Optimal quantum state estimation by no-signaling principle

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We obtain a simple derivation of the optimal quantum state estimation of a two-level system using the no-signaling principle. In particular, we show that the no-signaling principle determines the unique form of the guessing probability, independently to a given figure of merit such as the fidelity or the information gain. This proves that optimal measurements for a two-level quantum system is the same for almost all figures of merit.

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I. INTRODUCTION

Special relativity gives no-signaling principle that no information transfer can be faster-than-light. Quantum nonlocality appears to contradict the no-signaling principle. However, quantum nonlocality and no-signaling principle are in 'peaceful coexistence' [1].

For the coexistence, however, the no-signaling principle gives constraints on behavior of quantum systems. Interestingly, the bounds obtained by the no-signaling constraint are the same as those obtained purely by quantum mechanical methods. For example, there are, optimal quantum cloning [2], optimal unambiguous state discrimination [3], minimal error state discrimination [4–6], and maximum confidence state discrimination [7].

The purpose of this paper is add one in the list. Our topic is optimal state estimation for a single quantum bit (qubit). Massar and Popescu [8] showed that maximal average fidelity for a single qubit estimation is 2/3. Han suggested a way to derive known results by only spatial symmetry [9]. In this paper, we show how the known result is simply obtained by no-signaling principle. Moreover we show that, for any figure of merit, guessing-probability (distributions) are of the same form, $A\cos^2(\theta/2)+B\sin^2(\theta/2)$. Here A, B are constants and θ is angle between Bloch vector of prepared qubit and that of guessed one. This result actually confirms a conjecture in Refs. [10, 11] that optimal measurements are the same for any figure of merit.

II. QUANTUM STATE ESTIMATION

First let us describe the procedures of quantum state estimation more precisely. A player, Alice, randomly chooses a direction in 3-dimension. That is, she chooses a unit vector $\hat{r}$ with isotropic probability distribution. Then she prepares a qubit in a (pure) state with its Bloch vector $\hat{r}$, namely

$$\rho(\hat{r}) = \frac{1}{2}(\mathbb{1} + \hat{r}\cdot\vec{\sigma}) = |\hat{r}\rangle\langle\hat{r}|. \tag{1}$$

Here $\hat{r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, and $\sigma_x, \sigma_y, \sigma_z$ are Pauli operators. Then Alice sends a qubit $\rho(\hat{r})$ to another player, Bob. He knows all of Alice's procedures but doesn't know identity of the qubit of course. Then he makes a guess on the identity of the qubit by using all possible means including quantum measurement on the qubit. Bob's figure of merit (or score) is a function of state of sent qubit and that of guessed one. Usually, the closer the two states are, the higher the figure of merit is. Commonly used figures of merit are fidelity and information gain [10, 11]. Bob's task is to get maximal figures of merit on average.

It was shown that Bob's maximal average fidelity is 2/3 [8]. Bob's strategy achieving the maximum is simple [8]: He randomly chooses a unit vector $\hat{n}$ and performs a measurement $S_{\hat{n}}$ whose bases are $\rho(\hat{n}) = |\hat{n}\rangle\langle\hat{n}|$ and $\rho(-\hat{n}) = |-\hat{n}\rangle\langle-\hat{n}|$ on input qubit. Physically, $S_{\hat{n}}$ corresponds to Stern-Gerlach measurement in $\hat{n}$ direction if the qubit is in the spin of a particle. Next, if the measurement outcome is $\rho(\hat{n})$ ($\rho(-\hat{n})$), he makes a guess that Alice has sent a qubit in $\rho(\hat{n})$ ($\rho(-\hat{n})$) state. Let us consider guessing-probability $P(\hat{m}|\hat{r})$. Here $P(\hat{m}|\hat{r})\, d\Omega$ is probability that an outcome $\rho(\hat{r}')$ with unit vector $\hat{r}'$ around $\hat{m}$ within solid angle $d\Omega$ is obtained for an input qubit $\rho(\hat{r})$. It is not difficult to see that

$$P(\hat{m}|\hat{r}) = \frac{1}{2\pi}\cos^2\frac{\theta}{2} \;\; \propto |\langle\hat{m}|\hat{r}\rangle|^2, \tag{2}$$

where θ is angle between $\hat{m}$ and $\hat{r}$.

III. GUESSING-PROBABILITY HAS UNIQUE FORM.

Here we introduce a communication scenario between two remotely separated participants, Alice and Bob, to

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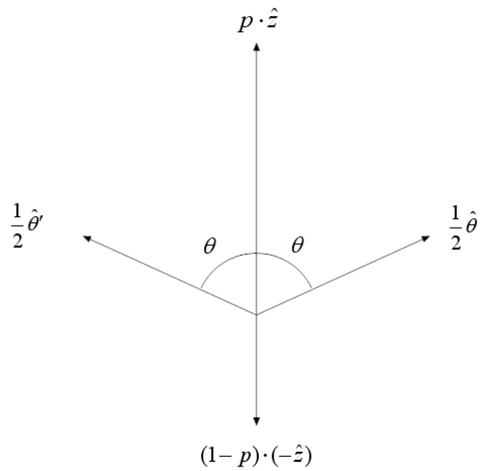

FIG. 1: Note $p\hat{z}+(1-p)(-\hat{z})=(1/2)\hat{\theta}+(1/2)(\hat{\theta}')$

which quantum state estimation can be incorporated, as in the case of minimal error state discrimination in Refs.[4–6]. Suppose Alice and Bob are sharing many copies of an entangled state,

$$|\psi\rangle = \sqrt{p}|0\rangle_A|\hat{z}\rangle_B + \sqrt{1-p}|1\rangle_A|-\hat{z}\rangle_B. \tag{3}$$

Here $|0\rangle$ and $|1\rangle$ are two orthogonal state of a qubit, A and B denote Alice and Bob. If Alice performs a measurement in $\{|0\rangle, |1\rangle\}$ basis, therefore, Bob is given a mixture of $|\hat{z}\rangle\langle\hat{z}|$ and $|-\hat{z}\rangle\langle-\hat{z}|$ with respective probabilities p and $1-p$. Then Bob's density operator is given by,

$$\rho_B = p|\hat{z}\rangle\langle\hat{z}| + (1-p)|-\hat{z}\rangle\langle-\hat{z}| = \frac{1}{2}\{\mathbb{1} + \vec{r}_B\cdot\vec{\sigma}\}, \tag{4}$$

where $\vec{r}_B = p\hat{z}+(1-p)(-\hat{z})$ (see Fig.1). Note that Bloch vector of a mixture is given by sum of Bloch vectors of pure states constituting the mixture with the corresponding probabilities as weighting factors. Let us consider Bob's Bloch vector, $\vec{r}_B = p\hat{z}+(1-p)(-\hat{z})$. Then consider a different decomposition of $\vec{r}_B$, $\vec{r}_B = (1/2)\hat{\theta}+(1/2)(\hat{\theta}')$. Here $\hat{\theta} = (0, \sin\theta, \cos\theta), \hat{\theta}' = (0, -\sin\theta, \cos\theta)$, and

$$\cos\theta = p-(1-p) = 2p-1. \tag{5}$$

This means that

$$\rho_B = \frac{1}{2}|\hat{\theta}\rangle\langle\hat{\theta}| + \frac{1}{2}|\hat{\theta}'\rangle\langle\hat{\theta}'| = \frac{1}{2}\{\mathbb{1} + \vec{r}_B\cdot\vec{\sigma}\}. \tag{6}$$

However, according to the Gisin-Hughston-Jozsa-Wootters theorem [12, 13], Alice can generate any decomposition of Bob's mixture by measuring her qubit in an appropriate basis. Therefore, in our case, either decomposition of Eq. (4) or that of Eq. (6) can be generated by Alice. This implies that the entangled state can be written as

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0'\rangle_A|\hat{\theta}\rangle_B + \frac{1}{\sqrt{2}}|1'\rangle_A|\hat{\theta}'\rangle_B, \tag{7}$$

where $\{|0'\rangle, |1'\rangle\}$ is another orthogonal basis. Therefore, Alice can generate decomposition of Eq. (4) (that of Eq. (6)) by performing measurement on her qubit in $\{|0\rangle, |1\rangle\}$ ($\{|0'\rangle, |1'\rangle\}$) basis.

However, if Bob can discriminate between the two decompositions, they can do faster-than-light communication: If Alice wants to send a message 0 (1), she repeatedly performs measurement in $\{|0\rangle, |1\rangle\}$ ($\{|0'\rangle, |1'\rangle\}$) basis. Then the decomposition of Eq. (4) (that of Eq. (6)) is generated at Bob's site. By discriminating the two decompositions, Bob can read out the message.

Now let us show that the guessing-probability has unique form by no-signaling principle. As we described above, in quantum state estimation Bob's task to make guesses on the input such that he gets maximal figure of merit on average. He is allowed to use all possible means including classical and quantum computers, and even humans. Let us consider, a 'black-box', a '(quantum) state estimator', which include everything needed for the estimation inside. For an input $\rho(\hat{r})$ the quantum-state-estimator gives just an outcome, its optimal guess, $\hat{m}$.

What we show is that, unless the guessing-probability $P(\hat{m}|\hat{r})$ is of the form $A\sin^2(\theta/2)+B\cos^2(\theta/2)$, where θ is angle between $\hat{z}$ and $\hat{\theta}$, faster-than-light communication is possible: By the fact that the two decompositions cannot be discriminated, it must be

$$\frac{1}{2}P(\hat{m}|\hat{\theta}) + \frac{1}{2}P(\hat{m}|\hat{\theta}') = pP(\hat{m}|\hat{z}) + (1-p)P(\hat{m}|-\hat{z}). \tag{8}$$

for all direction $\hat{m}$.

At this stage, we make a very plausible assumption that the estimator has an isotropy i.e. the guessing-probability $P(\hat{m}|\hat{r})$ is dependent only on angle between $\hat{m}$ and $\hat{r}$.

To get the functional form of $P(\hat{m}|\hat{\theta})$ most simply, we consider $\hat{m}=\hat{z}$ case

$$\frac{1}{2}P(\hat{z}|\hat{\theta}) + \frac{1}{2}P(\hat{z}|\hat{\theta}') = pP(\hat{z}|\hat{z}) + (1-p)P(\hat{z}|-\hat{z}). \tag{9}$$

If this Eq. (9) were not satisfied, Bob can discriminate the two decompositions by only observing how frequently the state estimator gives the outcome $\hat{z}$: More precisely, Bob counts frequency that the state estimator gives outcomes $\rho(\hat{r}')$ with unit vector $\hat{r}'$ around $\hat{z}$ within solid angle $d\Omega$. In the case of decomposition of Eq. (4), we can see that the frequency is $\{pP(\hat{z}|\hat{z})+(1-p)P(\hat{z}|-\hat{z})\}d\Omega$. In the case of decomposition of Eq.(6), the frequency is $\{(1/2)P(\hat{z}|\hat{\theta})+(1/2)P(\hat{z}|\hat{\theta}')\}d\Omega$.

However, by isotropy assumed in the above, we have

$$P(\hat{z}|\hat{\theta}) = P(\hat{z}|\hat{\theta}'). \tag{10}$$

By Eqs. (5), (9) and (10), and setting $P(\hat{z}|\hat{z}) \equiv A$, $P(\hat{z}|-\hat{z}) \equiv B$, we obtain

$$P(\hat{z}|\hat{\theta}) = A\cos^2\frac{\theta}{2} + B\sin^2\frac{\theta}{2}, \quad (11)$$

where θ is angle between $\hat{z}$ and $\hat{\theta}$.

We can also rewrite it as

$$P(\hat{z}|\hat{\theta}) = \alpha + \beta\cos\theta = \alpha + \beta\hat{z}\cdot\hat{\theta}. \quad (12)$$

By isotropy assumed above, we can generalize this result for any directions $\hat{m}$ and $\hat{\theta}$,

$$P(\hat{m}|\hat{\theta}) = \alpha + \beta\hat{m}\cdot\hat{\theta}. \quad (13)$$

Now, it is easy to show that this functional form generally satisfies Eq.(8). Therefore, we obtain general form of guessing-probability for state $\rho(\hat{\theta})$ as

$$P(\hat{m}|\hat{\theta}) = A\cos^2\frac{\theta}{2} + B\sin^2\frac{\theta}{2}, \quad (14)$$

where θ is angle between $\hat{m}$ and $\hat{\theta}$.

It is interesting that guessing-probability has a unique form regardless of figure of merit. If there is only a single guessing-probability, there is nothing to optimize. However, there are still infinitely many guessing-probabilitys depending on the constants A and B. Thus we should optimize it. When figure of merit is fidelity, it is easy to see that it is optimized when $B = 0$, obtaining $P(\hat{m}|\hat{\theta}) = A\cos^2(\theta/2)$. The actual measurement strategy which achieves the optimal one is the simple strategy described in section II. Using isotropy in our problem, we recover Eq. (2) after normalization. When the figure of merit is information-gain, it is optimized when either $B = 0$ or $A = 0$. guessing-probability in the former case is the same as the one when figure of merit is fidelity. Thus the optimal measurements are the same in this case. However, guessing-probability in the latter case is reversed, that is, 'a constant minus the guessing-probability of the former case'. However, the reversed one can be realized by the state estimator used in the former case. That is, when an outcome $\hat{r}$ is given, we adopt $-\hat{r}$ as true outcome. Thus, in the latter case, the optimal measurement is not different, too. It can be expected that optimal guessing-probability is the same for other figure of merits. In fact, we can see the followings. Almost all figures of merit have a property that the smaller the θ is, the higher the figure of merit is. As long as the property is satisfied, the optimal guessing-probability is obtained when $B = 0$. If guessing-probability are the same, clearly optimal measurement are also the same.

IV. CONCLUSION

We obtained a simple derivation of the optimal quantum state estimation of a two-level system using the no-signaling principle. In particular, we showed that the no-signaling principle determines the unique form of the guessing probability, independently to a given figure of merit such as the fidelity or the information gain. An optimal guessing-probability within the unique form can be realized by a simple actual measurement strategy. This proves that optimal measurements for a two-level quantum system is the same for all figures of merit, as long as the figure of merit satisfies a property that the smaller the θ is the higher the figure of merit is.

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