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Discrete basis of localized quantum states for the free particle

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Abstract

An infinite discrete series of solutions of the free Schrödinger equation in one dimension is constructed. These solutions are normalizable, expand the whole space of solutions, are spatially multi-localized and are eigenstates of a suitable defined number operator. Associated with these states, new sets of coherent states for the free particle are defined, representing traveling multi-localized wave packets. Some applications of these new families of states and a procedure to experimentally realize them are outlined.

1 Introduction

In the context of the quantum free particle, the eigenstates of the Hamiltonian, which are also eigenstates of the momentum operator, are not normalizable. These states, the plane waves, are fully delocalized. However, it is customary to expand any normalizable solution of the free Schrödinger equation in terms of plane waves, using the Fourier transform, building in this way wave packets which represent localized solutions. The simplest example is the Gaussian wave packet, which has the property of minimizing the uncertainty relations between the position and the momentum operator.

In [1] a transformation is proposed that maps states and operators from certain quantum systems to the free particle. This transformation is the quantum version of the Arnold transform (QAT), which in its original version [2] maps solutions of the classical equation of motion which is a non-homogeneous linear second order ordinary differential equation (LSODE) to solutions of the classical equation for the free particle.

In this paper we construct a discrete basis of the space of solutions of the quantum free particle in one dimension which is the map through the QAT of the eigenstates of the quantum harmonic oscillator. The first state of this basis is a Gaussian wave packet. They are not eigenstates of the free particle Hamiltonian, i.e. they are not stationary states, but, rather, eigenfunctions of a number operator $\hat{N}$ with discrete eigenvalues n. The lowest one, the Gaussian wave packet, is localized with arbitrary initial width L, which is related to the oscillator frequency and can be conveniently chosen, and has initial minimal uncertainty. The following ones are "multi-localized" in the sense that the n^{th}-order state presents n zeros and $n+1$ humps which spread out with time. This mimics the situation for the harmonic oscillator to such an extent that it is possible to build creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$. The number operator is going to be $\hat{N} \sim \hat{a}^\dagger \hat{a}$, although in the case of the free particle it is not the Hamiltonian. Going even further, we give a set of "coherent states" which are interpreted as traveling wave packets.

Although this construction could seem of a mere academic interest, it can be of physical relevance in Quantum Information Theory, using these states to transmit digital information. It might also be useful in describing scattering process in a discrete basis, instead of using plane waves.

The content of the letter is as follows. Section 2 is devoted to the construction of the discrete base of states by means of the Quantum Arnold transformation mapping the eigenstates of the harmonic oscillator to the free particle Hilbert space. Section 3 deals with the construction of coherent states and interpreting them as traveling wave packets. Section 4 presents some possible physical applications, and Section 5 proposes an experimental setting for producing these states. Finally, in an Appendix, an intuitive construction of these states without resorting to the QAT is presented.

2 A discrete basis of wave packets

Let $\mathcal{H}$ be the Hilbert space of solutions of the free particle Schrödinger equation, and $\mathcal{H}_{\text{HO}}$ the corresponding to the Harmonic oscillator. Since QAT involves a change of variables, we shall denote by $\psi(x,t) \in \mathcal{H}$ the free particle solutions and by $\varphi(x',t') \in \mathcal{H}_{\text{HO}}$ the harmonic oscillator ones. Then the QAT [1], or rather, its inverse, is given by:

$$\varphi(x',t') = \hat{A}^{-1}\psi(x,t) = \frac{1}{\sqrt{u_2(t')}} e^{\frac{i}{2}\frac{m}{\hbar}\frac{1}{W(t')}\frac{\dot{u}_2(t')}{u_2(t')}x'^2} \psi\Big(\frac{x'}{u_2(t')}, \frac{u_1(t')}{u_2(t')}\Big)\,, \tag{1}$$

where the classical Arnold transform is

$$\begin{aligned} A:\ & \mathbb{R}\times T' \longrightarrow \mathbb{R}\times T \\ & (x',t') \longmapsto\ (x,t) = A\big((x',t')\big) = \big(\tfrac{x'}{u_2(t')}, \tfrac{u_1(t')}{u_2(t')}\big)\,, \end{aligned} \tag{2}$$

with T and T' are open intervals of the real line containing $t=0$ and $t'=0$, respectively, $u_1(t')$ and $u_2(t')$ are two independent solutions of the LSODE (here dots means derivation with respect to t'):

$$\ddot{x}' + \dot{f}(t')\dot{x}' + \omega(t')^2 x' = 0\,, \tag{3}$$

and $W(t')$ is the Wronskian $W(t') = \dot{u}_1(t')u_2(t') - u_1(t')\dot{u}_2(t')$ of the two solutions. For the case of the harmonic oscillator $\dot{f} = 0$ and $\omega(t') = \omega$, and the two independent solutions can be chosen (see [1] for details) as $u_1(t') = \frac{1}{\omega}\sin(\omega t')$ and $u_2(t') = \cos(\omega t')$, with $W(t') = 1$. It can be checked that the change of variables results in:

$$\begin{aligned} t' &= \frac{1}{\omega}\arctan(\omega t) \\ x' &= \cos(\arctan(\omega t))x = \frac{x}{\sqrt{1+\omega^2 t^2}}\,. \end{aligned} \tag{4}$$

Applying now the QAT to the time-dependent harmonic oscillator eigenstates,

$$\varphi_n(x',t') = \frac{(\frac{m\omega}{\pi\hbar})^{\frac{1}{4}}}{\sqrt{2^n n!}}\, e^{-i\omega(n+\frac{1}{2})t'}\, e^{-\frac{m\omega}{2\hbar}x'^2} H_n(\sqrt{\frac{m\omega}{\hbar}}x')\,, \tag{5}$$

we obtain the following sets of states, solutions pf the Schrödinger equation for the free particle:

$$\psi_n(x,t) = \frac{(2\pi)^{-\frac{1}{4}}}{\sqrt{2^n n! L|\delta|}}\, e^{-\dfrac{x^2}{4L^2\delta}} \left(\frac{\delta^*}{|\delta|}\right)^{n+\frac{1}{2}} H_n(\frac{x}{\sqrt{2}L|\delta|})\,, \tag{6}$$

where, in order to obtain a more compact notation, we have introduced the quantities $L = \sqrt{\frac{\hbar}{2m\omega}}$, with dimensions of length, and $\tau = \frac{2mL^2}{\hbar} = \omega^{-1}$, with dimensions of time. We also denote by δ the complex, time dependent, adimensional expression $\delta = 1 + i\omega t = 1 + i\frac{\hbar t}{2mL^2} = 1 + it/\tau$.

The fact that these states are written in terms of the Hermite polynomials can be used to show that the set of states is a basis for the Hilbert space of solutions of the free Schrödinger equation, $L^2(\mathbb{R})$. In fact, at $t = 0$, $\psi_n(x,0)$ are the Hermite functions, which constitute a basis of $L^2(\mathbb{R})$. Since the time evolution is unitary, the set of states $\psi_n(x,t)$ is still a basis for any time t.

The first state of this basis, the one mapped from the harmonic oscillator vacuum state, is given by:

$$\psi_0(x,t) = \frac{(2\pi)^{-\frac{1}{4}}}{\sqrt{L|\delta|}} \left(\frac{\delta^*}{|\delta|}\right)^{\frac{1}{2}} e^{-\dfrac{x^2}{4L^2\delta}} = \frac{(2\pi)^{-\frac{1}{4}}}{\sqrt{L\delta}}\, e^{-\dfrac{x^2}{4L^2\delta}}\,, \tag{7}$$

which is nothing other than a Gaussian wave packet with center at the origin and width L. The parameter τ is the dispersion time of the Gaussian wave packet, (see, for instance, [3])

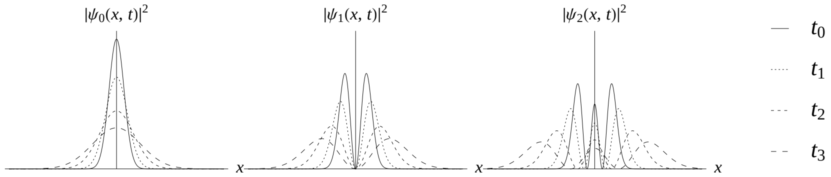

Figure 1: Spreading under time evolution of wave functions ψ_0, ψ_1 and ψ_2, with $t_k = k\tau$.

The QAT also allows to map operators from one Hilbert space to the other (see [1]), in such a way that ladder operators for the Harmonic oscillator can be mapped to ladder operators for the free particle that act as creation and annihilation operators for theses states:

$$\begin{aligned} \hat{a} &= L\delta\frac{\partial}{\partial x} + \frac{x}{2L} \\ \hat{a}^\dagger &= -L\delta^*\frac{\partial}{\partial x} + \frac{x}{2L}. \end{aligned} \tag{8}$$

The action of $\hat{a}$ and $\hat{a}^\dagger$ on these wave functions is the usual one:

$$\hat{a}\psi_n = \sqrt{n}\psi_{n-1}, \qquad \hat{a}^\dagger\psi_n = \sqrt{n+1}\psi_{n+1}. \tag{9}$$

Figure 1 shows some of these wave functions and how they evolve in time.

We see that the number of "parts", or *humps*, of the wave functions, determined by the number of zeros, is quantized, in the sense that there is one hump between two consecutive zeros. This property will be important for the physical applications discussed at the end of the paper. Another aspect of this quantum realization is shown when computing the uncertainties associated with each wave function. As a function of time, for each n, they read:

$$\Delta x_n \Delta p_n = (n + \frac{1}{2})\hbar|\delta|. \tag{10}$$

For $n = 0$ the time evolution of the uncertainty is the one which results from the usual Gaussian wave packet [3], and, among all, the minimal one.

The number operator associated with the creation and annihilation operators above will provide the position of the state in this grid of uncertainties. We can compute it (or map it from the number operator for the Harmonic oscillator) in the usual way:

$$\hat{N} = \frac{\hbar}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\,\hat{a}^\dagger\right) = \hbar\left[-|\delta|^2L^2\frac{\partial^2}{\partial x^2} + i\frac{t}{\tau}(x\frac{\partial}{\partial x} + \frac{1}{2}) + \frac{x^2}{4L^2}\right], \tag{11}$$

where we have added the factor $\hbar$ for convenience.

By making use of the Schrödinger equation, we can turn this operator into a first order one:

$$\hat{N} = \hbar \left[i|\delta|^2 \tau \frac{\partial}{\partial t} + i\frac{t}{\tau}(x\frac{\partial}{\partial x} + \frac{1}{2}) + \frac{x^2}{4L^2} \right] , \tag{12}$$

this expression being valid only on solutions of the Schrödinger equation. The action of this operator is such that:

$$\hat{N}\psi_n(x,t) = (n + \frac{1}{2})\hbar\psi_n(x,t), \tag{13}$$

thus reproducing the uncertainties given in Eq. 10 at time $t = 0$.

It is quite interesting to note that this operator belongs to the "maximal kinematical" symmetry of the free particle, i.e., the Schrödinger group [4]. This symmetry is the standard Galilean symmetry (with generators $\hat{P}^2 = 2m\hat{H}$, $\hat{P}$, $\hat{X}$ and the identity $\hat{I}$) together with spatial dilations ($\hat{XP}$) and non-relativistic "conformal" transformations ($\hat{X}^2$). These generators can be written:

$$\begin{array}{rclrcl} \hat{P} & = & -i\hbar\frac{\partial}{\partial x} & \hat{P}^2 & = & 2mi\hbar\frac{\partial}{\partial t} \\ \hat{X} & = & x + i\hbar\frac{t}{m}\frac{\partial}{\partial x} & \hat{XP} & = & -2i\hbar t\frac{\partial}{\partial t} - i\hbar x\frac{\partial}{\partial x} - \frac{i\hbar}{2} \\ \hat{X}^2 & = & 2i\frac{\hbar}{m}t^2\frac{\partial}{\partial t} + 2i\frac{\hbar}{m}tx\frac{\partial}{\partial x} + x^2 + \frac{i\hbar}{m}t\,, & & & \end{array} \tag{14}$$

providing the non-trivial commutation relations:

$$\left[\hat{X},\hat{P}\right] = i\hbar \tag{15}$$

$$\left[\hat{X},\hat{P}^2\right] = 2i\hbar\hat{P} \qquad \left[\hat{X},\hat{X}^2\right] = 0 \qquad \left[\hat{X},\hat{XP}\right] = i\hbar\hat{X} \tag{16}$$

$$\left[\hat{P},\hat{P}^2\right] = 0 \qquad \left[\hat{P},\hat{X}^2\right] = -2i\hbar\hat{X} \qquad \left[\hat{P},\hat{XP}\right] = -i\hbar\hat{P} \tag{17}$$

$$\left[\hat{X}^2,\hat{P}^2\right] = 4i\hbar\hat{XP} \qquad \left[\hat{X}^2,\hat{XP}\right] = 2i\hbar\hat{X}^2 \qquad \left[\hat{P}^2,\hat{XP}\right] = -2i\hbar\hat{P}^2\,. \tag{18}$$

It is easily checked that $\hat{N}$ is in this Lie algebra, its relation with the basis above being:

$$\hat{N} = \frac{\tau}{2m}\hat{P}^2 + \frac{m}{2\tau}\hat{X}^2\,. \tag{19}$$

From this expression it is clear that $\hat{N} = \frac{mL^2}{\hbar}\hat{H}_{\mathrm{HO}}$, where $\hat{H}_{\mathrm{HO}}$ is the operator corresponding to an harmonic oscillator of frequency $\omega = \frac{\hbar}{mL^2} = \frac{1}{\tau}$, but written in terms of constants of the motion of the free particle. See [1] for the relevance of quadratic operators like $\hat{N}$, but distinc from the Hamiltonian, for building basis of the Hilbert space (see also [5, 6, 7]).

3 Coherent states or traveling wave packets

As a natural consequence of the introduction of creation and annihilation operators, we construct a set of coherent states for the free particle as the eigenstates of the annihilation operator (they could also be obtained from the usual harmonic oscillator coherent states through the QAT). These states are of the form

$$\phi_a(x,t) = \frac{(2\pi)^{-\frac{1}{4}}}{\sqrt{L|\delta|}} \left(\frac{\delta^*}{|\delta|}\right)^{\frac{1}{2}} e^{-\dfrac{(x-x_0)^2 + x_0 v_0 t + i\tau v_0(v_0 t - 2x + x_0)}{4L^2\delta}}\,, \tag{20}$$

where a is the complex number

$$a = \frac{x_0}{2L} + i\frac{m v_0 L}{\hbar} = \frac{1}{2L}(x_0 + i v_0 \tau)\,, \tag{21}$$

and they verify:

$$\hat{a}\,\phi_a(x,t) = a\,\phi_a(x,t)\,. \tag{22}$$

These states can also be obtained by the action of a Galilean boost with parameter v_0 and a translation by x_0 on the vacuum Gaussian packet, and they constitute an over-complete set of the Hilbert space of the free particle. Coherent states represent traveling Gaussian wave packets, with mean momentum and initial position mv_0 and x_0, respectively. They are not eigenstates of the number operator, but its expectation values on these states are:

$$\langle\phi_a|\hat{N}|\phi_a\rangle = \hbar(|a|^2 + \frac{1}{2})\,. \tag{23}$$

Acting by Galilean boosts and translations on a fixed state of the basis, $\psi_n(x,t)$, a new over-complete set of states is obtained, with elements:

$$\begin{aligned}\phi_a^n(x,t) \quad &= \quad \frac{(2\pi)^{-\frac{1}{4}}}{\sqrt{2^n n! L|\delta|}} \left(\frac{\delta^*}{|\delta|}\right)^{n+\frac{1}{2}} H_n\left(\frac{x - x_0 - v_0 t}{\sqrt{2}L|\delta|}\right) \\ &\qquad e^{-\dfrac{(x-x_0)^2 + x_0 v_0 t + i\tau v_0(v_0 t - 2x + x_0)}{4L^2\delta}}\,,\end{aligned} \tag{24}$$

representing traveling multi-localized wave packets bearing $n+1$ humps, with mean momentum and initial position mv_0 and x_0, respectively, where a is given by (21). The fact that these sets are over-complete is a general property, for $t=0$, of coherent states of the Heisenberg-Weyl group (see [8]). Since the time evolution is unitary, this property is kept at any t. As in the case $n=0$, these states are not eigenstates of the number operator, but the expectation values are:

$$\langle\phi_a^n|\hat{N}|\phi_a^n\rangle = \hbar(|a|^2 + n + \frac{1}{2})\,. \tag{25}$$

Being a set of coherent states, the uncertainty relations of ϕ_a^n, $\forall a \in \mathbb{C}$, are the same as those of ϕ_n given in Eq. (10) (see [8]).

4 Physical applications

The theoretical relevance of these multi-localized traveling wave packets, as a discrete basis of normalized free particle states, is of no doubt. Similar constructions, like the Harmonic Oscillator (HO) method [9] or the Transformed Harmonic Oscillator (THO) method [10] have been proposed, mainly in nuclear physics, to describe the bound states and the continuum spectrum in a discrete basis. But there the construction is a mathematical tool for approximating the solutions, with no physical meaning. Our states, however, are physically meaningful (as traveling wave packets) and experimentally feasible (see next section).

Among the possible theoretical applications, we could think of expanding plane waves in terms of the discrete basis $\{\phi_a^n\}_{n=0}^{\infty}$, with fixed $a \in \mathbb{C}$, and describing scattering process in a discrete basis, or expanding arbitrary wave packets in a continuum over-complete set $\{\phi_a^n\}_{a\in\mathbb{C}}$, with fixed n, which could be discretized in a lattice $\mathbb{Z} \times \mathbb{Z}$ of points while keeping the over-complete character (they are over-complete for $t = 0$ if the area of the unit cell is smaller than $\hbar$, and again by the unitary time evolution they continue to be over-complete for any t, see [8]), to perform numerical computations.

It could also be applied to relativistic systems, particularly to the free particle in de-Sitter space-time, where the ordinary formulation of quantum theory does not find a natural physical vacuum [12]. In this sense, the generalization of our approach to the relativistic case would provide a hierarchy of states where the first state, the relativistic counterpart of the Gaussian wave packet, plays the role of a vacuum.

But the main point of our letter is that they could be of practical interest in the transmission of quantum information. As a possible application we can consider the transmission of digital information, encoded by the number of humps.

This encoding would be rather robust in the sense that the number of zeros is exactly preserved in the free evolution, and the number of humps is conserved even in the presence of small perturbations. Numerical calculations have been performed, simulating "noise" by square potentials (well or barriers), leading to the conclusion that this holds as long as the mean energy of the state is large compared with the scale of the noise and the wave packet is sharp enough in momentum space in such a way that the transmission coefficient can be considered a constant. Under these circumstances (see for instance [13]), the wave packet behaves as a plane wave and the effect of the barrier in the transmitted packet is an overall attenuation, preserving its shape, and a time delay which takes its maximum values for energies near the resonant ones (and where the transmission coefficient is one). As shown in [13], this result is valid for any bounded potential of compact support, provided that the width of the potential is small in the sense that the time to pass through the barrier is smaller than the dispersion time

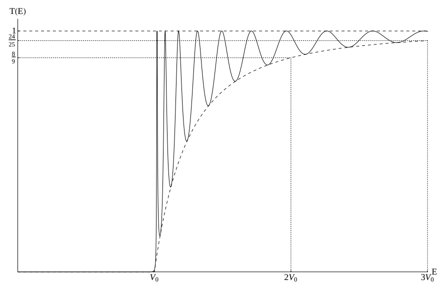

Figure 2: Transmission coefficient for the square barrier.

of the wave packet τ. Therefore, the conclusions obtained with the square potential can be generalized to any finite-range bounded potential.

In Figure 2, the transmission coefficient $T(E)$ for a square barrier as a function of the energy E of the incident plane wave is shown. The values of $T(E)$ for values of $E = 2V_0$ and $E = 3V_0$ have been singularized, where V_0 is the height of the barrier. For $E > 2V_0$, $\frac{8}{9} < T(E) \leq 1$, and for $E > 3V_0$, $\frac{24}{25} < T(E) \leq 1$. Therefore, if the wave packet has mean energy high enough, it penetrates the barrier without distortion and practically without attenuation, with only a time delay which can be appreciable for the resonant energies $E = V_0 + \frac{\hbar^2\pi^2}{2mb^2}n^2$, where $n = 1, 2, \ldots$, and b is the width of the barrier.

It should be stressed that, for $E < V_0$ the reflection is practically total (no transmission), and that for $E \approx V_0$, $T(E)$ varies very rapidly. Thus, the wave packet should be extremely narrow in momentum to avoid distortion. However, for $E > 2V_0$ it is enough to have $\Delta p \leq \frac{\hbar\pi}{2b}$ (half the period of the oscillations of $T(E)$).

5 Experimental realization

The preparation of this kind of discretized free states might be achieved by the use of a harmonic oscillator the potential of which is switched off at a given time. The vacuum state of this harmonic oscillator, when switched off, will provide the "vacuum" Gaussian wave packet with width $L = \sqrt{\frac{\hbar}{2m\omega}}$, where m is the mass of the particle and ω the frequency of the oscillator. Note that the dispersion time τ coincides with the inverse of the frequency of the oscillator. If the harmonic oscillator is in the n-th excited state, the $(n+1)$-hump state is obtained. To obtain traveling states, the initial state should be a coherent state $\phi_a(x,t)$ of eq. (20) for a one-hump traveling state or $\phi_a^n(x,t)$ for a $(n+1)$-hump traveling state. These coherent states can be obtained by acting with time-dependent classical forces on the harmonic oscillator according to Glauber [11, 8]. In fact, if the classical force is given by the potential $V(x) = -f(t)x$, and the initial state is the vacuum $|0\rangle$, then a standard coherent state $|a\rangle$ is obtained with $a = \frac{i}{\sqrt{2\omega}}\hat{f}(\omega)$, where $\hat{f}(\omega)$ is the Fourier component of $f(t)$ in the frequency ω of the oscillator.

To avoid the dispersion effect, the traveling time of these wave packets should be less than the dispersion time τ. This would seem a severe limit for the distances that the packets can travel being localized, but this is not the case. For instance, an electron with velocity $10^6\, m/s$ with $\Delta x = 0.1mm$ can travel a distance of 100 m while keeping localized (10^5 m for a proton), and this is more than enough for practical applications in Quantum Information theory.

Under the conditions commented in the previous section, these wave packets evolve without distortion even in the presence of perturbations. However, one could be interested, acting with appropriate potentials, in obtaining transitions between wave packets with different number of humps, in such a way that, for instance, a one-hump packet splits into a two-hump packet or a two-hump packet coalesces into a one-hump packet. This would open the door to performing quantum gates acting on q-bits realized with the one-hump and the two-hump states.

Finally, to detect this states and measure the number of humps, the number operator $\hat{N}$ could be used since its expectation value is directly related to the number of humps, see (25), once the initial position x_0 and the mean velocity v_0 are known.

It should be stressed that these states are physically observable and measurable. Let us consider, for instance, a two-hump wave packet $\phi^1_a(\vec{x}, t)$ in two or three dimensions with the humps in the transversal direction to that of the mean velocity $\vec{v}_0$ (the expressions in two and three dimensions are a straightforward generalization of those of one dimension). The separation of the two maxima of $|\phi^1_a|^2$ (see Fig. 1) is greater, in a factor 1.6, than the uncertainty in position Δx_1. Therefore the two humps should be measurable, and in fact, if this wave packet propagates in a bubble or wire chamber, two parallel, divergent tracks would be observed (if times $t << \tau$ are considered). For a three-hump wave packet, the separation among consecutive maxima (see Fig. 1) is smaller than the uncertainty in position, although the distance between the more separated maxima is greater than the uncertainty in position. This, together with the fact that the central maximum is smaller than the external ones, suggests that only two, overlapping thick tracks would be observed in a bubble or wire chamber. A similar behavior for a larger number of humps is expected.

Acknowledgments

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Appendix

Here we present an alternative construction of our discrete basis without resorting to the QAT, which is also more intuitive. In terms of the quantities defined at the beginning of Section 2, the Gaussian wave packet with center at the origin and width L can be writen in the form:

$$\psi_0(x,t) = \frac{(2\pi)^{-\frac{1}{4}}}{\sqrt{L\delta}} e^{-\frac{x^2}{4L^2\delta}} = \frac{(2\pi)^{-\frac{1}{4}}}{\sqrt{L|\delta|}} \left(\frac{\delta^*}{|\delta|}\right)^{\frac{1}{2}} e^{-\frac{x^2}{4L^2\delta}} \tag{26}$$

Then, we wonder which first order operator annihilates this state, i.e.

$$\hat{a}\,\psi_0(x,t) = 0\,. \tag{27}$$

The general form of such an operator would be $\hat{a} = f(x,t)\frac{\partial}{\partial x} + g(x,t)\frac{\partial}{\partial t} + h(x,t)$, and it is possible to choose

$$\hat{a} = L\delta\frac{\partial}{\partial x} + \frac{x}{2L}. \tag{28}$$

The adjoint of this operator is going to be the creation one:

$$\hat{a}^\dagger = -L\delta^*\frac{\partial}{\partial x} + \frac{x}{2L}. \tag{29}$$

Let us now check the action of $\hat{a}^\dagger$ on the vacuum state. This defines new states $\psi_n(x,t)$ up to normalization, by successively applying the creation operator. For example, we can compute the first state:

$$\psi_1(x,t) \equiv \hat{a}^\dagger\psi_0(x,t) = \frac{(2\pi)^{-\frac{1}{4}}}{\sqrt{L|\delta|}} \left(\frac{\delta^*}{|\delta|}\right)^{\frac{3}{2}} \frac{x}{L|\delta|} e^{-\frac{x^2}{4L^2\delta}}\,. \tag{30}$$

A general expression for this set of states is obtained by applying n times the creation operator and normalizing them, and can be cast into the compact form:

$$\psi_n(x,t) = \frac{(2\pi)^{-\frac{1}{4}}}{\sqrt{2^n n! L|\delta|}} e^{-\frac{x^2}{4L^2\delta}} \left(\frac{\delta^*}{|\delta|}\right)^{n+\frac{1}{2}} H_n\!\left(\frac{x}{\sqrt{2}L|\delta|}\right) \tag{31}$$

where H_n are the Hermite polynomials. $\psi_n(x,t)$ are solutions of the free Schrödinger equation for every integer n. We recover in this simple way the discrete basis for the free particle. All other quantities, like coherent states, can be computed directly without resorting to the QAT.

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