend on coefficients of the system and* φ^* *such that if* $\|\mathbf{v}_0\| \le \sigma_1$ *and* $\|\varphi_0 - \varphi^*\|_{H^2} \le \sigma_2$, *then problem* (1.5)–(1.7) *admits a unique global strong solution.*

Proof. Without loss of generality, we assume $\delta \le 1$. By C_i, $i = 1, 2, ...$ we denote constants that only depend on φ_* and on coefficients of the system. If $\|\mathbf{v}_0\|_{\mathbf{H}^1} \le 1$ and $\|\varphi_0 - \varphi^*\|_{H^4} \le 1$, it is not difficult to see that $\|\nabla \mathbf{v}_0\|^2 + \|\mathcal{Q}(0)\|^2 \le R$, where R depends only on φ^*. Fix this R, using Proposition 5.1, we can also fix the critical constant ε_0 determined by R. It follows from Lemma 3.1 and Lemma 3.5 that $\|\mathbf{v}(t)\|$ and $\|\varphi(t)\|_{H^3}$ are uniformly bounded (by a constant depending on φ^*). Since $\mathcal{E}$ is decreasing, we can see that

$$
\begin{aligned}
0 \le \mathcal{E}(0) - \mathcal{E}(t) &= \frac{1}{2}\|\mathbf{v}_0\|^2 - \frac{1}{2}\|\mathbf{v}(t)\|^2 + E(\varphi_0) - E(\varphi(t)) \le \frac{1}{2}\|\mathbf{v}_0\|^2 + E(\varphi_0) - E(\varphi(t)) \\
&\le \frac{1}{2}\|\mathbf{v}_0\|^2 + C_1\|\varphi(t) - \varphi_0\|_{H^2}. \qquad (5.3)
\end{aligned}
$$

First we require $\sigma_1 \le \min\left\{\frac{1}{2}\varepsilon_0^{\frac{1}{2}}, 1\right\}$. Let β denote the constant depending only on φ^* provided by Lemma 4.4. If we are able to prove

$$
\|\varphi(t) - \varphi_0\|_{H^2} < \varpi := \min\left\{\beta, \frac{\varepsilon_0}{2C_1}, \delta\right\}, \quad \forall t \ge 0, \qquad (5.4)
$$

then we can infer from (5.3) that

$$
\mathcal{E}(t) \ge \mathcal{E}(0) - \varepsilon_0, \quad \forall\, t \ge 0, \qquad (5.5)
$$

and our conclusion immediately follows from Proposition 5.1. We prove (5.4) by a contradiction argument. Assume $\sigma_2 \le \frac{\varpi}{4}$. Let t_* denote the smallest and finite time for which $\|\varphi(t_*) - \varphi^*\|_{H^2} \ge \varpi$. Without loss of generality, we can assume that $\mathcal{E}(t) > E(\varphi^*)$ $t \in [0, t_*)$. In fact, if there exists $t_{**} \in (0, t_*)$ such that $\mathcal{E}(t_{**}) = E(\varphi^*)$, since φ^* is the local minimizer and $\|\varphi(t_{**}) - \varphi^*\|_{H^2} < \varpi \le \delta$, we can see that $\mathbf{v}(t_{**}) = 0$ and $\varphi(t_{**}) = \varphi^{**}$, where φ^{**} is also a local minimizer (possibly different from φ^*) satisfying (4.4)–(4.5). Due to the uniqueness of strong solution, the evolution starting from t_{**} will be stationary and hence contradicting the definition of t_*. Thus, let $\mathcal{E}(t) > E(\varphi^*)$ for $t \in [0, t_*)$. We observe that the conditions in Lemma 4.4 are fulfilled with φ^*, on the interval $[0, t_*)$. In analogy with (4.13), we obtain

$$
-\frac{d}{dt}(\mathcal{E}(t) - E(\varphi^*))^\theta \ge C_2(\|\nabla \mathbf{v}\| + \|\mathcal{Q}\|), \quad \forall\, t \in [0, t_*), \qquad (5.6)
$$

where C_2 depends on θ, μ_4, λ. Using (4.12), we have

$$\begin{aligned}\int_0^{t_*} \|\varphi_t(t)\| dt &\le C_3(\mathcal{E}(0) - E(\varphi^*))^\theta \le C_3 \left(\frac{1}{2}\right)^\theta \|\mathbf{v}_0\|^{2\theta} + C_3|E(\varphi_0) - E(\varphi^*)|^\theta \\ &\le C_4\|\mathbf{v}_0\|^{2\theta} + C_5\|\varphi_0 - \varphi^*\|_{H^2}^\theta.\end{aligned}$$

As a result,

$$\begin{aligned}\|\varphi(t_*) - \varphi^*\|_{H^2} &\le \|\varphi(t^*) - \varphi_0\|_{H^2} + \|\varphi_0 - \varphi^*\|_{H^2} \\ &\le C\|\varphi(t^*) - \varphi_0\|_{H^3}^{\frac{2}{3}} \|\varphi(t^*) - \varphi_0\|^{\frac{1}{3}} + \|\varphi_0 - \varphi^*\|_{H^2} \\ &\le C(\|\varphi(t^*)\|_{H^3} + \|\varphi_0\|_{H^3})^{\frac{2}{3}} \left(\int_0^{t_*} \|\varphi_t(t)\| dt\right)^{\frac{1}{3}} + \|\varphi_0 - \varphi^*\|_{H^2} \\ &\le C_6 \left(\|\mathbf{v}_0\|^{\frac{2\theta}{3}} + \|\varphi_0 - \varphi^*\|_{H^2}^{\frac{\theta}{3}}\right) + \|\varphi_0 - \varphi^*\|_{H^2}, \end{aligned} \tag{5.7}$$

Taking

$$\sigma_1 \le \min\left\{\frac{1}{2}\varepsilon_0^{\frac{1}{2}}, 1, \left(\frac{\varpi}{4C_6}\right)^{\frac{3}{2\theta}}\right\}, \quad \sigma_2 \le \min\left\{\frac{\varpi}{4}, \left(\frac{\varpi}{4C_6}\right)^{\frac{3}{\theta}}\right\}, \tag{5.8}$$

we easily infer from (5.7) that $\|\varphi(t_*) - \varphi^*\|_{H^2} \le \frac{3}{4}\varpi < \varpi$, which leads to a contradiction with the definition of t^*. Hence, we have shown that (5.4) holds for all $t \ge 0$. The proof is complete. □

Remark 5.1. *The above proof indicates that if the (regular) initial data are properly close to certain local minimizer, then the global strong solution will remain in the neighborhood of this local minimizer for all time. The conclusion is also true for the case with absolute minimizer in Corollary 5.1. If the minimizer is isolate, then we obtain the stability of it.*

Lemma 5.2. *Suppose $n = 3$. Denote $\mathbf{A}_1(t) = \|\nabla \mathbf{v}(t)\|^2 + \|\mathcal{Q}(t)\|^2$ and $\tilde{\mathbf{A}}_1(t) = \mathbf{A}_1(t) + 1$. For any $\mu_4 > 0$, we have*

$$\begin{aligned}&\frac{d}{dt}\mathbf{A}_1(t) + \left(\frac{\mu_4}{2} - M_1\mu_4^{\frac{1}{2}}\tilde{\mathbf{A}}_1(t)\right)\|\Delta \mathbf{v}\|^2 + \left(\lambda K - M_2\mu_4^{-\frac{1}{2}}(1 + \mu_4^{-\frac{5}{2}})\tilde{\mathbf{A}}_1(t)\right)\|\Delta \mathcal{Q}\|^2 \\ \le\ & M_3(1 + \mu_4^{-3})\mathbf{A}_1(t),\end{aligned} \tag{5.9}$$

where M_1, M_2, M_3 are constants depending on $\|\mathbf{v}_0\|$, $\|\varphi_0\|_{H^2}$, $\mu_1, \mu_5, \lambda, K, \epsilon$, but not on μ_4.

Proof. We note that in the following calculation only the lower-order uniform estimates (3.1) are used. The possible relaxation on the viscosity μ_4 enable us to avoid using the $\mathbf{L}^\infty$-norm of $\nabla\varphi$, which was crucial in the proof of Lemma 3.6. In what follows, the generic constant C will only depend on $\|\mathbf{v}_0\|$, $\|\varphi_0\|_{H^2}$, μ_1, μ_5, λ, K, ϵ.

We revisit the terms on the right-hand side of (3.18).

$$\begin{aligned}\int_{\mathbb{T}^3} (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\Delta\mathbf{v} dx &\le \|\Delta \mathbf{v}\|\|\nabla \mathbf{v}\|_{\mathbf{L}^3}\|\mathbf{v}\|_{\mathbf{L}^6} \le C\|\Delta\mathbf{v}\|^{\frac{3}{2}}\|\nabla \mathbf{v}\|^{\frac{3}{2}} \\ &\le \mu_4^{\frac{1}{2}}\|\nabla\mathbf{v}\|^{\frac{4}{3}}\|\Delta\mathbf{v}\|^2 + C\mu_4^{-\frac{1}{2}}\|\nabla\mathbf{v}\|^2.\end{aligned}$$

$$I_1 + I_{2a} + I_{2b} + I_{2c} \le \frac{\mu_1}{4}\int_Q (d_i d_j \nabla_l D_{ij})^2 dx + C\|\nabla\mathbf{v}\|_{\mathbf{L}^3}^2\|\nabla\mathbf{d}\|_{\mathbf{L}^6}^2\|\mathbf{d}\|_{\mathbf{L}^\infty}^2,$$

where

$$
\begin{aligned}
&\|\nabla\mathbf{v}\|_{\mathbf{L}^3}^2\|\nabla\mathbf{d}\|_{\mathbf{L}^6}^2\|\mathbf{d}\|_{\mathbf{L}^\infty}^2 \le C\|\Delta\mathbf{v}\|^{\frac{3}{2}}\|\mathbf{v}\|^{\frac{1}{2}}\|\nabla\mathbf{d}\|_{\mathbf{L}^6}^2\|\mathbf{d}\|_{\mathbf{L}^\infty}^2\\
\le\ & C\|\Delta\mathbf{v}\|^{\frac{3}{2}}\|\mathbf{v}\|^{\frac{1}{2}}(\|\Delta^3\varphi\|^{\frac{1}{2}}\|\Delta\varphi\|^{\frac{3}{2}}+\|\Delta\varphi\|^2)(\|\Delta^2\varphi\|\|\nabla\varphi\|+\|\nabla\varphi\|^2)\\
\le\ & C\|\Delta\mathbf{v}\|^{\frac{3}{2}}\|\mathbf{v}\|^{\frac{1}{2}}(\|\Delta\mathcal{Q}\|^{\frac{1}{2}}+1)(\|\mathcal{Q}\|+1)\\
\le\ & \left[\frac{\mu_4}{24}+\mu_4^{\frac{1}{2}}(1+\|\mathcal{Q}\|^2)\right]\|\Delta\mathbf{v}\|^2+C\mu_4^{-\frac{1}{2}}(\|\mathcal{Q}\|^2+\|\nabla\mathbf{v}\|^2)\|\Delta\mathcal{Q}\|^2+C\mu_4^{-1}\|\nabla\mathbf{v}\|^2+C\mu_4^{-\frac{1}{2}}\|\mathcal{Q}\|^2.
\end{aligned}
$$

Next,

$$
\begin{aligned}
I_{2d} \le\ & C\|\mathbf{d}\|_{\mathbf{L}^\infty}^3\|\Delta\mathbf{d}\|\|\nabla\mathbf{v}\|_{\mathbf{L}^4}^2 \le C\|\nabla\varphi\|_{\mathbf{H}^2}^{\frac{3}{2}}\|\nabla\varphi\|_{\mathbf{H}^1}^{\frac{3}{2}}\|\nabla\Delta\varphi\|\|\nabla\mathbf{v}\|_{\mathbf{L}^4}^2\\
\le\ & C(\|\nabla\Delta\varphi\|^{\frac{5}{2}}+1)\|\Delta\mathbf{v}\|^{\frac{3}{2}}\|\nabla\mathbf{v}\|^{\frac{1}{2}} \le C(\|\mathcal{Q}\|^{\frac{1}{4}}+1)(\|\Delta\mathcal{Q}\|^{\frac{1}{2}}+1)\|\Delta\mathbf{v}\|^{\frac{3}{2}}\|\nabla\mathbf{v}\|^{\frac{1}{2}}\\
\le\ & \left[\frac{\mu_4}{24}+\mu_4^{\frac{1}{2}}(\|\nabla\mathbf{v}\|^{\frac{2}{3}}+\|\mathcal{Q}\|^{\frac{1}{3}})\right]\|\Delta\mathbf{v}\|^2+C(\mu_4^{-\frac{3}{2}}\|\mathcal{Q}\|+\mu_4^{-3}\|\nabla\mathbf{v}\|^2)\|\Delta\mathcal{Q}\|^2\\
& +C(\mu_4^{-\frac{3}{2}}+\mu_4^{-3})\|\nabla\mathbf{v}\|^2.
\end{aligned}
$$

$$
\begin{aligned}
I_3+I_4 \le\ & \|\Delta\mathbf{v}\|\|\nabla\mathbf{v}\|_{\mathbf{L}^3}\|\mathbf{d}\|_{\mathbf{L}^\infty}\|\nabla\mathbf{d}\|_{\mathbf{L}^6} \le \|\Delta\mathbf{v}\|^{\frac{3}{2}}\|\nabla\mathbf{v}\|^{\frac{1}{2}}\|\nabla\varphi\|_{\mathbf{H}^2}^{\frac{1}{2}}\|\nabla\varphi\|_{\mathbf{H}^1}^{\frac{1}{2}}\|\nabla\varphi\|_{\mathbf{H}^2}\\
\le\ & C(\|\nabla\Delta\varphi\|^{\frac{3}{2}}+1)\|\Delta\mathbf{v}\|^{\frac{3}{2}}\|\nabla\mathbf{v}\|^{\frac{1}{2}}.
\end{aligned}
$$

It is easy to see that I_3+I_4 can be bounded just like I_{2d}, because its order is lower.

$$
\begin{aligned}
I_5 =\ & \int_{\mathbb{T}^3}\Delta\mathbf{v}\cdot(\mathcal{Q}\mathbf{d})dx \le \|\Delta\mathbf{v}\|\|\mathcal{Q}\|\|\nabla\varphi\|_{\mathbf{L}^\infty} \le \|\Delta\mathbf{v}\|\|\mathcal{Q}\|(\|\mathcal{Q}\|^{\frac{1}{4}}+1)\\
\le\ & \left(\frac{\mu_4}{24}+\mu_4^{\frac{1}{2}}\|\mathcal{Q}\|^{\frac{1}{2}}\right)\|\Delta\mathbf{v}\|^2+C(\mu_4^{-\frac{1}{2}}+\mu_4^{-1})\|\mathcal{Q}\|^2.
\end{aligned}
$$

For the terms $J_1,...,J_5$ on the right-hand side of (3.21), we have

$$
\begin{aligned}
J_1 \le\ & K\|\nabla\varphi\|_{\mathbf{L}^\infty}\|\Delta\mathbf{v}\|\|\Delta\mathcal{Q}\|+C\|\Delta\mathcal{Q}\|\|\nabla\mathbf{v}\|_{\mathbf{L}^3}\|\varphi\|_{W^{2,6}}+C\|\Delta\mathcal{Q}\|\|\mathbf{v}\|_{\mathbf{L}^6}\|\varphi\|_{W^{3,3}}\\
\le\ & C(\|\mathcal{Q}\|^{\frac{1}{2}}+1)\|\Delta\mathbf{v}\|\|\Delta\mathcal{Q}\| \le \frac{\mu_4}{24}\|\Delta v\|^2+C\mu_4^{-1}(1+\|\mathcal{Q}\|)\|\Delta\mathcal{Q}\|^2.
\end{aligned}
$$

$$
\begin{aligned}
J_2+J_4 \le\ & C\|\nabla\mathcal{Q}\|(\|\nabla\varphi\|_{\mathbf{L}^\infty}^2+1)(\|\nabla\mathbf{v}\|\|\nabla\varphi\|_{\mathbf{L}^\infty}+\|\mathbf{v}\|_{\mathbf{L}^6}\|\nabla\nabla\varphi\|_{\mathbf{L}^3})\\
\le\ & C\|\nabla\mathcal{Q}\|\|\nabla\mathbf{v}\|(\|\mathcal{Q}\|^{\frac{3}{4}}+1)\\
\le\ & C(\|\Delta\mathcal{Q}\|^{\frac{1}{2}}\|\mathcal{Q}\|^{\frac{1}{2}}+\|\mathcal{Q}\|)\|\Delta\mathbf{v}\|^{\frac{1}{2}}\|\mathbf{v}\|^{\frac{1}{2}}\|\mathcal{Q}\|^{\frac{3}{4}}+C(\|\Delta\mathcal{Q}\|^{\frac{1}{2}}\|\mathcal{Q}\|^{\frac{1}{2}}+\|\mathcal{Q}\|)\|\nabla\mathbf{v}\|\\
\le\ & \left(\frac{\lambda K}{4}+C\mu_4^{-\frac{1}{2}}\right)\|\Delta\mathcal{Q}\|^2+\mu_4^{\frac{1}{2}}\|\mathcal{Q}\|\|\Delta\mathbf{v}\|^2+C(1+\mu_4^{-\frac{1}{6}})\|\mathcal{Q}\|^2+C\|\nabla\mathbf{v}\|^2.
\end{aligned}
$$

$$
J_3+J_5 \le C\|\nabla\mathcal{Q}\|_{\mathbf{L}^3}^2(\|\nabla\varphi\|_{\mathbf{L}^6}^2+1) \le C(\|\Delta\mathcal{Q}\|^{\frac{3}{2}}\|\mathcal{Q}\|^{\frac{1}{2}}+\|\mathcal{Q}\|^2) \le \frac{\lambda K}{4}\|\Delta\mathcal{Q}\|^2+C\|\mathcal{Q}\|^2.
$$

Collecting all the estimates and using the Young inequality, we can obtain that

$$
\begin{aligned}
& \frac{d}{dt}(\|\nabla\mathbf{v}\|^2+\|\mathcal{Q}\|^2)+\mu_1\int_Q(d_id_j\nabla_lD_{ij})^2dx+4\mu_5\int_Q(d_k\nabla_lD_{ki})^2dx\\
& +\left[\frac{\mu_4}{2}-C\mu_4^{\frac{1}{2}}(1+\|\nabla\mathbf{v}\|^2+\|\mathcal{Q}\|^2)\right]\|\Delta\mathbf{v}\|^2\\
& +\left[\lambda K-C\mu_4^{-\frac{1}{2}}(1+\mu_4^{-\frac{5}{2}})(1+\|\nabla\mathbf{v}\|^2+\|\mathcal{Q}\|^2)\right]\|\Delta\mathcal{Q}\|^2\\
\le\ & C(1+\mu_4^{-3})(\|\nabla\mathbf{v}\|^2+\|\mathcal{Q}\|^2),
\end{aligned}
$$

which yields (5.9). □

Based on Lemma 5.2, one can prove the existence and uniqueness of global strong solutions $(\mathbf{v}, \varphi)$ to our system provided that the viscosity μ_4 is properly large.

Theorem 5.3. *Suppose $n = 3$. For any $(\mathbf{v}_0, \phi_0) \in V \times H^4$, if $\mu_4 \geq \underline{\mu}_4(\mathbf{v}_0, \varphi_0)$ (cf. (5.11)), problem (1.5)–(1.7) admits a unique global strong solution.*

Proof. The crucial step is to obtain a uniform bound of $\mathbf{A}_1(t)$. Without loss of generality, we assume that $\mu_4 \geq 1$. Then we deduce from (5.9) that

$$\frac{d}{dt}\tilde{\mathbf{A}}_1(t) + \left(\frac{\mu_4}{2} - M_1\mu_4^{\frac{1}{2}}\tilde{\mathbf{A}}_1(t)\right)\|\Delta\mathbf{v}\|^2 + \left(\lambda K - 2M_2\mu_4^{-\frac{1}{2}}\tilde{\mathbf{A}}_1(t)\right)\|\Delta\mathcal{Q}\|^2 \leq 2M_3\tilde{\mathbf{A}}_1(t). \quad (5.10)$$

(3.2) yields that $\int_t^{t+1}\tilde{\mathbf{A}}_1(\tau)d\tau \leq \int_0^{+\infty}\mathbf{A}_1(\tau)d\tau + 1 \leq \max\left\{2, \frac{1}{\lambda}\right\}\mathcal{E}(0) + 1 =: \tilde{M}$. If the viscosity μ_4 satisfies the following relation

$$\mu_4 \geq \underline{\mu}_4 := \max\{1, \kappa^2\}, \quad \text{with } \kappa := \max\left\{2M_1, \frac{2M_2}{\lambda K}\right\}(\tilde{\mathbf{A}}_1(0) + 2M_3\tilde{M} + 2\tilde{M}). \quad (5.11)$$

then applying the classical method in [20], we can argue as in [32] to obtain that

$$\frac{\mu_4}{2} - M_1\mu_4^{\frac{1}{2}}\tilde{\mathbf{A}}_1(t) \geq 0, \quad \lambda K - 2M_2\mu_4^{-\frac{1}{2}}\tilde{\mathbf{A}}_1(t) \geq 0, \quad \forall\, t \geq 0.$$

The proof is complete. □

Finally, we study the long-time behavior of global solutions.

Lemma 5.3. *Let $n = 3$, the weak (or strong) solution $(\mathbf{v}, \varphi)$ to problem (1.5)–(1.7) has the following property:*

$$\lim_{t\to+\infty}(\|\nabla\mathbf{v}(t)\| + \|\mathcal{Q}(t)\|) = 0. \quad (5.12)$$

Proof. Since we are only concerning the behavior of $(\mathbf{v}, \varphi)$ for large time, due to the eventual regularity of weak solutions, we can reduce to the case of strong solutions by a finite shift of time. Then we can see that $\|\nabla\mathbf{v}(t)\|$ and $\|\mathcal{Q}(t)\|$ are uniformly bounded for $t \geq 0$. It follows from (5.1) that $\frac{d}{dt}\mathbf{A}(t) \leq C$ (similarly, from (5.9), we have $\frac{d}{dt}\mathbf{A}_1(t) \leq C$). Recalling that $\mathbf{A}(t), \mathbf{A}_1(t) \in L^1(0, +\infty)$ (cf. (3.2)), we arrive at the conclusion. □

Based on Lemma 5.3, we are able to prove the convergence to equilibrium result in $3D$. One can check the argument for $2D$ case in the previous section step by step. By applying corresponding Sobolev embedding theorems in $3D$, we can see that all calculations in Section 4.2 are valid. Hence, the details are omitted here.

Remark 5.2. *Since the set of equilibria can form a continuum, the global solution obtained in Corollary 5.1 or in Theorem 5.2 will converge to an equilibrium φ_∞ which is not necessarily the original minimizer φ^*. However, we can show that $E(\varphi_\infty) = E(\varphi^*)$. To see this, we recall the definition of ϖ in the proof of Theorem 5.2. Actually we showed that the solution $\varphi(t)$ will stay in the H^2-neighborhood of φ^* with radius less than β, so does φ_∞. Then, we can apply Lemma 4.4 with $\psi = \varphi^*$ and $\varphi = \varphi_\infty$ obtaining that $|E(\varphi_\infty) - E(\varphi^*)|^{1-\theta} \leq \|-K\Delta^2\varphi_\infty + \nabla\cdot f(\nabla\varphi_\infty)\|_A = 0$.*

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