# A UNIQUENESS THEOREM FOR MEROMORPHIC MAPPINGS WITH TWO FAMILIES OF HYPERPLANES

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**Abstract**

In this paper, we extend the uniqueness theorem for meromorphic mappings to the case where the family of hyperplanes depends on the meromorphic mapping and where the meromorphic mappings may be degenerate.

## 1 Introduction

The uniqueness problem of meromorphic mappings under a condition on the inverse images of divisors was first studied by Nevanlinna [6]. He showed that for two nonconstant meromorphic functions $f$ and $g$ on the complex plane $\mathbb{C}$, if they have the same inverse images for five distinct values, then $f \equiv g$. In 1975, Fujimoto [3] generalized Nevanlinna's result to the case of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$. He showed that for two linearly nondegenerate meromorphic mappings $f$ and $g$ of $\mathbb{C}^m$ into $\mathbb{C}P^n$, if they have the same inverse images counted with multiplicities for $(3n+2)$ hyperplanes in general position in $\mathbb{C}P^n$, then $f \equiv g$.

In 1983, Smiley [9] showed that

**Theorem 1.** *Let $f, g$ be linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$. Let $\{H_j\}_{j=1}^q$ $(q \geq 3n+2)$ be hyperplanes in $\mathbb{C}P^n$ in general position. Assume that*

*a)* $f^{-1}(H_j) = g^{-1}(H_j)$ , *for all* $1 \leq j \leq q$ *(as sets),*

*b)* $\dim \big(f^{-1}(H_i) \cap f^{-1}(H_j)\big) \leq m-2$ *for all* $1 \leq i < j \leq q$ ,

*c)* $f = g$ *on* $\bigcup_{j=1}^q f^{-1}(H_j)$ .

*Then* $f \equiv g$.

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In 2006 Thai-Quang [11] generalized this result of Smiley to the case where $q \geq 3n+1$ and $n \geq 2$. In 2009, Dethloff-Tan [2] showed that for every nonnegative integer $c$ there exists a positive integer $N(c)$ depending only on $c$ such that Theorem 1 remains valid if $q \geq (3n+2-c)$ and $n \geq N(c)$. They also showed that the coefficient of $n$ in the formula of $q$ can be replaced by a number which is smaller than 3 for all $n >> 0$. Furthermore, they established a uniqueness theorem for the case of $2n+3$ hyperplanes and multiplicities are truncated by $n$. At the same time, they strongly generalized many uniqueness theorems of previous authors such as Fujimoto [4], Ji [5] and Stoll [10]. Recently, by using again the technique of Dethloff-Tan [2], Chen-Yan [1] showed that the assumption "multiplicities are truncated by $n$" in the result of Dethloff-Tan can be replaced by "multiplicities are truncated by 1". In [8], Quang examined the uniqueness problem for the case of $2n+2$ hyperplanes.

We would like to note that so far, all results on the uniqueness problem have still been restricted to the case where meromorphic mappings are sharing a common family of hyperplanes. The purpose of this paper is to introduce a uniqueness theorem for the case where the family of hyperplanes depends on the meromorphic mapping. We also will allow that the meromorphic mappings may be degenerate. For this purpose we introduce some new techniques which can also be used to obtain simpler proofs for many other uniqueness theorems.

We shall prove the following uniqueness theorem:

**Theorem 2.** *Let $f, g$ be nonconstant meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$. Let $\{H_j\}_{j=1}^q$ and $\{L_j\}_{j=1}^q$ $(q > 2n+2)$ be families of hyperplanes in $\mathbb{C}P^n$ in general position. Assume that*

*a)* $f^{-1}(H_j) = g^{-1}(L_j)$ *for all* $1 \leq j \leq q$,

*b)* $\dim\big(f^{-1}(H_i) \cap f^{-1}(H_j)\big) \leq m-2$ *for all* $1 \leq i < j \leq q$,

*c)* $\frac{(f,H_i)}{(g,L_i)} = \frac{(f,H_j)}{(g,L_j)}$ *on* $\bigcup_{k=1}^q f^{-1}(H_k) \setminus \big(f^{-1}(H_i) \cup f^{-1}(H_j)\big)$ *for all* $1 \leq i < j \leq q$.

*Then the following assertions hold :*

*i)* $\dim\langle Imf\rangle = \dim\langle Img\rangle \overset{Def.}{=:} p$,

*where for a subset $X \subset \mathbb{C}P^n$, we denote by $\langle X\rangle$ the smallest projective subspace of $\mathbb{C}P^n$ containing $X$.*

*ii)* *If*

$$(*)\quad q > \frac{2n+3-p+\sqrt{(2n+3-p)^2+8(p-1)(2n-p+1)}}{2}\ (\geq 2n+2),$$

*then*

$$\frac{(f,H_1)}{(g,L_1)} \equiv \cdots \equiv \frac{(f,H_q)}{(g,L_q)}\,.$$

*Furthermore, there exists a linear projective transformation $\mathcal{L}$ of $\mathbb{C}P^n$ into itself such that $\mathcal{L}(f) \equiv g$ and $\mathcal{L}(H_j \cap \langle Imf \rangle) = L_j \cap \mathcal{L}(\langle Imf \rangle)$ for all $j \in \{1,\dots,q\}$.*

**Remark.** 1.) In Theorem 2 condition $c)$ is well defined since, by condition $a)$, $\frac{(f,H_i)}{(g,L_i)}$ is a (nonvanishing) holomorphic function outside $f^{-1}(H_i)$.

2.) The condition $(*)$ is satisfied in the following cases:

+) $q \geq 2n+3$ and $p \in \{1,2,n-1,n\}, n \in \mathbb{Z}^+$.

+) $q \geq 2n+p+1$ and $p \in \{2,3,\dots,n\}, n \in \mathbb{Z}^+$.

3.) If there exists a subset $\{j_0,\dots,j_n\} \subset \{1,\dots,q\}$ such that $H_{j_i} \equiv L_{j_i}$ for all $i \in \{0,\dots,n\}$, then the proof of Theorem 2 implies that $f \equiv g$.

4.) For the special case where $f, g$ are linearly nondegenerate (i.e. $p = n$) and $H_j \equiv L_j$ , from Theorem 2 we get again the results of Dethloff-Tan [2] and Chen-Yan [1].

## 2 Preliminaries

We set $\|z\| := \big(|z_1|^2 + \cdots + |z_m|^2\big)^{1/2}$ for $z = (z_1,\dots,z_m) \in \mathbb{C}^m$ and define

$$B(r) := \big\{z \in \mathbb{C}^m : \|z\| < r\big\}, \qquad S(r) := \big\{z \in \mathbb{C}^m : \|z\| = r\big\}$$

for all $0 < r < \infty$. Define

$$d^c := \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial), \quad \upsilon := \big(dd^c\|z\|^2\big)^{m-1}$$
$$\sigma := d^c \log\|z\|^2 \wedge \big(dd^c \log\|z\|^2\big)^{m-1}.$$

Let $F$ be a nonzero holomorphic function on $\mathbb{C}^m$. For each $a \in \mathbb{C}^m$, expanding $F$ as $F = \sum P_i(z-a)$ with homogeneous polynomials $P_i$ of degree $i$ around $a$, we define

$$\nu_F(a) := \min\big\{i : P_i \not\equiv 0\big\}.$$

Let $\varphi$ be a nonzero meromorphic function on $\mathbb{C}^m$. We define the zero divisor $\nu_\varphi$ as follows: For each $z \in \mathbb{C}^m$, we choose nonzero holomorphic functions $F$ and $G$ on a neighborhood $U$ of $z$ such that $\varphi = F/G$ on $U$ and $\dim\big(F^{-1}(0) \cap G^{-1}(0)\big) \leqslant m-2$. Then we put $\nu_\varphi(z) := \nu_F(z)$.

Let $\nu$ be a divisor in $\mathbb{C}^m$ and $k$ be positive integer or $+\infty$. Set $|\nu| := \overline{\big\{z:\ \nu(z) \neq 0\big\}}$ and $\nu^{[k]}(z) := \min\{\nu(z), k\}$.

The truncated counting function of $\nu$ is defined by

$$N^{[k]}(r,\nu) := \int_1^r \frac{n^{[k]}(t)}{t^{2m-1}}dt \quad (1 < r < +\infty),$$

where

$$n^{[k]}(t) = \begin{cases} \displaystyle\int_{|\nu|\cap B(t)} \nu^{[k]} \cdot \upsilon & \text{for } m \geqslant 2, \\ \displaystyle\sum_{|z|\leqslant t} \nu^{[k]}(z) & \text{for } m = 1. \end{cases}$$

We simply write $N(r,\nu)$ for $N^{[+\infty]}(r,\nu)$.

For a nonzero meromorphic function $\varphi$ on $\mathbb{C}^m$, we set $N_\varphi^{[k]}(r) := N^{[k]}(r,\nu_\varphi)$ and $N_\varphi(r) := N^{[+\infty]}(r,\nu_\varphi)$. We have the following Jensen's formula:

$$N_\varphi(r) - N_{\frac{1}{\varphi}}(r) = \int_{S(r)} \log|\varphi|\sigma - \int_{S(1)} \log|\varphi|\sigma.$$

Let $f : \mathbb{C}^m \longrightarrow \mathbb{C}P^n$ be a meromorphic mapping. For an arbitrary fixed homogeneous coordinate system $(w_0 : \cdots : w_n)$ in $\mathbb{C}P^n$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, which means that each $f_i$ is a holomorphic function on $\mathbb{C}^m$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ outside the analytic set $\{f_0 = \cdots = f_n = 0\}$ of codimension $\geqslant 2$. Set $\|f\| = \big(|f_0|^2 + \cdots + |f_n|^2\big)^{1/2}$.
The characteristic function $T_f(r)$ of $f$ is defined by

$$T_f(r) := \int_{S(r)} \log\|f\|\sigma - \int_{S(1)} \log\|f\|\sigma, \quad 1 < r < +\infty.$$

For a meromorphic function $\varphi$ on $\mathbb{C}^m$, the characteristic function $T_\varphi(r)$ of $\varphi$ is defined by considering $\varphi$ as a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^1$.

We state the First Main Theorem and the Second Main Theorem in Value Distribution Theory: For a hyperplane $H: a_0w_0 + \cdots + a_nw_n = 0$ in $\mathbb{C}P^n$ with $\mathrm{Im}f \not\subseteq H$, we put $(f,H) = a_0f_0 + \cdots + a_nf_n$, where $(f_0 : \cdots : f_n)$ is a reduced representation of $f$.

**First Main Theorem.** *Let $f$ be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$, and $H$ be a hyperplane in $\mathbb{C}P^n$ such that $(f,H) \not\equiv 0$. Then*

$$N_{(f,H)}(r) \leqslant T_f(r) + O(1) \quad \text{for all } r > 1.$$

Let $n, N, q$ be positive integers with $q \geq 2N - n + 1$ and $N \geq n$. We say that hyperplanes $H_1, \ldots, H_q$ in $\mathbb{C}P^n$ are in $N$-subgeneral position if $\cap_{i=0}^N H_{j_i} = \varnothing$ for every subset $\{j_0, \ldots, j_N\} \subset \{1, \ldots, q\}$.

**Cartan-Nochka Second Main Theorem ([7], Theorem 3.1).** *Let $f$ be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $H_1, \ldots, H_q$ hyperplanes in $\mathbb{C}P^n$ in $N$-subgeneral position ($q \geq 2N - n + 1$). Then*

$$(q - 2N + n - 1)T_f(r) \leqslant \sum_{j=1}^{q} N^{[n]}_{(f,H_j)}(r) + o\big(T_f(r)\big)$$

*for all $r$ except for a subset $E$ of $(1, +\infty)$ of finite Lebesgue measure.*

# 3 Proof of Theorem 3

We first remark that $f^{-1}(H_j) = g^{-1}(L_j) \neq \mathbb{C}P^n$ for all $j \in \{1, \ldots, q\}$, and that therefore $\{H_j \cap \langle \mathrm{Im}f \rangle\}_{j=1}^q$ (respectively $\{L_j \cap \langle \mathrm{Im}g \rangle\}_{j=1}^q$) are hyperplanes in $\langle \mathrm{Im}f \rangle$ (respectively $\langle \mathrm{Im}g \rangle$) in $n-$subgeneral position: Indeed, otherwise there exists $t \in \{1, \ldots, q\}$ such that $f^{-1}(H_t) = \mathbb{C}P^n$. Then by the assumption $b)$ we have $\dim f^{-1}(H_j) \leq m - 2$ for all $j \in \{1, \ldots, q\} \setminus \{t\}$. Therefore, $f^{-1}(H_j) = \varnothing$ for all $j \in \{1, \ldots, q\} \setminus \{t\}$. Then $\langle \mathrm{Im}f \rangle \not\subset H_j$ for all $j \in \{1, \ldots, q\} \setminus \{t\}$. Thus, $\{H_j \cap \langle \mathrm{Im}f \rangle\}_{\substack{j=1\\ j\neq t}}^q$ are hyperplanes in $\langle \mathrm{Im}f \rangle$ in $n$-subgeneral position.

By the Cartan-Nochka Second Main Theorem, we have

$$(q - 2n + \dim\langle \mathrm{Im}f \rangle - 2)T_f(r) \leq \sum_{\substack{j=1\\ j\neq t}}^{q} N^{[\dim\langle \mathrm{Im}f \rangle]}_{(f,H_j)}(r) + o(T_f(r)) = o(T_f(r)).$$

This is a contradiction to the fact that $q > 2n+2$.

Since $\{H_j\}_{j=1}^{n+1}$ and $\{L_j\}_{j=1}^{n+1}$ are families of hyperplanes in general position, $\tilde{f} := \big((f,H_1) : \cdots : (f,H_{n+1})\big)$ and $\tilde{g} := \big((g,L_1) : \cdots : (g,L_{n+1})\big)$ are reduced representations of meromorphic mappings $\tilde{f}$ and $\tilde{g}$ respectively of $\mathbb{C}^m$ into $\mathbb{C}P^n$. Furthermore, $\dim\langle \mathrm{Im} f\rangle = \dim\langle \mathrm{Im}\tilde{f}\rangle$, $\dim\langle \mathrm{Im} g\rangle = \dim\langle \mathrm{Im}\tilde{g}\rangle$, $T_{\tilde{f}}(r) = T_f(r) + O(1)$ and $T_{\tilde{g}}(r) = T_g(r) + O(1)$.

By assumptions $a)$ and $c)$ we that

$$\tilde{f} = \tilde{g} \quad \text{on } \cup_{j=1}^{q} f^{-1}(H_j). \tag{3.1}$$

We now prove that

$$\dim\langle \mathrm{Im} f\rangle = \dim\langle \mathrm{Im} g\rangle \overset{\text{Def.}}{=} p. \tag{3.2}$$

This is equivalent to prove that $\dim\langle \mathrm{Im}\tilde{f}\rangle = \dim\langle \mathrm{Im}\tilde{g}\rangle$. Therefore, it suffices to show that for any hyperplane $H$ in $\mathbb{C}P^n$ then

$$(H,\tilde{f}) \equiv 0 \quad \text{if and only if} \quad (H,\tilde{g}) \equiv 0.$$

Suppose that the above assertion does not hold. Without loss of the generality, we may assume that there exists a hyperplane $H$ such that $(H,\tilde{f}) \not\equiv 0$ and $(H,\tilde{g}) \equiv 0$. Then by (3.1) we have

$$(\tilde{f}, H) = 0 \quad \text{on } \cup_{j=1}^{q} f^{-1}(H_j). \tag{3.3}$$

By (3.3) and by the First Main Theorem and the Cartan-Nochka Second Main Theorem we have

$$\begin{aligned}
(q - 2n + \dim\langle \mathrm{Im} f\rangle - 1)T_f(r) + O(1) &\le \sum_{j=1}^{q} N_{(f,H_j)}^{[\dim\langle \mathrm{Im} f\rangle]}(r) + o(T_f(r)) \\
&\le \dim\langle \mathrm{Im} f\rangle \sum_{j=1}^{q} N_{(f,H_j)}^{[1]}(r) + o(T_f(r)) \\
&\overset{(3.3)}{\le} \dim\langle \mathrm{Im} f\rangle N_{(\tilde{f},H)}(r) + o(T_f(r)) \\
&\le \dim\langle \mathrm{Im} f\rangle T_{\tilde{f}}(r) + o(T_f(r)) \\
&= \dim\langle \mathrm{Im} f\rangle T_f(r) + o(T_f(r)).
\end{aligned}$$

This is a contradiction to the fact that $q > 2n+2$. We complete the proof of (3.2).

Now we prove that

$$\frac{(f,H_1)}{(g,L_1)} \equiv \cdots \equiv \frac{(f,H_q)}{(g,L_q)}. \tag{3.4}$$

We distinguish the following two cases:

**Case 1:** There exists a subset $J := \{j_0, \dots, j_n\} \subset \{1, \dots, q\}$ such that

$$\frac{(f,H_{j_0})}{(g,L_{j_0})} \equiv \cdots \equiv \frac{(f,H_{j_n})}{(g,L_{j_n})} \overset{\text{Def.}}{\equiv} u\,.$$

We have $\mathrm{Pole}(u) \cup \mathrm{Zero}(u) \subset f^{-1}(H_{j_0}) \cap f^{-1}(H_{j_1})$, which is an analytic set of codimension at least 2 by assumption $b$). Hence, $\mathrm{Pole}(u) \cup \mathrm{Zero}(u) = \varnothing$.

Since $H_{j_0}, ..., H_{j_n}$ are hyperplanes in general position, $F := \big((f,H_{j_0}) : \cdots : (f,H_{j_n})\big)$ is the reduced representation of a meromorphic mapping $F$ of $\mathbb{C}^m$ into $\mathbb{C}P^n$. Still by the same reason $T_F(r) = T_f(r) + O(1)$.

Suppose that (3.4) does not hold. Then, there exists $i_0 \in \{1, \dots, q\} \setminus \{j_0, \dots, j_n\}$ such that

$$\frac{(f,H_{i_0})}{(g,L_{i_0})} \not\equiv u. \tag{3.5}$$

Since the families $\{H_j\}_{j=1}^q$ and $\{L_j\}_{j=1}^q$ are in general position, there exist hyperplanes $H^{i_0} : a_0\omega_0 + \cdots + a_n\omega_n = 0,\ L^{i_0} : b_0\omega_0 + \cdots + b_n\omega_n = 0$ in $\mathbb{C}P^n$ such that $(f,H_{i_0}) \equiv (F,H^{i_0})$, and $(g,L_{i_0}) \equiv b_0(g,L_{j_0}) + \cdots + b_n(g,L_{j_n}) \equiv \frac{(F,L^{i_0})}{u}$. Therefore, by (3.5) we have

$$\frac{(F,H^{i_0})}{(F,L^{i_0})} \equiv \frac{(f,H_{i_0})}{u(g,L_{i_0})} \not\equiv 1.$$

By assumption $c$) and since $\mathrm{Pole}(u) \cup \mathrm{Zero}(u) = \varnothing$, we have $u = \frac{(f,H_{j_0})}{(g,L_{j_0})} = \frac{(f,H_{i_0})}{(g,L_{i_0})} = u\frac{(F,H^{i_0})}{(F,L^{i_0})}$ on $\big(\bigcup_{k=1}^q f^{-1}(H_k)\big) \setminus \big(f^{-1}(H_{i_0}) \cup f^{-1}(H_{j_0})\big)$ and $u = \frac{(f,H_{j_1})}{(g,L_{j_1})} = \frac{(f,H_{i_0})}{(g,L_{i_0})} = u\frac{(F,H^{i_0})}{(F,L^{i_0})}$ on $\big(\bigcup_{k=1}^q f^{-1}(H_k)\big) \setminus \big(f^{-1}(H_{i_0}) \cup f^{-1}(H_{j_1})\big)$. Then $\frac{(F,H^{i_0})}{(F,L^{i_0})} = 1$ on $\big(\bigcup_{k=1}^q f^{-1}(H_k)\big) \setminus f^{-1}(H_{i_0})$.

Therefore,

$$\begin{aligned}\sum_{k=1, k\neq i_0}^{q} N^{[1]}_{(f,H_k)}(r) &\leq N_{\frac{(F,H^{i_0})}{(F,L^{i_0})}-1}(r) \\ &\leq T_{\frac{(F,H^{i_0})}{(F,L^{i_0})}}(r) + O(1) \leq T_F(r) + O(1) = T_f(r) + O(1).\end{aligned}$$

Therefore, by the Cartan-Nochka Second Main Theorem we have

$$T_f(r) + O(1) \geq \sum_{k=1, k\neq i_0}^{q} N^{[1]}_{(f,H_k)}(r) \geq \sum_{k=1,k\neq i_0}^{q} \frac{1}{p} N^{[p]}_{(f,H_k)}(r)$$
$$\geq \frac{q-2n+p-2}{p} T_f(r) - o(T_f(r)).$$

This implies that $q \leq 2n+2$. This is a contradiction. Hence, we get (3.4) in this case.

**Case 2:** For any subset $J \subset \{1,\ldots,q\}$ with $\#J = n+1$, there exists a pair $i, j \in J$ such that

$$\frac{(f,H_i)}{(g,L_i)} \not\equiv \frac{(f,H_j)}{(g,L_j)}.$$

We introduce an equivalence relation on $L := \{1,\cdots,q\}$ as follows: $i \sim j$ if and only if

$$\det \begin{pmatrix} (f,H_i) & (f,H_j) \\ (g,L_i) & (g,L_j) \end{pmatrix} \equiv 0.$$

Set $\{L_1,\cdots,L_s\} = L/\sim$. It is clear that $\sharp L_k \leq n$ for all $k \in \{1,\cdots,s\}$. Without loss of generality, we may assume that $L_k := \{i_{k-1}+1,\cdots,i_k\}$ $(k \in \{1,\cdots,s\})$ where $0 = i_0 < \cdots < i_s = q$.
We define the map $\sigma : \{1,\cdots,q\} \to \{1,\cdots,q\}$ by

$$\sigma(i) = \begin{cases} i+n & \text{if } i+n \leq q, \\ i+n-q & \text{if } i+n > q. \end{cases}$$

It is easy to see that $\sigma$ is bijective and $|\sigma(i) - i| \geq n$ (note that $q > 2n+2$). This implies that $i$ and $\sigma(i)$ belong to distinct sets of $\{L_1,\cdots,L_s\}$. This implies that for all $i \in \{1,\ldots,q\}$,

$$P_i := \det \begin{pmatrix} (f,H_i) & (f,H_{\sigma(i)}) \\ (g,L_i) & (g,L_{\sigma(i)}) \end{pmatrix} \not\equiv 0.$$

By the assumption and by the definition of function $P_i$, we have

$$\nu_{P_i} \geq \min\{\nu_{(f,H_i)}, \nu_{(g,L_i)}\} + \min\{\nu_{(f,H_{\sigma(i)})}, \nu_{(g,L_{\sigma(i)})}\} + \sum_{\substack{j=1 \\ j\neq i,\sigma(i)}}^{q} \nu^{[1]}_{(f,H_j)} \qquad (3.6)$$

outside an analytic set of codimension $\geq 2$.
On the other hand, since $f^{-1}(H_k) = g^{-1}(L_k)$ we have

$$\begin{aligned}\min\{\nu_{(f,H_k)}, \nu_{(g,L_k)}\} &\geq \min\{\nu_{(f,H_k)}, p\} + \min\{\nu_{(g,L_k)}, p\} - p\min\{\nu_{(f,H_k)}, 1\} \\ &= \nu^{[p]}_{(f,H_k)} + \nu^{[p]}_{(g,L_k)} - p\nu^{[1]}_{(f,H_k)}\end{aligned}$$

for $k \in \{i, \sigma(i)\}$.
Therefore, by (3.6) we have

$$\begin{aligned}\nu_{P_i} \geq \nu^{[p]}_{(f,H_i)} &+ \nu^{[p]}_{(g,L_i)} + \nu^{[p]}_{(f,H_{\sigma(i)})} + \nu^{[p]}_{(g,L_{\sigma(i)})} \\ &- p\nu^{[1]}_{(f,H_i)} - p\nu^{[1]}_{(f,H_{\sigma(i)})} + \sum_{\substack{j=1 \\ j\neq i,\sigma(i)}}^{q} \nu^{[1]}_{(f,H_j)}\end{aligned}$$

outside an analytic set of codimension $\geq 2$.
Then for all $i \in \{1, \ldots, q\}$ we have

$$\begin{aligned}N_{P_i}(r) \geq N^{[p]}_{(f,H_i)}(r) &+ N^{[p]}_{(g,L_i)}(r) + N^{[p]}_{(f,H_{\sigma(i)})}(r) + N^{[p]}_{(g,L_{\sigma(i)})}(r) \\ &- pN^{[1]}_{(f,H_i)}(r) - pN^{[1]}_{(f,H_{\sigma(i)})}(r) + \sum_{\substack{j=1 \\ j\neq i,\sigma(i)}}^{q} N^{[1]}_{(f,H_j)}(r). \quad (3.7)\end{aligned}$$

On the other hand, by Jensen's formula

$$\begin{aligned}N_{P_i}(r) &= \int_{S(r)} \log|P_i|\sigma + O(1) \\ &\leq \int_{S(r)} \log(|(f,H_i)|^2 + |(f,H_{\sigma(i)})|^2)^{\frac{1}{2}}\sigma \\ &\qquad + \int_{S(r)} \log(|(g,L_i)|^2 + |(g,L_{\sigma(i)})|^2)^{\frac{1}{2}}\sigma + O(1) \\ &\leq T_f(r) + T_g(r) + O(1).\end{aligned}$$

Therefore, by (3.7) for all $i \in \{1, \ldots, q\}$ we have

$$\begin{aligned}N^{[p]}_{(f,H_i)}(r) &+ N^{[p]}_{(g,L_i)}(r) + N^{[p]}_{(f,H_{\sigma(i)})}(r) + N^{[p]}_{(g,L_{\sigma(i)})}(r) \\ &- pN^{[1]}_{(f,H_i)}(r) - pN^{[1]}_{(f,H_{\sigma(i)})}(r) + \sum_{\substack{j=1 \\ j\neq i,\sigma(i)}}^{q} N^{[1]}_{(f,H_j)}(r) \\ &\leq T_f(r) + T_g(r) + O(1). \quad (3.8)\end{aligned}$$

By summing-up of both sides of the above inequality for all $i \in \{1, \ldots, q\}$, we have

$$2\sum_{j=1}^{q}\big(N^{[p]}_{(f,H_j)}(r)+N^{[p]}_{(g,L_j)}(r)\big)+(q-2p-2)\sum_{j=1}^{q}N^{[1]}_{(f,H_j)}(r) \le q\big(T_f(r)+T_g(r)\big)+O(1). \tag{3.9}$$

Therefore, since $f^{-1}(H_j)=g^{-1}(L_j)$ we have

$$2\sum_{j=1}^{q}\big(N^{[p]}_{(f,H_j)}(r)+N^{[p]}_{(g,L_j)}(r)\big)+\frac{q-2p-2}{2}\sum_{j=1}^{q}\big(N^{[1]}_{(f,H_j)}(r)+N^{[1]}_{(g,L_j)}(r)\big) \le q\big(T_f(r)+T_g(r)\big)+O(1). \tag{3.10}$$

Then

$$\big(2+\frac{q-2p-2}{2p}\big)\sum_{j=1}^{q}\big(N^{[p]}_{(f,H_j)}(r)+N^{[p]}_{(g,L_j)}(r)\big)\le q\big(T_f(r)+T_g(r)\big)+O(1). \tag{3.11}$$

By (3.11) and by the Cartan-Nochka Second Main Theorem we have

$$\frac{(q+2p-2)(q-2n+p-1)}{2p}\big(T_f(r)+T_g(r)\big)\le q\big(T_f(r)+T_g(r)\big)+o\big(T_f(r)+T_g(r)\big).$$

It follows that $(q+2p-2)(q-2n+p-1)\le 2pq$. Then $q^2-(2n+3-p)q-2(p-1)(2n+1-p)\le 0$. This is a contradiction to condition $(*)$ of Theorem 2. Thus we have completed the proof of (3.4).

Assume that $H_j : a_{j0}\omega_0+\cdots+a_{jn}\omega_n = 0$, $L_j : b_{j0}\omega_0+\cdots+b_{jn}\omega_n = 0 \quad (j=1,\ldots,q)$.
Set

$$A := \begin{pmatrix} a_{10} & \ldots & a_{1n}\\ a_{20} & \ldots & a_{2n}\\ \vdots & \ddots & \vdots\\ a_{(n+1)0} & \ldots & a_{(n+1)n}\end{pmatrix}, B := \begin{pmatrix} b_{10} & \ldots & b_{1n}\\ b_{20} & \ldots & b_{2n}\\ \vdots & \ddots & \vdots\\ b_{(n+1)0} & \ldots & b_{(n+1)n}\end{pmatrix}, \text{ and } \mathcal{L}=B^{-1}\cdot A.$$

By (3.4), we have $A(f)\equiv B(g)$, so we get $\mathcal{L}(f)\equiv g$.

Set $H_j^* = (a_{j0}, \ldots, a_{jn}) \in \mathbb{C}^{n+1}$, $L_j^* = (b_{j0}, \ldots, b_{jn}) \in \mathbb{C}^{n+1}$. We write $H_j^* = \alpha_{j1}H_1^* + \cdots + \alpha_{j(n+1)}H_{n+1}^*$ and $L_j^* = \beta_{j1}L_1^* + \cdots + \beta_{j(n+1)}L_{n+1}^*$.
By (3.4) we have

$$\frac{\alpha_{j1}(f, H_1) + \cdots + \alpha_{j(n+1)}(f, H_{n+1})}{\beta_{j1}(g, L_1) + \cdots + \beta_{j(n+1)}(g, L_{n+1})} \equiv \frac{(f, H_1)}{(g, L_1)} \equiv \cdots \equiv \frac{(f, H_{n+1})}{(g, L_{n+1})}$$

for all $j \in \{1, \ldots, q\}$.
This implies that

$$(\alpha_{j1} - \beta_{j1})(f, H_1) + \cdots + (\alpha_{j(n+1)} - \beta_{j(n+1)})(f, H_{n+1}) \equiv 0 \tag{3.12}$$

for all $j \in \{1, \ldots, q\}$.
On the other hand $f : \mathbb{C}^m \longrightarrow \langle \mathrm{Im} f \rangle$ is linearly nondegenerate and $\{H_j\}_{j=1}^{n+1}$ are in general position in $\mathbb{C}P^n$. Thus, by (3.12) we have

$$(\alpha_{j1} - \beta_{j1})(\omega, H_1) + \cdots + (\alpha_{j(n+1)} - \beta_{j(n+1)})(\omega, H_{n+1}) = 0 \tag{3.13}$$

for all $\omega \in \langle \mathrm{Im} f \rangle$ for all $j \in \{1, \ldots, q\}$.

Let hyperplanes $\alpha_j : \alpha_{j1}\omega_0 + \cdots + \alpha_{j(n+1)}\omega_n = 0$ and $\beta_j : \beta_{j1}\omega_0 + \cdots + \beta_{j(n+1)}\omega_n = 0 \quad (j = 1, \ldots, q)$.
By (3.13) we have

$$(A(\omega), \alpha_j) = (A(\omega), \beta_j) \tag{3.14}$$

for all $\omega \in \langle \mathrm{Im} f \rangle$ and $j \in \{1, \ldots, q\}$.

For any $j \in \{1, \ldots, q\}$ and for any $\omega \in \langle \mathrm{Im} f \rangle$ we have

$$\begin{aligned}
(\omega, H_j) &= \alpha_{j1}(\omega, H_1) + \cdots + \alpha_{j(n+1)}(\omega, H_{n+1}) \\
&= (A(\omega), \alpha_j) \\
&\overset{(3.14)}{=} (A(\omega), \beta_j) \\
&= (B \cdot \mathcal{L}(\omega), \beta_j) \\
&= \beta_{j1}(\mathcal{L}(\omega), L_1) + \cdots + \beta_{j(n+1)}(\mathcal{L}(\omega), L_{n+1}) \\
&= (\mathcal{L}(\omega), L_j).
\end{aligned}$$

This implies that $\mathcal{L}(\langle \mathrm{Im} f \rangle \cap H_j) = L_j \cap \mathcal{L}(\langle \mathrm{Im} f \rangle)$ for all $j \in \{1, \ldots, q\}$, which completes the proof of Theorem 2. □

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