

# CONSTRUCTING UNIVERSAL ABELIAN COVERS OF GRAPH MANIFOLDS

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ABSTRACT. To a rational homology sphere graph manifold one can associate a weighted tree invariant called splice diagram. It was shown in [Ped10a] that the splice diagram determines the universal abelian cover of the manifold. We will in this article turn the proof of this in to an algorithm to explicitly construct the universal abelian cover from the splice diagram.

## 1. INTRODUCTION

Graph manifolds is an important class of 3-manifolds, they are defined as the manifolds which have only Seifert fibered pieces in their JSJ-decomposition. They are also the 3-manifolds which are boundaries of plumbed 4-manifolds, and a very used method to represent a graph manifold  $M$  is by giving a plumbing diagram of a 4-manifold  $X$  such that  $M = \partial X$ . Neumann gave a complete calculus for changing  $X$  but keeping  $M$  fixed in [Neu81], and when we are going to construct the universal abelian cover in section 3 we are going to do this by constructing a plumbing diagram of it. It should also be noted that graph manifolds are also the manifolds that have no hyperbolic pieces in their geometric decomposition.

If one restrict to rational homology spheres (from know on QHS) then one have a graph invariant of graph manifolds called splice diagrams. They where original introduced in [EN85] and [Sie80] for integer homology sphere graph manifolds, and were then later generalized by Neumann and Wahl to QHS's in [NW02], and used extensively in [NW05b] and [NW05a].

The simplest example of a graph manifold is of course a Seifert fibered manifold, and if one restrict to QHS's, Neumann found a nice construction of the universal abelian cover in [Neu83a], namely as the link of a Brieskorn complete intersection defined by the collection of the first of the two Seifert invariants associated to the singular fibers. This is exactly the information given by the splice diagram of Seifert fibered manifolds, and Neumann and Wahl used the splice diagram to generalize the Brieskorn complete intersections provided the splice diagram satisfy what they called the semigroup condition, to what they called *splice diagram equations*. Under a further restriction on the given manifold they were able to prove that the link of the splice diagram equations is the universal abelian cover in [NW05a].

This indicates that the splice diagram might determine the universal abelian cover, and in [Ped10a] I was able to prove the following theorem:

**Theorem 1.1.** *Let  $M_1$  and  $M_2$  be two QHS graph manifold there have the same splice diagram. Let  $\widetilde{M}_i \rightarrow M_i$  be the universal abelian covers. Then  $\widetilde{M}_1$  and  $\widetilde{M}_2$  are homeomorphic.*

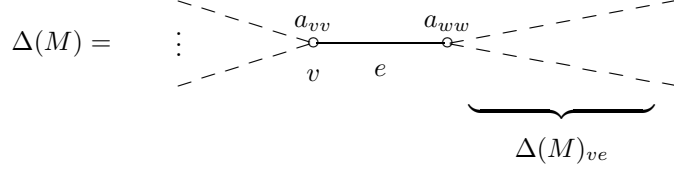
The proof consist of inductively constructing the universal abelian cover from the splice diagram, and the purpose of this article is to extract an algorithm for constructing the universal abelian cover from the proof. I will hence not prove

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**Proposition 2.2.** *Let  $v$  be a node in  $\Gamma(M)$ . Then the sign  $\varepsilon$  at  $v$  is  $\varepsilon = -\text{sign}(a_{vv})$ , where  $a_{vv}$  is the entry of  $A(M)^{-1}$  corresponding to the node  $v$ .*

The most important information the splice diagram do not encode of the manifold is  $|H_1(M)|$ , but that and the splice diagram do determine the rational euler number of any of the Seifert fibered pieces of  $M$ , by the following proposition from [Ped10a].

**Proposition 2.3.** *Let  $v$  be a node in a splice diagram decorated as in Fig. 1 below with  $r_i \neq 0$  for  $i \neq 1$ , and let  $e_v$  be the rational euler number of  $M_v$ . Then*

$$(1) \quad e_v = -d \left( \frac{\varepsilon s_1}{N D_1 \prod_{j=2}^k r_k} + \sum_{i=2}^k \frac{\varepsilon_i M_i}{r_i D_i} \right)$$

where  $d = |H_1(M)|$ ,  $N = \prod_{j=1}^k n_j$ ,  $M_i = \prod_{j=1}^i m_{ij}$ , and  $D_i$  is the edge determinant associated to the edge between  $v$  and  $v_i$ .

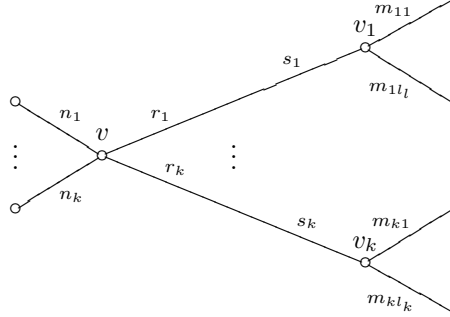


Figure 1

Note that this does give a formula for  $e_v/d$  from  $\Gamma$ , which we will need later.

In the algorithm for constructing the universal abelian cover of  $M$  from  $\Gamma(M)$ , one number associated to each end of an edge in  $\Gamma(M)$  is going to be very important, the *ideal generator*, which is constructed the following way. Let  $v$  and  $w$  be two vertices of  $\Gamma(M)$  then we define the *linking number* of  $v$  and  $w$   $l_{vw}$  as the product of all edge weights adjacent to but not on the shortest path from  $v$  to  $w$ . We define  $l'_{vw}$  the same way, but omitting weights adjacent to  $v$  and  $w$ . If  $e$  is an edge adjacent to  $v$ , we then let  $\Gamma_{ve}$  be the connected component of  $\Gamma(M) - e$  not containing  $v$ . And define the following ideal of  $\mathbb{Z}$

$$I_{ve} = \langle l'_{vw} \mid w \text{ a leaf in } \Gamma_{ve} \rangle.$$

Then we define the ideal generator  $\bar{d}_{ve}$  associated to  $v$  and  $e$  to be the positive generator of  $I_{ve}$ .

**Definition 2.4.** A splice diagram  $\Gamma$  satisfy the *ideal condition* if the ideal generator  $\bar{d}_{ve}$  divides the edge weight  $d_{ve}$

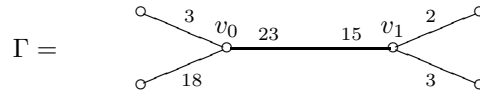
**Proposition 2.5.** *Let  $M$  be a  $\mathbb{Q}$ HS graph manifold, then  $\Gamma(M)$  satisfy the ideal condition.*

This proposition follows from the following topological description of the ideal generator from Appendix 1 of [NW05a].

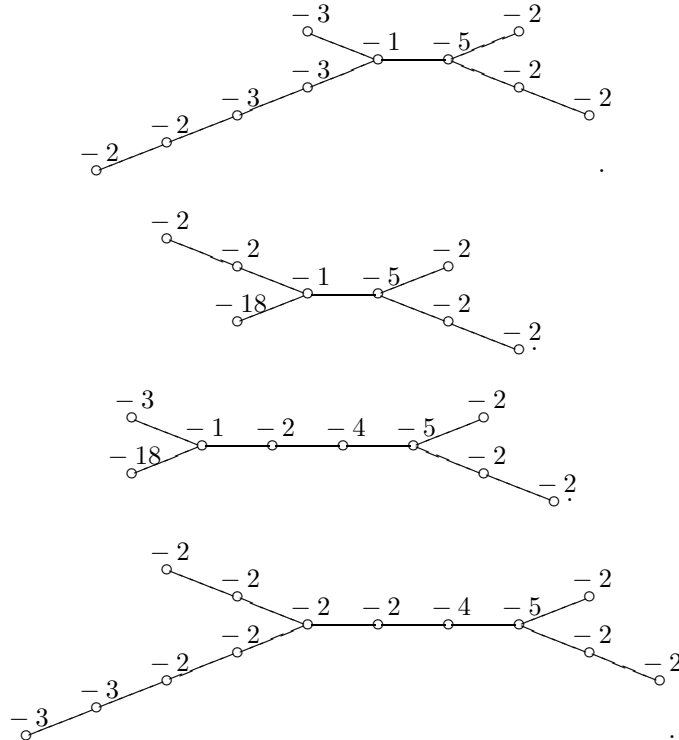
**Theorem 2.6.** *The ideal generator  $\bar{d}_{ve}$  is  $|H_1(M_{ve}/K)|$ , where  $K$  is the knot given as the core of the solid torus glued into  $M'_{ve}$  to construct  $M_{ve}$ .*

### 3. CONSTRUCTION THE UNIVERSAL ABELIAN COVER: AN EXAMPLE

In this section we are going to see how the algorithm used in the proof of Theorem [Ped10a] can be used to construct the universal abelian cover  $\widetilde{M}$  of a graph manifold  $M$ , from the splice diagram  $\Gamma(M)$ . We are going to specify  $\widetilde{M}$  by constructing a plumbing diagram  $\Delta$  for  $\widetilde{M}$ . To illustrate the construction we are going to use the following example

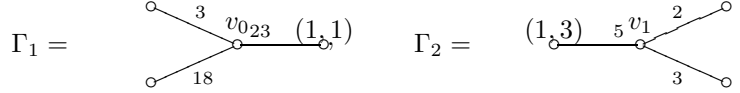


There are four different manifolds who has  $\Gamma$  as their splice diagram, and several more non manifold graph orbifolds. By Theorem [Ped10a]  $\Gamma$  is the splice diagram of a singularity link, it then follows from [Ped10b] that  $\widetilde{M}$  is a rational homology sphere. The example is also interesting since non of the manifolds with  $\Gamma$  as their splice diagram satisfy the congruence condition of Neumann and Wahl see [NW05a], but there are non manifold orbifolds with splice diagram  $\Gamma$  which satisfy the orbifold congruence condition see [Ped10c]. Below is plumbing diagrams for the four manifolds with  $\Gamma$  as their splice diagram:



**3.1. Constructing the Building Blocks.** The inductive procedure in the construction of the universal abelian cover works by taking an edge  $e$  between two nodes of  $\Gamma$  and make a new non connected splice diagram  $\Gamma_e$  where  $e$ , has been replaced with two leaves. So starting with the edge called  $e_1$  and going through this process of cutting the edge until we have cut the last edge between two nodes  $e_{N-1}$ , we get that  $\Gamma_{e_{N-1}}$  is a collection of one node splice diagrams  $\Gamma_{e_{N-1}} = \{\Gamma_i\}_{i=1}^N$ . For each of theses one node splice diagram  $\Gamma_i$  one then takes a number of copies of a specific manifold  $M_i$ , and use the information from the  $\Gamma_{e_j}$ 's to glue the pieces together. So the first step is to determine this manifolds  $\{M_i\}_{i=1}^N$ , which are the building blocks of the universal abelian cover.

First lets see how the  $\Gamma_{e_j}$ 's are going to look. Each time we cut an edge  $e$  between the nodes  $w_1$  and  $w_2$  in  $\Gamma$ , we divide every edge weight  $d_{ve'}$  such that  $w_1$  or  $w_2$  is in  $\Gamma_{ve'}$ , by the ideal generator  $\bar{d}_{w_i e}$  of the edge weight  $d_{w_i e}$  such that  $v$  is not in  $\Gamma_{w_i e}$ . In our example we only have two edge weights where this is true with respect to  $e$  namely  $d_{v_0 e} = 23$  and  $d_{v_1 e} = 15$ , and  $\bar{d}_{v_0 e} = 1$  and  $\bar{d}_{v_1 e} = 3$ . So the two one node splice diagrams  $\Gamma_1$  and  $\Gamma_2$  are going to look like



The pair added to the new leaves is recording of the following information which is going to be used when the gluing are made: the first number specifies which number in the sequence of cutting this is, in this case the first, and the second number is the ideal generator associated to the weight before cutting.

Next we want to find the building block  $M_i$  associated to each of the  $\Gamma_i$ 's. To do this we have to separate the  $\Gamma_i$ 's into two types, the first are the once that do not have an edge weight of 0, and the second are the once that, remember at most one weight adjacent to a node can be 0.

In the first case we use the following theorem

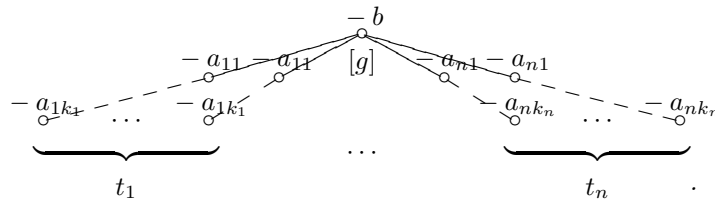
**Theorem 3.1.** *Let  $M$  be a rational homology orbifold fibration  $S^1$ -fibration over a orbifold surface, with Seifert invariants  $(\alpha_1, \beta_1), \dots, (\alpha_n, \beta_n)$ . Then the universal abelian cover of  $M$  is the link of the Brieskorn complete intersection  $\Sigma(\alpha_1, \dots, \alpha_n)$ .*

The way one construct the manifolds after cutting an edge may result in graph orbifolds instead of just graph manifolds, as explained in the proof of 6.3 in [Ped10a], and hence we need this theorem for  $S^1$  orbifold fibrations. Neumann only proves this theorem for Seifert fibered manifolds in [Neu83a] and [Neu83b], but the proof given in [Neu83b] also work in the general case of an orbifold  $S^1$ -fibration. The value of  $\varepsilon$  does not matter, since reversing the orientation of a Seifert fibered manifold only changes the  $\beta_i$ 's not the  $\alpha_i$ 's and hence only change the splice diagrams by replacing  $\varepsilon$  with  $-\varepsilon$ .

So in our example  $M_1$  is the link of  $\Sigma(3, 18, 23)$  and  $M_2$  is the link of  $\Sigma(2, 3, 5)$ .

Next we use the following theorem to get plumbing diagrams for the  $M_i$ 's.

**Theorem 3.2.** *Let  $M$  be the link of the Brieskorn complete intersection  $\Sigma(\alpha_1, \dots, \alpha_n)$ . A plumbing diagram for  $M$  is given by*



The values of  $g$ , and the  $t_i$ 's, are given by

$$(2) \quad t_i = \frac{\prod_{j \neq i} (\alpha_j)}{\text{lcm}_{j \neq i} (\alpha_j)}$$

$$(3) \quad g = \frac{1}{2} \left( 2 + \frac{(n-2) \prod_i \alpha_i}{\text{lcm}_i (\alpha_i)} - \sum_{i=1}^n t_i \right).$$

Then one calculates numbers  $p_1, \dots, p_n$  as

$$(4) \quad p_i = \frac{\text{lcm}_j (\alpha_j)}{\text{lcm}_{j \neq i} (\alpha_j)},$$

and find numbers  $q_1, \dots, q_n$  as the smallest possible solutions to the equations

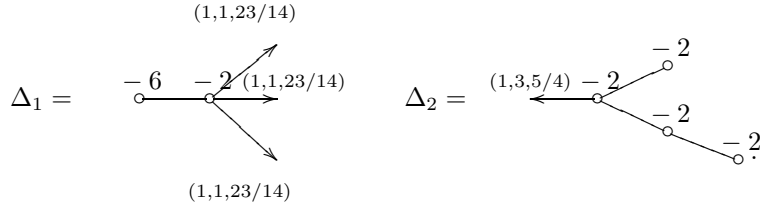
$$(5) \quad \frac{\text{lcm}_j (\alpha_j)}{\alpha_i} q_i \equiv -1 \pmod{p_i}.$$

Then the  $a_{ij}$ 's, are given as the continued fraction  $p_i/q_i = [a_{i1}, \dots, a_{ik_i}]$ . Finally  $b$  is given by

$$(6) \quad b = \frac{\prod_i \alpha_i + \text{lcm}_i (\alpha_i) \sum_i q_i \prod_{j \neq i} \alpha_j}{(\text{lcm}_i \alpha_i)^2}.$$

Before we use this theorem to make a plumbing diagram  $\Delta_i$  for the  $M_i$ , notice that to make the gluing we have to remove some solid tori from the  $M_i$ 's to make the gluing, so we need to record this data in  $\Delta_i$ . Some leaves in  $\Gamma_i$  have a pair of integers attached. These leaves corresponds to the tori in  $M$  we cut along when we created  $\Gamma_i$ . Since  $M_i$  is the universal abelian cover of any graph orbifold with  $\Gamma_i$  as its splice diagram, several singular fibers sits above the singular fiber corresponding to these leaves. It is a neighborhood of each of these singular fibers we have to remove. So if  $\alpha_j$  is a edge weight in  $\Gamma_i$  to a leaf with a pair attached, the the  $t_j$  singular fibers above the leaf, corresponds to all the strings with the weights  $-a_{j1}, \dots, -a_{jn_j}$ . So in the plumbing diagram for  $M_i$  we replace these string with an arrow, and add a triple which consist of the pair attached to the leaf and  $p_j/q_j = [a_{j1}, \dots, a_{jk_j}]$  to each of the arrows.

Using these theorems on our example we get the following plumbing diagrams



The second case is not as easy, the proof of Theorem 6.3 in [Ped10a] give a construction in this case, but it might not be a Seifert fibered manifold, an I have at the present no simple way to find a plumbing diagram for the building blocks in this case.

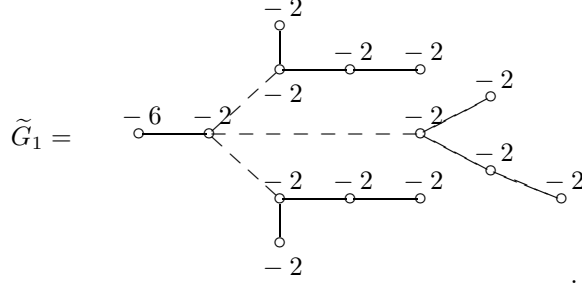
**3.2. Gluing the Building Blocks.** The only thing that remains to construct the universal abelian cover is to glue together the building blocks  $M_i$ , this will be done by using the plumbing diagrams  $\Delta_i$  to create a plumbing diagram  $\Delta$  for  $\widetilde{M}$ .

Start by taking two of the  $\Delta_i$ 's and create a plumbing diagram  $G_1$ , then we take an other of the  $\Delta_i$ 's and glue this to  $G_1$  to create  $G_2$ , we continue this process until all the  $\Delta_i$ 's has been used, and then  $\Delta = G_{N-1}$  where  $G_{N-1}$  is the last created plumbing diagram.

Now the order we glue the  $\Delta_i$ 's together in is important, this is why we added a triple at the arrows. We start by taking  $\Delta_i$  which have at least one arrow that has

a triple  $(N - 1, d_i, r_i)$ , where  $N - 1$  is the highest value for value for the first number in the triple. And that  $\Delta_j$  such at least one arrow that has a triple  $(N - 1, d_j, r_j)$ . By the method we constructed the  $\Delta_i$ 's there are exactly two satisfying this. Then we take  $d_i$  copies of  $\Delta_i$  and  $d_j$  copies of  $\Delta_j$ . We the create an intermediate  $\tilde{G}_1$  by for each of the copies of  $\Delta_i$  replace each arrow with the triple  $(N - 1, d_i, r_i)$  with a dashed line to a copy of  $\Delta_j$  replacing the arrow with the triple  $(N - 1, d_j, r_j)$ , such that a copy of  $\Delta_i$  is only connected to a copy of  $\Delta_j$  once. This will create a connected weighted graph  $\tilde{G}_1$ , with no arrows which has first number in the triple equal to  $N - 1$ .

Lets see how this is done in our example. We only have two  $\Delta_i$ 's, so we start by gluing  $\Delta_1$  to  $\Delta_2$ . The triples are  $(1, 1, 23/14)$  and  $(1, 3, 5/4)$  so we start by taking one copy  $\Delta_1$  and 3 copies of  $\Delta_2$ , replace each of the arrows in the copy of  $\Delta_1$  with a dashed line to one of the copies of  $\Delta_2$  replacing its arrow. We then get



The next step is to replace the dashed lines by a string to create  $G_1$ . First by symmetry all the strings are going to be equal, so we only have to calculate one string. To do this we use that there are two different ways to calculate the rational euler number of the Seifert fibered piece corresponding to a node in  $G_1$ , one using  $G_1$  and one given by the splice diagram by a formula derived at the end of the proof of Theorem 6.3 in [Ped10a].

Choose a node  $v$  of  $\tilde{G}_1$  which is attached to a dashed line, then the rational euler number is given by  $b + \sum_e q_e/p_e$ , where the sum is taking over all edges adjacent to  $v$  (including the dashed lines),  $(p_e, q_e)$  is the Seifert pair associated to the string and  $b$  is the weight at  $v$ . Now there are four types of different edges attached to  $v$  and we need to see how to get  $(p_e, q_e)$  from each type of the edge.

First there are the edge that starts a string that ends at a valence one vertex. Form these strings we get  $(p_e, q_e)$  from the continued fraction associated to the string i.e.  $p_e/q_e = [a_{e1}, \dots, a_{ek_e}]$ .

Second types of edges are on string that leads to other nodes (when one makes  $G_1$  these do not exist, but they can be there when we are going to make  $G_2$ ). We again gain the Seifert pair from the continued fraction, this time from the string between  $v$  and the other node.

The third type of edges are the arrows, their we gain  $(p_e, q_e)$  from the triple  $(n_e, d_e, r_e)$  attached to the arrow as  $p_e/q_e = r_e$ .

The last type of edges are the dashed lines, here we do not find the Seifert pair since we are trying to make an equation to do just that. But notice that all the dashed lines have the same Seifert pair  $(p, q)$ , hence we get the following equation

$$(7) \quad e_v = d' \frac{q}{p} + b + \sum_e \frac{q_e}{p_e}.$$

Where the sum is taken over all edges at  $v$  except the dashed lines, and  $d'$  is the number of dashed lines at  $v$ . Notice that if  $v$  is a node sitting above the  $\Delta_i$  piece, then  $d' = d_j$ .

Returning to our example, if we use the left most node as  $v$ , the equation becomes

$$e_v = 3\frac{p}{q} - 2 + \frac{1}{6} = 3\frac{p}{q} - \frac{11}{6}.$$

Now  $G_1 \cup (\bigcup_{l \neq i, j} \Delta_l)$  is a plumbing diagram for any manifold with splice diagram  $\Gamma_{e_{N-2}}$ , hence it is  $\Gamma_{e_{N-2}}$  we need to use when we make the other calculation of  $e_v$ .

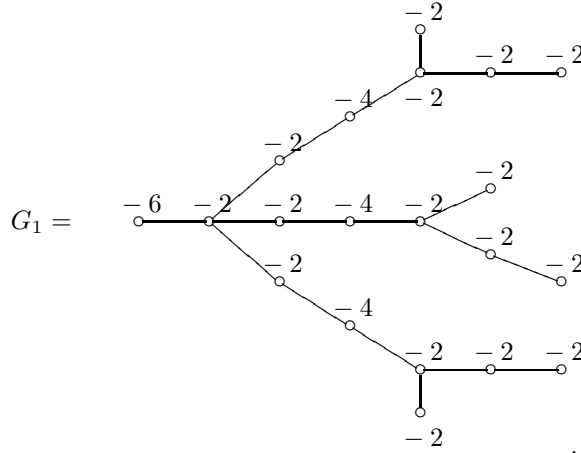
From the end of the proof of Theorem 6.3 in [Ped10a] one gets that if  $v$  sits above the  $\Delta_i$  piece

$$(8) \quad e_v = \frac{\lambda^2}{d_i} \tilde{e}_v / d.$$

Where  $\lambda = \prod m_j / \text{lcm}(m_1/\bar{d}_1, \dots, m_k/\bar{d}_k)$  where the  $m_j$ 's are the edge weight adjacent to the node corresponding to  $v$  in  $\Gamma_{e_{N-2}}$  and the  $\bar{d}_j$ 's are the ideal generator associated to the edges.  $\tilde{e}_v$  is the rational euler number of the the node corresponding to  $v$  in any graph orbifold  $\tilde{M}$  whit  $\Gamma(\tilde{M}) = \Gamma_{e_{N-2}}$  and  $d = |H_1^{orb}(\tilde{M})|$ , now neither of these numbers are determined by  $\Gamma_{e_{N-2}}$ , but proposition 2.3 gives a formula for  $\tilde{e}_v/d$  only using  $\Gamma_{e_{N-2}}$ .

In our example  $\Gamma_{e_{N-2}} = \Gamma$ , so using this we find that  $\lambda = 3$  and  $\tilde{e}_v/d = -5/378$ , so  $e_v = -5/42$ .

Now one find  $p/q$  by combining the equations (7) and (8), which in our example gives  $p/q = 4/7$ . Remember this is the continued fraction associated to the string replacing the dashed lines when seen from  $v$ , if we have used the node in the other end of the line we would have found  $p/q'$  where  $qq' \equiv -1 \pmod{p}$ . Replacing all the dashed lines with the strings corresponding to the continued fractions on gets  $G_1$  from  $\tilde{G}_1$ . This becomes the following plumbing diagram in our example

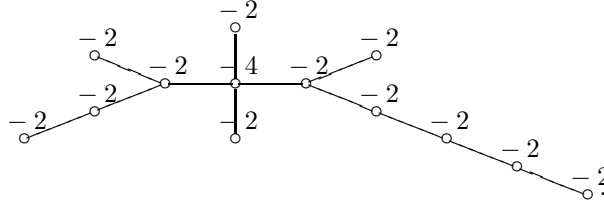


If  $\Gamma_{e_{N-2}} \neq \Gamma$  then one adds  $G_1$  to the collection of  $\Delta_i$ 's not used, and repeat the process by taking the to plumbing diagrams of this collection which have arrows which triple start with  $N - 2$ . One continues this process until all the  $\Delta_i$ 's have been used, and the final  $G_{N-1}$  is then a plumbing diagram for the universal abelian cover  $\tilde{M}$  of  $M$ .

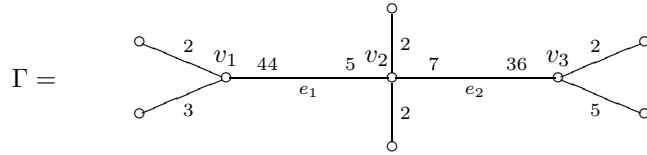
We will finish by giving a couple of other examples of the use of the algorithm, but will leave the details of the calculation to the readers.



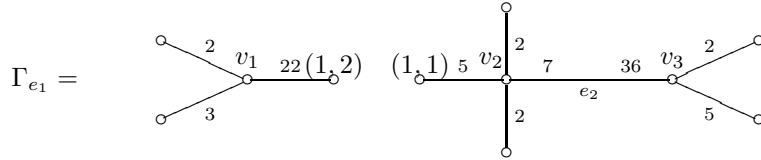
**Example 3.3.** Let  $M$  be the manifold defined by the following plumbing diagram:



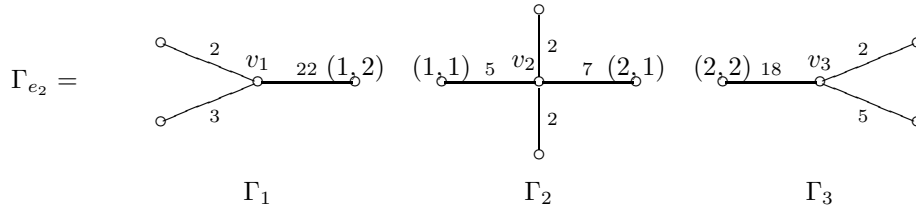
Its splice diagram is



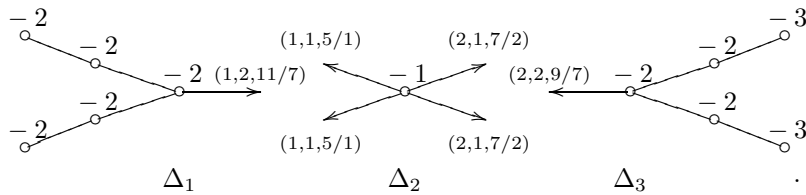
If we first cut along the edge called  $e_1$  we get



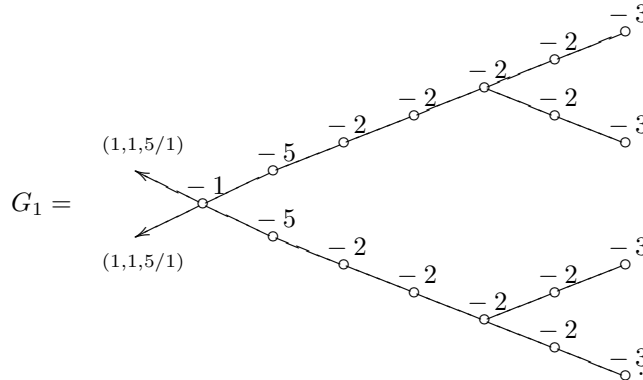
And cutting along  $e_2$  the gives us



Next one determines the 3 building blocks and get the following plumbing diagrams

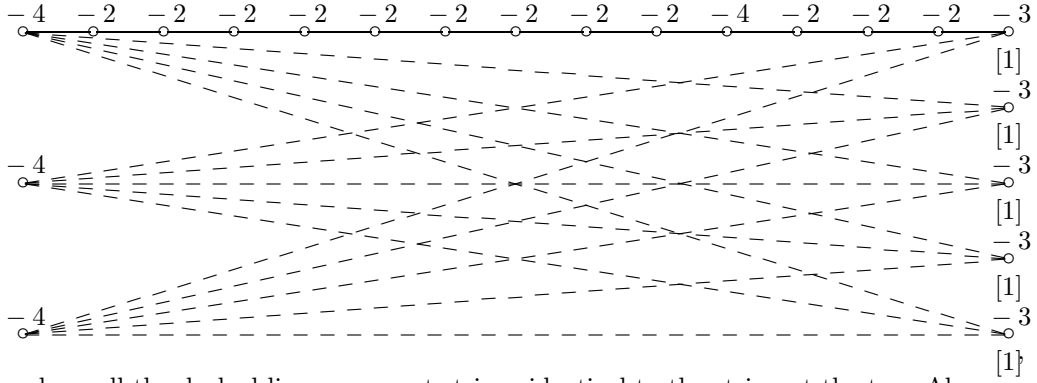


One first glue the one copy of  $\Delta_2$  to two copies of  $\Delta_3$  and get after calculating the strings





so to create the plumbing diagram  $G$  of the universal abelian cover, we glue 3 copies of  $\Delta_1$  to 5 copies of  $\Delta_2$  calculate the string and get



where all the dashed lines represent strings identical to the string at the top. Also remember that the graph is not a planar graph so any intersection between the strings represented by the dashed lines do not represent intersections in  $G$ , just crossings arising by a planar projection of  $G$  which is what we see here.

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