Entanglement transitions in random definite particle states

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Entanglement within qubits are studied for the subspace of definite particle states or definite number of up spins. A transition from an algebraic decay of entanglement within two qubits ($\sim 1/N^2$) with the total number N of qubits, to an exponential one when the number of particles is increased from two to three is studied in detail. In particular the probability that the concurrence is non-zero is calculated using statistical methods and shown to agree with numerical simulations. Further entanglement within a block of m qubits is studied using the log-negativity measure which indicates that a transition from algebraic to exponential decay occurs when the number of particles exceeds m.

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Entanglement has been extensively investigated in the recent past, as it is a critical resource for quantum information processing [1]. One model of quantum computation, the one-way quantum computing, relies explicitly on entanglement. The resource of entanglement is not at all rare, a random pure quantum state is typically highly entangled [2, 3]. In fact there is so much entanglement in typical random pure states that recent studies [4, 5] find them not to be useful for one-way quantum computation. This motivates the question of studying subsets of states with a control over the amount of entanglement available.

It is well known that most of the entanglement in many body quantum systems is multipartite. In random pure states of N qubits, we need to consider blocks whose total size is at least larger than $N/2$, for them to be entangled [6]. This being the case, entanglement in smaller blocks is nearly impossible to observe. Previous studies have shown how rare it is to have two qubits entangled in a many qubit random pure state [6, 7]. In this Letter it is shown that there is a surprising connection between the number of up-spins or particles present in definite particle states and entanglement. Thus producing definite particle random states, as defined below, may allow control over the type of entanglement that is desired. For instance if two qubit entanglement is to be obtained, it is shown that typical three-particle states will render this nearly impossible to achieve. The border between probable and improbable is described by a transition from an algebraic to an exponential decay, which is typically obtained at phase transitions. Further, the approach presented in this letter might shed light on methods that are applicable to a wider class of problems in the area of quantum complex systems.

Random pure states belong to the ensemble of states that are uniformly sampled from the Hilbert space, with the only constraint being normalization. Such states arise for instance in mesoscopic systems [8], nuclear physics [9] etc. and have been modeled as eigenfunctions of random matrices from the usual Gaussian ensembles. There have been studies that explore how to efficiently produce operators with statistical properties of random matrices [10]. Classically chaotic systems have long been known to exhibit such states in their quantum limit, and studies of entanglement in quantum chaotic systems often take recourse to random states [11, 12].

A definite particle state is a random pure state in a subspace formed by the basis vectors of the Hilbert space, which, when expressed in the spin-z basis, have a fixed number, say l, of "ones", or spin ups. Clearly many Hamiltonian systems including spin models (such as the quantum spin-glass, or the disordered Heisenberg lattices) are potential places where such states can occur as eigenstates. The number of particles allows to add complexity to the states in a systematic manner, and interesting properties for entanglement unfold in the process. The ensemble of interest is taken to be the one where the vectors in this subspace are such that they are all equally likely, subject only to the constraint of normalization.

A previous study of entanglement in random one-particle states showed that the averaged concurrence between any two qubits scales as $1/N$ [13]. Thus with increasing number of qubits entanglement between any two still remains considerable, although decreasing, in contrast to a full random state. In this Letter it is shown that for random two-particle states the average entanglement between qubits scales as $1/N^2$, while for three-particle states this becomes exponentially small, as it goes as $\exp(-N\ln(N))$. Thus when the number of particles exceeds two a transition is seen in the entanglement between two qubits. It maybe noted that for full random states it is not precisely known how such an entanglement scales with the number of qubits.

It is possible to generalize the results of concurrence between two qubits to entanglement within the block A having m qubits of the system for instance by studying the log-negativity measure [14]. Numerical evidence points to the plausible result that the entanglement decays with N algebraically if the number of particles in the subspace (l) is less than or equal to the block-length (m). Once again quite surprisingly the decay of entanglement becomes exponential when the number of particles exceeds the block-length.

A definite l-particle state is best written by grouping states with a given number of particles present in one block, say A, and its complementary block, say B. Let the number of qubits in block A be m and let $l \geq m$. Label the states by the number of particles (or total spin S_z) in subsets A and B to write:

$$\begin{aligned} |\psi\rangle &= \sum_{j=1}^{\binom{N-m}{l}} c_{1j}^{(0)} |0\rangle_A |l\rangle_B^j + \sum_{j=1}^{\binom{N-m}{l-1}} \sum_{i=1}^{\binom{m}{1}} c_{ij}^{(1)} |1\rangle_A^i |l-1\rangle_B^j \\ &+ \ldots + \sum_{j=1}^{\binom{N-m}{l-m}} c_{1j}^{(m)} |m\rangle_A |l-m\rangle_B^j. \end{aligned} \tag{1}$$

The reduced density matrix of the subsystem A denoted ρ_A, which is the state of the block of qubits we are interested in studying, then has a block structure with square blocks of sizes $\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}$. These blocks correspond to having a given number of particles, k, in the subsystem A and can be identified with one of the terms in the expression for the state. Further, each of these blocks can be written as $G_k = Q_k Q_k^\dagger$ where Q_k is a matrix whose entries are the coefficients $c_{ij}^{(k)}$ of the state. The condition that trace of a density matrix is unity implies that $\sum_k \mathrm{Tr}(Q_k Q_k^\dagger) = 1$. To construct the ensemble of l-particle states, draw all the $\mathcal{N} = \binom{N}{l}$ coefficients $c_{ij}^{(k)}$ from the normal distribution $N(0,1)$ and normalize them so that the trace condition is met. This is equivalent to choosing them uniformly with the only constraint being normalization [15].

In the important case of the reduced density matrix of a block A with $m = 2$ qubits in a pure state of N qubits and l particles can be written as:

$$\rho_A = \begin{pmatrix} a_{00} & 0 & 0 & 0 \\ 0 & a_{11} & a_{12} & 0 \\ 0 & a_{12}^* & a_{22} & 0 \\ 0 & 0 & 0 & a_{33} \end{pmatrix}, \tag{2}$$

where, $a_{00} = \sum_{i=1}^{\mu_0} (c_{1i}^{(0)})^2$, $\quad a_{33} = \sum_{i=1}^{\mu_2} (c_{1i}^{(2)})^2$, and

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{pmatrix} = Q_1 Q_1^\dagger,\ Q_1 = \begin{pmatrix} c_{11}^{(1)} & \cdots & c_{1\mu_1}^{(1)} \\ c_{21}^{(1)} & \cdots & c_{2\mu_1^{(1)}} \end{pmatrix}. \tag{3}$$

Here $\mu_i = \binom{N-2}{l-i}$, $i = 0, 1, 2$. The results presented in this work deal with real coefficients, $c_{ij}^{(k)*} = c_{ij}^{(k)}$, a situation that would be relevant for example for systems with time reversal symmetry. The central features, including the scaling, remain the same in the complex case. Also note that while the above expressions have been written when $l \geq m$, it is straightforward to write the same in the other case.

Concurrence [16] is a measure of entanglement present between two qubits such as those in the subsytem A. The above structure in Eq. (2), greatly simplifies the expression for concurrence [17]

$$C = 2 \max(|a_{12}| - \sqrt{a_{00} a_{33}}, 0), \tag{4}$$

and this allows for analytical estimates to be made, in contrast to the case of a full random state.

Due to the large number of coefficients $c_{ij}^{(k)}$ involved, it is a good approximation to assume that the normalization constraint is only important to set their scale and that they are otherwise independent. This implies that these are *i.i.d.* random variables drawn from the normal distribution $N(0, 1/\mathcal{N})$.

The approach to finding the mean concurrence will be to first estimate the probability that it will be nonzero. The term a_{12} involves a correlation between two strings of normally distributed numbers, each of length μ_1, while a_{00} maybe taken to be effectively its average and considered to be non-fluctuating. The following approximation then ensues:

$$\Pr(C > 0) \approx \Pr(|a_{12}| - \sqrt{\langle a_{00} \rangle} \sqrt{a_{33}} > 0). \tag{5}$$

The distribution of $|a_{12}|$, P_{12}, is of central importance and can obtained from, for example, the probability density function of one of the marginals of the Wishart distribution for correlation matrices. Suppressing the calculation, the result is

$$P_{12}\left(|a_{12}| = x\right) = 2\mathcal{N} \frac{K_\nu(\mathcal{N}x)}{\sqrt{\pi}\Gamma(\mu_1/2)} \left(\frac{\mathcal{N}x}{2}\right)^\nu, \tag{6}$$

where $K_\nu(x)$ is the modified Bessel function of the second kind, and $\nu = (\mu_1 - 1)/2$. The distribution of $\sqrt{a_{33}}$, P_{33}, follows from that of (square root of) a chi-square distribution with μ_2 degrees of freedom:

$$P_{33}\left(\sqrt{a_{33}} = y\right) = \frac{\mathcal{N}^{\mu_2/2}}{2^{\mu_2/2-1}\Gamma(\mu_2/2)} y^{\mu_2 - 1} e^{-\mathcal{N}y^2/2}. \tag{7}$$

Thus in view of the approximation above it follows that:

$$\Pr(C > 0) = \int_0^\infty P_{33}(y) \int_{\sqrt{\langle a_{00} \rangle} y}^\infty P_{12}(x)\, dx\, dy. \tag{8}$$

This can be evaluated by steps which are outlined here: (a) change variable y to $\sqrt{2y/\mathcal{N}}$, and $\mathcal{N}x$ to x; (b) use the integral representation $K_\nu(x) = \int_0^\infty e^{-x \cosh t} \cosh(\nu t)\, dt$, and change variable from $x \cosh t$ to x. This leads to the exact expression (given the approximation in Eq. (5))

$$\begin{aligned} \Pr(C > 0) = \beta \int_0^\infty & \frac{\cosh(\nu t)}{\cosh^{\nu+1} t} \\ & \int_0^\infty y^{\frac{\mu_2}{2} - 1} e^{-y}\, \Gamma\left(\nu + 1, \sqrt{\gamma y} \cosh t\right)\, dy\, dt. \end{aligned} \tag{9}$$

Here $\beta = 2^{-\nu+1}/\sqrt{\pi}\Gamma(\mu_1/2)\Gamma(\mu_2/2)$, and $\gamma = 2\langle a_{00}\rangle\mathcal{N}$. While further simplification is possible, for example by expanding e^{-y}, it is expedient to seek a non-trivial upper bound that reveals the nature of the decay with N, the number of qubits. A careful examination of the integrands indicate that this can be most easily achieved by using $e^{-y} < 1$ and thus effectively removing the exponential from the integral. The remaining integrals can be done exactly to give the first inequality below, while the second follows from standard inequalities for ratios of gamma functions:

$$\begin{aligned}\Pr(C>0) &< \frac{2^{\mu_2}}{\gamma^{\mu_2/2}\sqrt{\pi}}\frac{\Gamma\left(\frac{\mu_1+\mu_2}{2}\right)}{\Gamma\left(\frac{\mu_1}{2}\right)}\frac{\Gamma(\frac{\mu_2+1}{2})}{\Gamma(\frac{\mu_2}{2}+1)} \\ &< \frac{1}{\sqrt{\pi}}\left(\frac{2\mu_1\eta}{\gamma}\right)^{\frac{\mu_2}{2}}\frac{1}{\sqrt{\frac{\mu_2}{2}+\frac{1}{4}}}, \end{aligned} \qquad (10)$$

where $\eta = 1+(\mu_2-2)/(2\mu_1)$. Note that when the number of particles is much less than the number of qubits, $\eta \approx 1$. For the case of two particle states, $l = 2$, the first inequality yields

$$\Pr(C>0) < \frac{2\sqrt{2}}{\pi}\frac{1}{\sqrt{N}}, \qquad (11)$$

as $\mu_1 = N-2$ and $\mu_2 = 1$. The inequality is valid for large N. especially as the value of $\langle a_{00}\rangle$ is taken to be 1. As a matter of fact that this is an excellent estimate itself is seen from Fig. (1).

The two particle case is of special interest and can be essentially derived from simpler formulae, if it is observed that the fluctuations in a_{33}, arising from a single realization of the random variables, is more than the others. Note that: $a_{00} \sim \mu_0/\mathcal{N} \sim 1$, $a_{33} \sim \mu_2/\mathcal{N} \sim 1/N^2$, and $|a_{12}|^2 \sim \mu_1/\mathcal{N}^2 \sim 4/N^3$. Hence typically the concurrence will indeed be zero. Replacing the average values for the fluctuating a_{00} and $|a_{12}|$ results in $\Pr(C>0) \approx \Pr(\sqrt{a_{33}} < \sqrt{\frac{2}{\pi}\frac{2}{N^{3/2}}}) = \frac{2\sqrt{2}}{\pi}\frac{1}{\sqrt{N}}$, coinciding with the upper bound just derived.The average value of $|a_{12}|$ is used, rather than the (square root of the) average of $|a_{12}|^2$; the exact distribution can be used to show that $\langle|a_{12}|^2\rangle = \frac{\pi}{2}(\langle|a_{12}|\rangle^2)$. Thus for two particle states the probability of concurrence being positive decreases algebraically, in contrast to the one-particle case when $P(C>0) = 1$, as $a_{33} = 0$.

For $l = 3$, three particle states, a completely different behavior is obtained as $\mu_1 \sim N^2/2$, $\mu_2 \sim N$, and $\gamma \sim N^3/3$ which result in

$$\Pr(C>0) < \sqrt{\frac{2}{\pi N}}\exp\left(-\frac{N}{2}\log(N/3)\right). \qquad (12)$$

Unlike the two-particle case the probability that the concurrence is positive decreases at least exponentially with the number of qubits, see Fig. (2). Another new feature is that it is quite essential to take into account the fluctuations in *both* $|a_{12}|$ and in a_{33}. Ignoring say the fluctuations in a_{33} results in much smaller estimates of the probability than what is found.

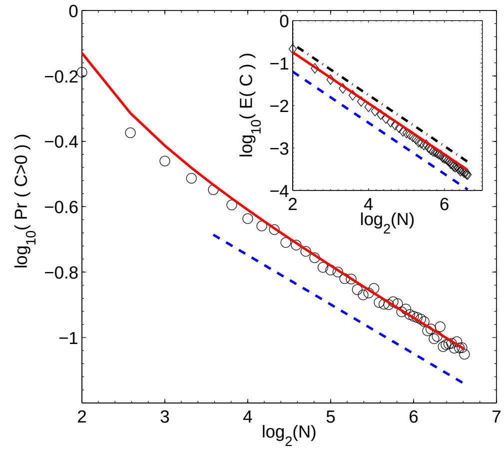

FIG. 1: The scaling of $\Pr(C>0)$ for random two particle states with N qubits. The dashed line is of slope $-1/2$, the circles are from numerical simulations, while the solid line is the estimate in Eq. (11). Inset shows the average concurrence, the dashed line is of slope -2, the dashed-dot line is the upper-bound while the solid line is the estimate in Eq. (13).

When $l > 2$, but still much less than N, the upper-bound in Eq. (10) does not estimate the probability accurately. While it can be made tighter, this is indeed a good bound as it is simple, decreases with N, and shows the advertized transition in the entanglement as one particle is added to a two particle state. It will be seen that the entanglement hitherto shared between two qubits will now be available for three-body and multi-party entanglement.

If $p = l/N$ is of order 1 (and less than 1/2), the states are “macroscopically” occupied; employing the approximation that $\binom{N}{Np} \sim e^{SN}$ where $S = -p\ln(p) - (1-p)\ln(1-p)$ is the binary entropy corresponding to probability p, results in the upper bound $\Pr(C>0) < d_1 e^{-SN/2}\, e^{-d_2 e^{SN}}$, where d_1 and d_2 are positive constants of order 1. However the upper-bound in Eq. (10) has to be used with caution as it can be rendered trivial if $(2\mu_1\eta/\gamma) > 1$, and consequently d_2 becomes negative. Thus for $N = 10$ qubits and $l = 5$ particles the upper-bound ≈ 2.2 is trivial while for $l = 4$ it is $\approx 1.5\times 10^{-5}$. While $N = 12$, $l = 6$ results in a trivial bound, $l = 5$ results in $\Pr(C>0) < 3.3\times 10^{-15}$. Similarly when $N = 14$ and $l = 6$, the upper-bound is $\approx 1.6\times 10^{-43}$, it is improbable that two qubits will be entangled.

The mean concurrence, $\mathbb{E}(C)$ is now estimated. In the two particle case for instance

$$\mathbb{E}(C) \sim 2\langle|a_{12}|\rangle\Pr(C>0) \sim \frac{16}{\pi^{3/2}N^2}. \qquad (13)$$

A more general estimate is possible as $\mathbb{E}(C) = \mathbb{E}[2(x -$

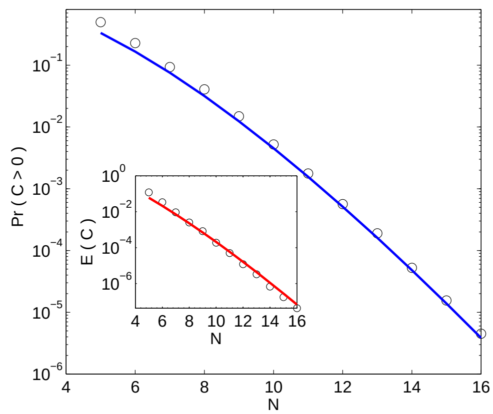

FIG. 2: The probability $\Pr(C > 0)$ for three particle states, and the average concurrence as number of qubits N is changed. Note that the y-axes are on a logarithmic scale. The circles are from numerical simulations while the solid line in the case of $\Pr(C > 0)$ is from an exact numerical evaluation of Eq. (8).

$\sqrt{\langle a_{00}\rangle y})\Theta(x - \sqrt{\langle a_{00}\rangle y})]$. Using the distribution $P_{12}(x)P_{33}(y)$ and following the same steps as outlined for the probability above it follows that $\mathbb{E}(C) <$

$$\frac{2^{\mu_2+2}}{\mathcal{N}\gamma^{\mu_2/2}\sqrt{\pi}}\frac{\Gamma\left(\frac{\mu_1+\mu_2+1}{2}\right)}{\Gamma\left(\frac{\mu_1}{2}\right)} < \frac{2\sqrt{\gamma}}{\mathcal{N}\sqrt{\pi}}\left(\frac{2\mu_1\eta'}{\gamma}\right)^{\frac{\mu_2+1}{2}} \qquad (14)$$

where $\eta' = 1 + (\mu_2 - 1)/(2\mu_1)$. In the two particle case this gives $\mathbb{E}(C) < 8/\sqrt{\pi}N^2$, which is quite close to the estimate above. The exponential decay for three or more particles is manifest. The mean concurrences are shown in the insets of Figs. (1),(2).

The vanishingly small two qubit entanglement for more than $l = 2$ goes into multiparty entanglement. A measure of entanglement that can be easily extended to a subsystem having more than two qubits is the log-negativity [14] and is given by $E_{LN}(\rho^{AB}) = \log(||\rho^{\Gamma}_{AB}||)$, where $||\rho^{\Gamma}||$ is the trace norm of the partial transpose matrix ρ^{Γ} [18]. Studying again block length of 2, numerical results not shown here indicate that entanglement between two qubits as measured by log-negativity decays algebraically (as $1/N^3$ in contrast to the $1/N^2$ for concurrence) for the case of two particles but becomes exponential when the particle number is increased to three. A similar behavior is exhibited for the entanglement between a qubit and the other pair when a block of 3 qubits is considered. Algebraic decay of the log-negativity for $l \leq 3$ is replaced by exponential decay for $l > 3$, see Fig. (3). Results not presented here indicate a similar behavior for block lengths of 4; however the numerics becomes considerably more difficult thereon.

In summary this Letter has given definitive evidence of an entanglement transition between two qubits as the number of particles is increased to three, while using log-

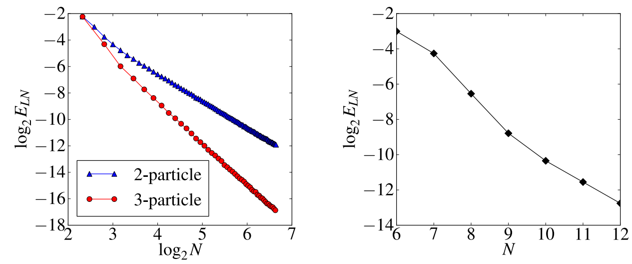

FIG. 3: Scaling of log-negativity in a block of 3 qubits, with the total number N of qubits for (left) two- and three-particle and (right) four-particle states .

negativity it is indicated that the following generalization would hold: the entanglement content in m qubits decays algebraically with N, the number of qubits, if the number of particles $l \leq m$, and exponentially if $l > m$. Of course throughout this work $l \leq [N/2]$.

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