# On the conjugacy of pure imaginary elements of quaternion algebras and Cayley algebras

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**Abstract**

In the algebras $\boldsymbol{H}$, $\boldsymbol{H}'$, $\boldsymbol{H}^C$, $\mathfrak{C}$, $\mathfrak{C}'$ and $\mathfrak{C}^C$, we show some results on the conjugacy of two pure imaginary non-zero elements with same norm.

## Introduction

It is known that any automorphism $\alpha$ of the field $\boldsymbol{H}$ of quaternions is an inner automorphism and any two pure imaginary quaternions $a$, $b$ with the same norm are conjugate, that is, there exists a pure imaginary non-zero element $p \in \boldsymbol{H}$ such that $b = pap^{-1}$ (Theorem 6.(1)). In the case of the split quaternion algebra $\boldsymbol{H}'$ (resp. the complex quaternion algebra $\boldsymbol{H}^C$), any two pure imaginary non-zero elements $a, b$ with the same norm are conjugate, that is, there exists an invertible element $p \in \boldsymbol{H}'$ (resp. $\boldsymbol{H}^C$) such that $b = pap^{-1}$ (Theorem 6.(2)). In the case of the division Cayley algebra $\mathfrak{C}$, any two pure imaginary elements $a, b$ with the same norm are conjugate, that is, there exists a pure imaginary non-zero element $p \in \mathfrak{C}$ such that $b = pap^{-1}$ (Theorem 3.(1)). In the case of the split Cayley algebra $\mathfrak{C}'$ (resp. the complex Cayley algebra $\mathfrak{C}^C$), we need twice conjugate operations, that is, for any two pure imaginary non-zero elements $a, b$ with the same norm, there exist invertible pure imaginary elements $p, q \in \mathfrak{C}'$ (resp. $\mathfrak{C}^C$) such that $b = q(pap^{-1})q^{-1}$ (Theorem 3.(2)). We obtain the above results in the constructive manner, by using concrete elements.

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## 0. Preliminaries

Let $\boldsymbol{H}$ be the field of quaternions with the canonical $\boldsymbol{R}$-basis $\{1, e_1, e_2, e_3\}$ with the usual multiplication defined by

$$ {e_1}^2 = {e_2}^2 = {e_3}^2 = -1, $$
$$ e_1e_2 = e_3 = -e_2e_1, \quad e_2e_3 = e_1 = -e_3e_2, \quad e_3e_1 = e_2 = -e_1e_3, $$

and let $\boldsymbol{H}^C$ be the complexfication algebra of $\boldsymbol{H}$: $\boldsymbol{H}^C = \boldsymbol{H} \oplus i\boldsymbol{H}$. The algebra $\boldsymbol{H}'$ of split quaternions is defined as follows. $\boldsymbol{H}'$ is the algebra with $\boldsymbol{R}$-basis

$\{1,\, e_1{}',\, e_2,\, e_3{}'\}$ with the multiplication defined by

$$e_1{}'^2 = e_3{}'^2 = 1, e_2{}^2 = -1,$$
$$e_1{}'e_2 = e_3{}' = -e_2 e_1{}', \quad e_2 e_3{}' = e_1{}' = -e_3{}'e_2, \quad e_3{}'e_1{}' = -e_2 = -e_1{}'e_3{}'.$$

Next, let $\mathfrak{C} = \boldsymbol{H} \oplus \boldsymbol{H} e_4$ (resp. $\mathfrak{C}' = \boldsymbol{H}' \oplus \boldsymbol{H}' e_4$) be the division Cayley algebra (resp. the split Cayley algebra) over $\boldsymbol{R}$ with the multiplication

$$(m_1 + n_1 e_4)(m_2 + n_2 e_4) = (m_1 m_2 - \overline{n_2} n_1) + (n_1 \overline{m_2} + n_2 m_1) e_4,$$

for $m_1 + n_1 e_4, m_2 + n_2 e_4 \in \mathfrak{C}$ (resp. $\mathfrak{C}'$), where $\overline{m}$ is the conjugate element of $m \in \boldsymbol{H}$ (resp. $\boldsymbol{H}'$), and let $e_5 = e_1 e_4, e_6 = -e_2 e_4, e_7 = e_3 e_4$ (resp. $e_5{}' = e_1{}'e_4, e_6 = -e_2 e_4, e_7{}' = e_3{}'e_4$). The complex Cayley algebra $\mathfrak{C}^C$ is defined as the complexification of $\mathfrak{C}$: $\mathfrak{C}^C = \mathfrak{C} \oplus i\mathfrak{C}$. In $\mathfrak{C}$, $\mathfrak{C}'$ and $\mathfrak{C}^C$, the conjugation is defined by $\overline{m + n e_4} = \overline{m} - n e_4$. In the algebras $K = \boldsymbol{H}$, $\boldsymbol{H}'$, $\boldsymbol{H}^C$, $\mathfrak{C}$ , $\mathfrak{C}'$ and $\mathfrak{C}^C$ above, the inner product $(a, b)$ and the norm $\boldsymbol{N}(a)$ by

$$(a, b) = \frac{1}{2}(a\overline{b} + b\overline{a}), \quad \boldsymbol{N}(a) = (a, a) = a\overline{a}.$$

Note that if $a \in K$ satisfies $\boldsymbol{N}(a) \neq 0$, then $a$ is invertible and the inverse element $a^{-1}$ of $a$ is given by $a^{-1} = \overline{a}/\boldsymbol{N}(a)$. Finally, we use the following notation

$$K_0 = \{a \in K \mid \overline{a} = -a\}, \qquad K_0{}^* = \{a \in K_0 \mid a \neq 0\},$$
$$K^\times = \{a \in K \mid \boldsymbol{N}(a) \neq 0\}, \quad K_0{}^\times = \{a \in K_0 \mid \boldsymbol{N}(a) \neq 0\}.$$

For $a \in K_0$, $\boldsymbol{N}(a)$ is nothing but $-a^2$.

## 1. Cases of Cayley algebras

**Lemma 1.** (1) *For any $a \in (\mathfrak{C})_0$, there exists $p \in (\mathfrak{C})_0{}^*$ such that $pap^{-1} = -a$.*
(2) *Let $K = \mathfrak{C}', \mathfrak{C}^C$. For any $a \in K_0$, there exists $p \in K_0{}^\times$ such that $pap^{-1} = -a$.*

**Proof.** We may assume that $a \neq 0$.
(1) Case $a \in (\mathfrak{C})_0$. Express $a = a_1 e_1 + \cdots + a_7 e_7, a_k \in \boldsymbol{R}$. Then, at least one element $p$ of the following elements

$$a_2 e_1 - a_1 e_2, \quad a_3 e_2 - a_2 e_3, \quad a_5 e_4 - a_4 e_5, \quad a_7 e_6 - a_6 e_7$$

satisfies $pap^{-1} = -a$. Indeed, each element $p$ above satisfies $pa = -ap$, and the norm $\boldsymbol{N}(p)$ are

$$a_2{}^2 + a_1{}^2, \quad a_3{}^2 + a_2{}^2, \quad a_5{}^2 + a_4{}^2, \quad a_7{}^2 + a_6{}^2.$$

If $\boldsymbol{N}(p) = 0$ for all $p$, then we have $a_1 = \cdots = a_7 = 0$, that is, $a = 0$, which contradicts the assumption $a \neq 0$.

(2)-(i) Case $a \in (\mathfrak{C}')_0$. Express $a = a_1e_1{}' + a_2e_2 + a_3e_3{}' + a_4e_4 + a_5e_5{}' + a_6e_6 + a_7e_7{}', a_k \in \boldsymbol{R}$. Then, at least one element $p$ of the following elements

$$a_4e_2 - a_2e_4, \quad a_6e_4 - a_4e_6, \quad a_3e_1{}' - a_1e_3{}', \quad a_7e_5{}' - a_5e_7{}'$$

satisfies $pap^{-1} = -a$. Indeed, each element $p$ above satisfies $pa = -ap$, and the norm $\boldsymbol{N}(p)$ are

$${a_4}^2 + {a_2}^2, \quad {a_6}^2 + {a_4}^2, \quad -{a_3}^2 - {a_1}^2, \quad -{a_7}^2 - {a_5}^2.$$

If $\boldsymbol{N}(p) = 0$ for all $p$, then we have $a_1 = \cdots = a_7 = 0$, that is, $a = 0$, which contradicts the assumption $a \neq 0$.
(2)-(ii) Case $a \in (\mathfrak{C}^C)_0$. Express $a = a_1e_1 + \cdots + a_7e_7, a_k \in C$. Then, at least one element $p$ of the following elements

$$a_2e_1 - a_1e_2, \quad a_3e_2 - a_2e_3, \quad a_1e_3 - a_3e_1,$$
$$a_4e_3 - a_3e_4, \quad a_5e_4 - a_4e_5, \quad a_6e_5 - a_5e_6, \quad a_7e_6 - a_6e_7$$

satisfies $pap^{-1} = -a$. Indeed, each element $p$ above satisfies $pa = -ap$, and the norm $\boldsymbol{N}(p)$ are

$${a_2}^2 + {a_1}^2, \quad {a_3}^2 + {a_2}^2, \quad {a_1}^2 + {a_3}^2,$$
$${a_4}^2 + {a_3}^2, \quad {a_5}^2 + {a_4}^2, \quad {a_6}^2 + {a_5}^2, \quad {a_7}^2 + {a_6}^2.$$

If $\boldsymbol{N}(p) = 0$ for all $p$, then we have $a_1 = \cdots = a_7 = 0$, that is, $a = 0$, which contradicts the assumption $a \neq 0$.

**Lemma 2.** *Let $K = \mathfrak{C}', \mathfrak{C}^C$. For any $a, b \in {K_0}^*$ such that $(a, b) = 0$ and $\boldsymbol{N}(a) + \boldsymbol{N}(b) = 0$, there exists $p \in {K_0}^\times$ (which depends on $a$ and $b$) such that $\boldsymbol{N}(pap^{-1} + b) \neq 0$.*

**Proof.** (1) Case $K = \mathfrak{C}^C$. Express $a = a_1e_1 + \cdots + a_7e_7, b = b_1e_1 + \cdots + b_7e_7, a_k, b_k \in C$. (1)-(i) When $a_kb_k \neq 0$ for some $k$. Let $p = e_k$. It holds $\boldsymbol{N}(p) = 1 \neq 0$ and

$$\begin{aligned} &\boldsymbol{N}(pap^{-1} + b) = \boldsymbol{N}(a) + \boldsymbol{N}(b) + 2(pap^{-1}, b) \\ &= 0 - 2(a_1b_1 + \cdots + a_{k-1}b_{k-1} - a_kb_k + a_{k+1}b_{k+1} + \cdots + \boldsymbol{a}_7b_7) \\ &= 4a_kb_k + 2(a, b) = 4a_kb_k \neq 0. \end{aligned}$$

(1)-(ii) When $a_kb_k = 0$ for all $k = 1, 2, \cdots, 7$. There exist $k, l$ such that $k \neq l$, $a_k \neq 0, b_l \neq 0$. In this case, note that $a_l = b_k = 0$. Now, let $p = e_k + e_l$. Then, $\boldsymbol{N}(p) = 2$ and $\boldsymbol{N}(pap^{-1} + b) = \boldsymbol{N}(a) + \boldsymbol{N}(b) + 2(a_kb_l - (a, b)) = 0 + 2(a_kb_l - 0) = 2a_kb_l \neq 0$.

(2) Case $K = \mathfrak{C}'$. Express $a = a_1e_1{}' + a_2e_2 + a_3e_3{}' + a_4e_4 + a_5e_5{}' + a_6e_6 + a_7e_7{}', b = b_1e_1{}' + b_2e_2 + b_3e_3{}' + b_4e_4 + b_5e_5{}' + b_6e_6 + b_7e_7{}', a_k, b_k \in \boldsymbol{R}$.

(2)-(i) When $a_kb_k \neq 0$ for some $k$. If $k \in \{2, 4, 6\}$ (resp. $k \in \{1, 3, 5, 7\}$), then let $p = e_k$ (resp. $e_k{}'$). Then $\boldsymbol{N}(p) = 1$ (resp. $-1$) $\neq 0$ and $\boldsymbol{N}(pap^{-1} + b) = 4a_kb_k$ (resp. $-4a_kb_k$) $\neq 0$ in the same way as 1-(i).

(2)-(ii) When $a_k b_k = 0$ for all $k = 1, 2, \cdots, 7$. There exist $k, l \in \{2, 4, 6\}$ such that $k \neq l$, $a_k \neq 0$, $b_l \neq 0$. In this case, note that $a_l = b_k = 0$. Now, let $p = e_k + e_l$. Then, $\boldsymbol{N}(p) = 2$ and $\boldsymbol{N}(pap^{-1} + b) = \boldsymbol{N}(a) + \boldsymbol{N}(b) + 2(a_k b_l - (a, b)) = 0 + 2(a_k b_l - 0) = 2a_k b_l \neq 0$.

**Theorem 3.** (1) *For any $a, b \in (\mathfrak{C})_0{}^*$ such that $\boldsymbol{N}(a) = \boldsymbol{N}(b)$, there exists $p \in (\mathfrak{C})_0{}^*$ such that $pap^{-1} = b$.*

(2) *Let $K = \mathfrak{C}', \mathfrak{C}^C$. For any $a, b \in K_0{}^*$ such that $\boldsymbol{N}(a) = \boldsymbol{N}(b)$, there exist $p, q \in K_0{}^\times$ such that $q(pap^{-1})q^{-1} = b$.*

**Proof.** (1)-(i) Case $\boldsymbol{N}(a + b) \neq 0$. Let $p = a + b$, then we have

$$\begin{aligned} pap^{-1} &= (a + b)a(a + b)^{-1} = (a^2 + ba)(a + b)^{-1} \\ &= (b^2 + ba)(a + b)^{-1} = b(b + a)(a + b)^{-1} = b. \end{aligned}$$

(1)-(ii)Case $\boldsymbol{N}(a + b) = 0$. This implies $b = -a$. Then, there exists $p \in (\mathfrak{C})_0{}^*$ such that $pap^{-1} = -a = b$, by Lemma 1(1).

(2)-(i) Case $\boldsymbol{N}(a + b) \neq 0$. Let $p = a + b$. Then we have $pap^{-1} = b$ in the same way as (1)-(i).

(2)-(ii) Case $\boldsymbol{N}(a - b) \neq 0$. For given $b$, there exists $q \in K_0{}^\times$ such that $qbq^{-1} = -b$ (Lemma 1.(2)). Now, let $p = a - b$, then we have

$$\begin{aligned} q(pap^{-1})q^{-1} &= q((a - b)a(a - b)^{-1})q^{-1} = q((a^2 - ba)(a - b)^{-1})q^{-1} \\ &= q((b^2 - ba)(a - b)^{-1})q^{-1} = q(b(b - a)(a - b)^{-1})q^{-1} \\ &= q(-b)q^{-1} = b. \end{aligned}$$

(2)-(iii) Case $\boldsymbol{N}(a + b) = \boldsymbol{N}(a - b) = 0$. This implies $(a, b) = 0$ and $\boldsymbol{N}(a) + \boldsymbol{N}(b) = 0$. Then we get $\boldsymbol{N}(a) = \boldsymbol{N}(b) = 0$, that is, $a^2 = b^2 = 0$. For $a, b \in K_0{}^*$, we can choose $p \in K_0{}^\times$ such that $\boldsymbol{N}(pap^{-1} + b) \neq 0$, by Lemma 2. Let $q = pap^{-1} + b$. Then, noting that $(pap^{-1})^2 = -\boldsymbol{N}(pap^{-1}) = 0$, we have

$$\begin{aligned} q(pap^{-1})q^{-1} &= (pap^{-1} + b)(pap^{-1})(pap^{-1} + b)^{-1} \\ &= ((pap^{-1})^2 + b(pap^{-1}))(pap^{-1} + b)^{-1} \\ &= (b^2 + b(pap^{-1}))(pap^{-1} + b)^{-1} \\ &= (b(b + pap^{-1}))(pap^{-1} + b)^{-1} = b. \end{aligned}$$

**Remark**. In the split Cayley algebra $\mathfrak{C}'$ (resp. the complex Cayley algebra $\mathfrak{C}^C$), there exist two elements $a, b \in (\mathfrak{C}')_0{}^*$ (resp. $(\mathfrak{C}^C)_0{}^*$) with the same norm $\boldsymbol{N}(a) = \boldsymbol{N}(b)$ such that $b$ can not be expressed by $pap^{-1}$ for any $p \in \mathfrak{C}'$ (resp. $\mathfrak{C}^C$).

For example, in $\mathfrak{C}'$, let $a = 4e_1{}' + 5e_2 + 3e_3{}' - 5e_4 + 4e_5{}' + 3e_7{}'$, $b = 3e_2 + 4e_6 + 5e_7{}'$. Then, $\boldsymbol{N}(a) = \boldsymbol{N}(b) = 0$. If $p = x_0 + x_1 e_1{}' + x_2 e_2 + x_3 e_3{}' + x_4 e_4 +$

$x_5{e_5}' + x_6e_6 + x_7{e_7}' \in \mathfrak{C}'$ satisfies $pa = bp$, then

$$\begin{cases} \quad\quad +4x_1 -2x_2 +3x_3 +5x_4 +4x_5 +4x_6 -2x_7 = 0 \\ 4x_0 \quad\quad +3x_2 -8x_3 +4x_4 +5x_5 +8x_6 -4x_7 = 0 \\ 2x_0 +3x_1 \quad\quad -4x_3 -4x_4 -8x_5 -5x_6 +4x_7 = 0 \\ 3x_0 +8x_1 -4x_2 \quad\quad +8x_4 +4x_5 -4x_6 +5x_7 = 0 \\ -5x_0 +4x_1 +4x_2 +8x_3 \quad\quad -4x_5 -8x_6 -3x_7 = 0 \\ 4x_0 -5x_1 -8x_2 -4x_3 -4x_4 \quad\quad +3x_6 +8x_7 = 0 \\ -4x_0 +8x_1 +5x_2 -4x_3 +8x_4 +3x_5 \quad\quad -4x_7 = 0 \\ -2x_0 +4x_1 +4x_2 -5x_3 -3x_4 -8x_5 -4x_6 \quad\quad = 0 \end{cases}.$$

Hence, $p$ must be

$$p = s(104{e_1}' + 40e_2 + 3{e_3}' - 165e_4 + 132{e_5}' + 24{e_7}')$$
$$+t(-46{e_1}' - 8e_2 + 3{e_3}' + 75e_4 - 60{e_5}' + 6e_6)$$

with arbitrary parameters, $s, t \in \boldsymbol{R}$. However, for any $s, t \in \boldsymbol{R}$, it holds $\boldsymbol{N}(p) = 0$. Therefore, there does not exist $p \in \mathfrak{C}'$ such that $pap^{-1} = b$.

Next, in $\mathfrak{C}^C$, let $a = 4ie_1 + 5e_2 + 3ie_3 - 5e_4 + 4ie_5 + 3ie_7$, $b = 3e_2 + 4e_6 + 5ie_7$. Then $\boldsymbol{N}(a) = \boldsymbol{N}(b) = 0$. If $p = x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7 \in \mathfrak{C}^C$ satisfies $pa = bp$, then

$$\begin{cases} \quad\quad -4ix_1 -2x_2 -3ix_3 +5x_4 -4ix_5 +4x_6 +2ix_7 = 0 \\ 4ix_0 \quad\quad +3ix_2 -8x_3 +4ix_4 +5x_5 +8ix_6 -4x_7 = 0 \\ 2x_0 -3ix_1 \quad\quad +4ix_3 -4x_4 +8ix_5 -5x_6 -4ix_7 = 0 \\ 3ix_0 +8x_1 -4ix_2 \quad\quad -8ix_4 +4x_5 -4ix_6 +5x_7 = 0 \\ -5x_0 -4ix_1 +4x_2 -8ix_3 \quad\quad +4ix_5 -8x_6 +3ix_7 = 0 \\ 4ix_0 -5x_1 -8ix_2 -4x_3 -4ix_4 \quad\quad +3ix_6 +8x_7 = 0 \\ -4x_0 -8ix_1 +5x_2 +4ix_3 +8x_4 -3ix_5 \quad\quad +4ix_7 = 0 \\ -2ix_0 +4x_1 +4ix_2 -5x_3 -3ix_4 -8x_5 -4ix_6 \quad\quad = 0 \end{cases}.$$

Hence, $p$ must be

$$p = s(104e_1 - 40ie_2 + 3e_3 + 165ie_4 + 132e_5 + 24e_7)$$
$$+t(-46ie_1 - 8e_2 + 3ie_3 + 75e_4 - 60ie_5 + 6e_6)$$

with arbitrary parameters, $s, t \in C$. However, for any $s, t \in C$, it holds $\boldsymbol{N}(p) = 0$. Therefore, there does not exist $p \in \mathfrak{C}'$ such that $pap^{-1} = b$.

## 2. Cases of quaternion algebras

**Lemma 4.** (1) *For any $a \in (\boldsymbol{H})_0$, there exists $p \in (\boldsymbol{H})_0{}^*$ such that $pap^{-1} = -a$.*
(2) *Let $K = \boldsymbol{H}'$, $\boldsymbol{H}^C$. For any $a \in K_0$, there exists $p \in {K_0}^\times$ such that $pap^{-1} = -a$.*

**Proof.** Since Cayley algebras $\mathfrak{C}$, $\mathfrak{C}'$ and $\mathfrak{C}^C$ naturally contain quaternion algebras $\boldsymbol{H}$, $\boldsymbol{H}'$ and $\boldsymbol{H}^C$ respetively, this lemma has already been shown by Lemma 1.

**Lemma 5.** *Let* $K = \boldsymbol{H}', \boldsymbol{H}^C$. *For any* $a, b \in {K_0}^*$ *such that* $(a, b) = 0$ *and* $\boldsymbol{N}(a) + \boldsymbol{N}(b) = 0$, *there exists* $p \in {K_0}^{\times}$ (*which depends on* $a$ *and* $b$) *such that* $\boldsymbol{N}(pap^{-1} + b) \neq 0$.

**Proof.** Since Cayley algebras $\mathfrak{C}'$ and $\mathfrak{C}^C$ naturally contain quaternion algebras $\boldsymbol{H}'$ and $\boldsymbol{H}^C$ respetively, this lemma has already been shown by Lemma 2.

**Theorem 6.** (1) *For any* $a, b \in {(\boldsymbol{H})_0}^*$ *such that* $\boldsymbol{N}(a) = \boldsymbol{N}(b)$, *there exists* $p \in {(\boldsymbol{H})_0}^*$ *such that* $pap^{-1} = b$.

(2) *Let* $K = \boldsymbol{H}', \boldsymbol{H}^C$. *For any* $a, b \in {K_0}^*$ *such that* $\boldsymbol{N}(a) = \boldsymbol{N}(b)$, *there exists* $p \in K^{\times}$ *such that* $pap^{-1} = b$.

**Proof.** Since Cayley algebras $\mathfrak{C}$, $\mathfrak{C}'$ and $\mathfrak{C}^C$ naturally contain quaternion algebras $\boldsymbol{H}$, $\boldsymbol{H}'$ and $\boldsymbol{H}^C$ respetively, this is the particular case of Theorem 3. In the case of (2), since the associativity is valid in $\boldsymbol{H}'$ and $\boldsymbol{H}^C$, Theorem 3.(2) implies (2).

# References


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