A SPECTRAL SEQUENCE FOR THE HOCHSCHILD COHOMOLOGY OF A COCONNECTIVE DGA

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Abstract. A spectral sequence for the computation of the Hochschild cohomology of a coconnective dga over a field is presented. This spectral sequence has a similar flavour to the spectral sequence presented in [4] for the computation of the loop homology of a closed orientable manifold.

1. Introduction

Given an differential graded algebra A over a field k, the Hochschild cohomology of A with coefficients in the differential graded A-bimodule A is

$$HH^*(A) = \mathrm{Ext}^*_{A\otimes_k A^{\mathrm{op}}}(A, A)$$

Before describing the spectral sequence let us give a few definitions. A differential graded algebra (a *dga*) A over a field k will be called *coconnective* if $A^n = 0$ for $n < 0$ and $H^0(A) = k$. The dga A is *augmented* if it has a morphism of k-dgas $\mathfrak{a} : A \to k$. It is of *finite type* if $H^n(A)$ is a finite dimensional k-vector space for every n and it is *bounded* if $H^n(A) \neq 0$ only for a finite number of values of n. For an augmented k-dga A there is a well known map of graded algebras $\chi : HH^*(A) \to \mathrm{Ext}^*_A(k, k)$ which we shall call the *shearing map*. We will give several equivalent definitions of the shearing map in Section 7. The dga $A \otimes_k A^{\mathrm{op}}$ will be denoted by A^e. Note there is also a well known isomorphism $\mathrm{Ext}^*_A(k, k) \cong \mathrm{Ext}^*_{A^e}(A, k)$ which will also be described in Section 7.

1.1. **Theorem.** *Let A be a coconnective augmented dga over a field k. Then there exists a conditionally convergent multiplicative spectral sequence*

$$E^2_{p,q} = Ext^{-q}_{A^e}(A, H^{-p}(A)) \implies HH^{-p-q}(A)$$

Under the isomorphism $Ext^*_A(k, k) \cong Ext^*_{A^e}(A, k)$ the infinite cycles in $E^2_{0,*}$ can be identified with the image of the shearing map $\chi : HH^*(A) \to Ext^*_A(k, k)$. When A is bounded the spectral sequence has strong convergence. When A is of finite type then this spectral sequence takes the form

$$E^2_{p,q} = H^{-p}(A) \otimes_k Ext^{-q}_A(k, k) \implies HH^{-p-q}(A)$$

with the obvious multiplicative structure on $H^{-p}(A) \otimes_k Ext^{-q}_A(k, k)$.

Several remarks are in order. First, a multiplicative spectral sequence means there is a multiplication defined on each E^r-term for which the differential d^r is a derivation, the multiplication on E^{r+1} is the one induced from E^r in the obvious way and the resulting multiplication on E^∞ agrees with the multiplication on the associated graded object of $HH^*(A)$. Note that we use the convergence conditions for spectral sequences as defined by Boardman in [2].

Second, we did not specify the multiplicative structure on the E^2-term of this spectral sequence in the case where A is not of finite type. Roughly speaking, one can consider A

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as a coalgebra with respect to the derived tensor product $\otimes^{\mathbf{L}}_A$. On the other side we have an appropriate pairing $H^n(A) \otimes^{\mathbf{L}}_A H^m(A) \to H^{n+m}(A)$ induced by the multiplication on A. Together these yield the multiplicative structure on the E^2-term. The precise description of this multiplication is given in Section 6. When A is of finite type both multiplicative structures on the E^2-term agree under the appropriate isomorphism, this is done in Lemma 8.1.

Last, the grading of this spectral sequence is homological. Hence the differential on the E^r-term is $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$, and the spectral sequence lies in the second quadrant. This has the unfortunate consequence of yielding minus signs on the E^2-term description. The choice of homological grading was motivated by topological examples discussed below in 1.3.

The main consequence of this spectral sequence is the following result.

1.2. **Theorem.** *Let k be a field of characteristic $p > 0$. Let A be a coconnective augmented dga over k such that $H^*(A)$ is finite dimensional and $Ext^*_A(k,k)$ is finitely generated over a central Noetherian sub-algebra. Then $Ext^*_A(k,k)$ is finitely generated over the image of the shearing map and $HH^*(A)$ is Noetherian.*

This theorem is used in [1] to show a structural property of the derived category of A, where A is the dga of singular cochains (with coefficients in k) on a space whose fundamental group is a finite p-group.

1.3. **Relation to other work.** As noted in the abstract, the spectral sequence presented here bears some resemblance the spectral sequence for string homology of Cohen, Jones and Yan presented in [4] which we now describe. Let M be a closed orientable manifold, let k be a commutative ring and let A be the dga of singular cochains with coefficients in k. The E^2-term of the spectral sequence of [4] is $E^2_{p,q} \cong H^{-p}(M, H_q(\Omega M))$, where ΩM is the based loop space of M. Assuming also that M is simply connected then this E^2-term is isomorphic to $H^{-p}(A) \otimes_k \mathrm{Ext}^{-q}_A(k,k)$.

The spectral sequence of [4] converges to the homology of the free (unbased) loops on M, properly desuspended, which they call the *loop homology*. By results of Cohen and Jones [3] this loop homology is isomorphic to the Hochschild cohomology of A, this is an isomorphism of graded algebras when the loop homology is given the Chas-Sullivan product. Thus, when M is simply connected, both the spectral sequence given here and that of [4] have the same E^2 and E^∞ terms and both are multiplicative. One wonders if these two spectral sequences are the same when M is simply connected. We will not deal with this question.

In [11], Felix, Thomas and Vigué-Poirrier consider a map $I : HH^*(A) \to \mathrm{Ext}^*_{A^e}(A,k)$ where A is the dga of singular cochains with coefficients in k on a simply connected closed oriented manifold. They give a model for I, and using this model get several results concerning the kernel and image of I. In Lemma 7.4 it is shown that I is equal to the shearing map defined here, and using the spectral sequence presented here we recover one of their results.

1.4. **On the choice of setting and method of proof.** In [5] and [6] Dugger gives a systematic treatment of the construction of multiplicative spectral sequences for topological spaces and for spectra. To mimic Dugger's work, we have have found it easier to use Quillen model category machinery. This also highlights the only significant difference between the construction here and Dugger's treatment - in this paper we are forced to use the bar construction since we cannot assume that our filtration consists of cofibrant objects.

Roughly speaking, one gets a multiplicative structure on maps from a comonoid to a monoid in any monoidal category. In [6] Dugger shows how an appropriate filtration of the comonoid yields a multiplicative spectral sequence. Here we filter the monoid as Dugger does in [5]. Viewed in this way, the construction of the spectral sequence is a simple translation of classical constructions from topology.

1.5. **Organization of this paper.** We start by presenting some consequences of the spectral sequence in Section 2. In Sections 3 and 4 we set the necessary model category structure and the differential graded tools we will use. In Section 5 the spectral sequence is constructed. Section 6 establishes the multiplicative properties of the spectral sequence. Section 7 gives various descriptions of the shearing map. Finally, in Section 8 we identify the E^2-term of the spectral sequence.

The proof of Theorem 1.1 is spread throughout this paper. Lemma 5.3 shows the existence of the spectral sequence and its convergence properties. Proposition 6.15 gives the multiplicative property. Lemma 8.1 identifies the E^2-term of the spectral sequence when A is of finite type. Finally, Lemma 8.2 proves the statement regarding the image of the shearing map.

1.6. **Notation and terminology.** A chain complex X is described by a pair $(X^\natural, d^X)$ where $X^\natural$ is the underlying graded abelian group and d^X is the differential. Note that we adhere to the convention that subscript grading is homological degree while superscript grading is cohomological. Thus $X_n = X^{-n}$ and the differential lowers the homological degree (or raises the cohomological degree) $d : X_n \to X_{n-1}$. The tensor product of two chain complexes over k will be denoted simply by $\otimes$.

Throughout k is a field and A is a coconnective augmented dga over k whose augmentation map $A \to k$ is denoted by $\mathfrak{a}$. The opposite dga is denoted by A^{op} and A^e is the dga $A \otimes A^{\mathrm{op}}$. The derived category of differential graded left A-modules will be denoted by $\mathbf{D}(A)$. By an *equivalence* of chain complexes we mean a quasi isomorphism, as usual such morphisms are denoted by $\sim$.

We will refer to morphisms of chain complexes and differential graded A-modules as *maps* (the relevant categories are described in Section 3). This will serve to distinguish maps from the morphisms in the corresponding derived categories.

2. Consequences of the spectral sequence

Three equivalent definitions of the shearing map will be given in Section 7, but we give a quick review of one here. There is a way to assign to each element x in the Hochschild cohomology $HH^*(A)$ a natural transformation $\zeta(x) : 1_{\mathbf{D}(A)} \to \Sigma^n 1_{\mathbf{D}(A)}$, see 7.11 for details. Roughly speaking, given a morphism $x : A \to \Sigma^n A$ in $\mathbf{D}(A^e)$ we have the natural transformation $\zeta(x)_M = x \otimes^{\mathbf{L}}_A M$. This assignment preserves addition and turns multiplication to composition of natural transformations. We can now define the shearing map $\chi : HH^*(A) \to \operatorname{Ext}^*_A(k,k)$ by $\chi(x) = \zeta(x)_k$.

From the definition it is immediate that that the image of the shearing map χ lies in the centre of $\operatorname{Ext}^*_A(k,k)$. In Lemma 7.4 we show that our definition of the shearing map agrees with that of the morphism I from [11], thereby recovering their result that the image of I is central. We also recover the following result of [11, 4.1 - Theorem 7]

2.1. **Theorem.** *Let A be a coconnective bounded dga over a field k such that $H^n(A) = 0$ for $n > d$. Then the kernel of the shearing map is nilpotent of nilpotency index less than or equal to d. If, in addition, $H^1(A) = 0$ then the nilpotency index is less than or equal to $d/2$.*

Proof. Since A is bounded the spectral sequence has strong convergence. It also implies that all the elements in $E^r_{p,q}$ are nilpotent whenever $p \neq 0$. Theorem 1.1 implies that $\oplus_{p<0} E^\infty_{p,*}$ is isomorphic to the kernel of the shearing map. To be precise, E^∞ is isomorphic to the associated graded object of $HH^*(A)$ coming from some filtration, and under this isomorphism $\oplus_{p<0} E^\infty_{p,*}$ can be identified with the kernel of the shearing map.

Nevertheless, this means that the kernel of the shearing map is nilpotent and has nilpotency index which is less than or equal to the nilpotency index of the ideal $\oplus_{p<0} E^\infty_{p,*} \subset E^2_{*,*}$. □

As noted earlier, the main consequence of the spectral sequence is Theorem 1.2 which we now prove.

Proof of Theorem 1.2. Suppose A is coconnective and bounded. Denote by B the graded algebra $\mathrm{Ext}^*_A(k,k)$. We are also assuming B is finitely generated as a module over a central Noetherian sub-algebra N. We mean *central* in the graded commutative sense, thus if $n \in N$ and $x \in B$ then $nx = (-1)^{|n||x|}xn$, but this will play no part in what follows.

Since N is Noetherian then by Noether Normalization it contains a polynomial sub-algebra $P = k[x_1, ..., x_n]$ such that N is finitely generated over P. Thus the degree of each x_i must be even. By identifying B with $E^2_{0,*}$ we shall now consider P as a sub-algebra of E^2. By the description of the multiplication on the E^2-term when A is of finite type we see that P is central in E^2.

We now employ an argument of Quillen. By the Leibnitz rule and the fact that x_i is central and of even degree we see that x_i^p is a cycle for d^2, for every i. Continuing in this fashion we see that $x_i^{p^r}$ is a cycle for the differential d^r on E^r. Since A is bounded the spectral sequence collapses in some finite stage, say R. Hence $x_1^{p^R}, ..., x_n^{p^R}$ are all infinite cycles and so in the image of the shearing map. Clearly B is finitely generated as a module over the sub-algebra $P' = k[x_1^{p^R}, ..., x_n^{p^R}]$.

Since $H^*(A)$ is finite dimensional we see that E^2 is finitely generated as a module over P'. Hence E^2 is a Noetherian P'-module. In addition, because the elements of P' are infinite cycles we see, by the Leibnitz rule, that the differential d^2 is a morphism of P'-modules. Hence the kernel of d^2 is a Noetherian P'-module and so is this kernel's quotient E^3. Continuing by induction we end up showing that $E^R = E^\infty$ is a Noetherian P'-module. This shows that $HH^*(A)$ is finitely generated as a module over a polynomial sub-algebra and hence Noetherian. □

3. Model category preliminaries

3.1. **The category of dg-k-modules.** By a *k-module* we mean a differential graded k-module, in other words a $\mathbb{Z}$-graded chain complex of k-vector spaces. Following Dwyer [9], a k-module concentrated in degree 0 will be referred to as a *discrete* k-module. A map of k-modules is just a chain map.

The category of k-modules has a cofibrantly generated model category structure described by Hovey in [12, 2.3]. In this model category structure the weak equivalences are quasi isomorphisms and the fibrations are degreewise surjections. In addition, this is a symmetric monoidal model category [12, Proposition 4.2.13], with the monoidal structure being the usual tensor product $\otimes$ of chain complexes over k. A monoid with respect to $\otimes$ is simply a dga.

Note that we define the suspension of a k-module X to be $\Sigma X = (\Sigma k) \otimes X$.

3.2. **The category of A-modules.** Fix a dga A over a field k. Since A is a monoid with respect to $\otimes$ one can define the category of left A-modules as in [15]. We shall describe these explicitly. An *A-module* a differential graded *left* A-module. A morphism of A-modules $f : M \to N$ is a morphism of chain complexes of degree zero which commutes with the action of A. The resulting category of A-modules is clearly an abelian category.

The category of A-modules has a Quillen model category structure where the weak equivalences are quasi-isomorphisms and fibrations are degreewise surjections, see [15, Theorem 4.1]. Hence every object is fibrant in this model category structure. The *derived category of A-modules*, denoted $\mathbf{D}(A)$, is the homotopy category of this Quillen model category. As is well known, $\mathbf{D}(A)$ is a triangulated category and a short exact sequence of A-modules induces an exact triangle in $\mathbf{D}(A)$, since it is a homotopy fibration sequence. We define $\mathrm{Ext}^n_A(X, Y)$ to be $\hom_{\mathbf{D}(A)}(\Sigma^{-n}X, Y)$.

Since $\otimes$ is symmetric, given a dga A we can define its opposite A^{op}, and $A^e = A \otimes A^{\mathrm{op}}$ is also a dga. An A-bimodule is simply an A^e-module, and thus the derived category of A-bimodules is $\mathbf{D}(A^e)$. When considering A as an A^e-module we always mean the obvious bimodule structure on A (there can be other bimodule structures, e.g. $A \otimes k \cong A$, but they will play no part in this paper).

The category of A-bimodules has a tensor product $- \otimes_A -$. This tensor product is not symmetric and its unit A is not cofibrant. Nevertheless there is a monoidal structure $- \otimes^{\mathbf{L}}_A -$ on the derived category of bimodules.

3.3. **Cones and cylinders.** The *cone* of a map $X \to Y$ of A-modules is the A-module $\mathbf{C}f = (\Sigma X^\natural \oplus Y^\natural, d)$ where $d = (-d^X, f + d^Y)$. There is a natural map $c_f : Y \to \mathbf{C}f$ which splits the map $Y \to \mathrm{coker} f$ and the cokernel of $Y \to \mathbf{C}f$ is ΣX. We write $\mathbf{C}X$ for $\mathbf{C}1_X$ and c_X for c_{1_X}. In fact, $\mathbf{C}X = (\mathbf{C}k) \otimes X$ and the map $\mathbf{C}X \to \Sigma X$ is simply $(\mathbf{C}k \to \Sigma k) \otimes X$.

From [15] and [12, 2.3] we learn that the generating cofibrations for A-modules are $\mathcal{I} = \{\Sigma^n A \to \mathbf{C}\Sigma^n A\}$ and the generating acyclic cofibrations are $\mathcal{J} = \{0 \to \mathbf{C}\Sigma^n A\}$. From this one easily sees that for any cofibrant A-module X, the map $X \to \mathbf{C}X$ is a cofibration. Since $\mathbf{C}f$ is the pushout of $\mathbf{C}X \leftarrow X \xrightarrow{f} Y$ we see that $Y \to \mathbf{C}f$ is a cofibration whenever X is cofibrant.

3.4. *Remark.* It is easy to see that an A-module X is in $\mathcal{I}$-cell [12, Definition 2.1.9] if and only if X is a semi-free A-module [10]. Hence whenever X is cofibrant then X is a retract of a semi-free A-module.

Given an A-module X we define the *cylinder* of X to be $X \wedge I = \mathbf{C}(X \xrightarrow{(1,1)} X \oplus X)$. There are obvious maps $X \oplus X \to X \wedge I \to X$. It is a simple exercise to show that whenever X is cofibrant then $X \wedge I$ is a very good cylinder object for X [8, Definition 4.2], i.e. $X \oplus X \to X \wedge I$ is a cofibration and $X \wedge I \to X$ is an acyclic fibration.

4. Differential graded preliminaries

4.1. **Realization of simplicial A-modules.** The construction we name realization is simply the homotopy colimit of a simplicial A-module. There are other well known choices for this homotopy colimit, we have chosen one whose good properties are easy to prove.

4.2. **Definition.** Let $X_{(\bullet)}$ be a simplicial A-module, thus each $X_{(n)}$ is an A-module with the face maps $d_i : X_{(n)} \to X_{(n-1)}$, $0 \le i \le n$ being A-module morphisms (we will only concern ourselves with the face maps). Define the *realization* of $X_{(\bullet)}$ to be the A-module

$|X_{(\bullet)}|$ whose underlying graded $A^\natural$-module is

$$\bigoplus_p (\Sigma^p X_{(p)})^\natural$$

with differential ∂ given by

$$\partial|_{\Sigma^p X^\natural_{(p)}} = (-1)^p \partial^{X_{(p)}} + \sum_{i=0}^{p} (-1)^i d_i$$

It is not difficult to see that $|X_{(\bullet)}|$ is an A-module. Moreover, realization is a functor from simplicial A-modules to A-modules.

This realization is none other than the total complex of the double complex generated from $X_{(\bullet)}$ where the additional differential is simply $\sum_{i=0}^{p}(-1)^i d_i$.

4.3. **Definition.** Given an A-module M the *constant simplicial A-module* $M_{(\bullet)}$ is given by $M_{(n)} = M$ for all n and all maps are the identity map. It is easy to see that the realization of the constant simplicial A-module $M_{(\bullet)}$ has a natural equivalence $|M_{(\bullet)}| \xrightarrow{\simeq} M$.

4.4. **The bar construction.** The (unnormalized two-sided) bar construction provides a model for the derived tensor product of a right A-module with a left A-module. We recall this construction next.

4.5. **Definition.** Let M be a right A-module and let N be a left A-module. The simplicial A-module $B_{(\bullet)}(M, A, N)$ is

$$B_{(t)}(M, A, N) = M \otimes_k A^{\otimes t} \otimes_k N$$

where $A^{\otimes t} = \underbrace{A \otimes_k \cdots \otimes_k A}_{t \text{ times}}$. The face maps are

$$d_i = \begin{cases} \eta_M \otimes 1_A^{t-1} \otimes 1_N & i = 0 \\ 1_M \otimes 1_A^{i-1} \otimes \mu \otimes 1_A^{t-i-1} \otimes 1_N & 0 < i < t \\ 1_M \otimes 1_A^{t-1} \otimes \eta_N & i = t \end{cases}$$

where $\mu : A \otimes A \to A$ is the multiplication map and η_M and η_N are the module structure maps $\eta_M : M \otimes A \to A$ and $\eta_N : A \otimes N \to N$. We have no use for the degeneracy maps and so we forgo their description. The *bar construction* is the realization $|B_{(\bullet)}(M, A, N)|$ which we shall denote by $M \boxtimes N$.

When M and N are A-bimodules then $M \boxtimes N$ is again an A-bimodule, with the left A-action coming from the left A-action on M and the right A-action coming from the right A-action on N. The following is well known.

4.6. **Lemma.** *Let M be a right A-module and let N be a left A-module, then $M \boxtimes N$ is a model for the derived tensor product $M \otimes^{\mathbf{L}}_A N$ and there exists a natural map $M \boxtimes N \to M \otimes_A N$.*

Proof. We shall only specify the natural map mentioned in the lemma. Note that $M \otimes_A N$ is the coequalizer of $d_0, d_1 : M \otimes A \otimes N \rightrightarrows M \otimes N$. It is now obvious how to define a natural map of simplicial modules

$$B_{(\bullet)}(M, A, N) \to (M \otimes_A N)_{(\bullet)}$$

The natural map we are after is then the composition

$$|B_{(\bullet)}(M, A, N)| \to |(M \otimes_A N)_{(\bullet)}| \xrightarrow{\simeq} M \otimes_A N$$

□

4.7. *Remark.* A generic element of $M \boxtimes N$ is denoted by $m[a]n$ where $m \in M$, $[a] = [a_1|\cdots|a_p] \in A^{\otimes p}$ and $n \in N$. The natural map $M \boxtimes N \to M \otimes_A N$ is then

$$m[a]n \;\mapsto\; \begin{cases} m \otimes n & p = 0 \\ 0 & \text{otherwise} \end{cases}$$

4.8. **Properties of the bar construction.** We show that the bar construction is associative and has maps which can play the role of unit maps, although they are not isomorphisms.

4.9. **Definition.** Let L, M and N be A^e-modules. There is a natural isomorphism of A^e-modules $\alpha : (L \boxtimes M) \boxtimes N \to L \boxtimes (M \boxtimes N)$ which we now describe. A generic element of $(L \boxtimes M) \boxtimes N$ is denoted by $(l[a]m)[b]n$ where $l \in L$, $m \in M$, $n \in N$ and $[a] = [a_1|\cdots|a_p] \in A^{\otimes p}$ and $[b] = [b_1|\cdots|b_q] \in A^{\otimes q}$. Note that the degree of $[a]$ is $\deg a_1 + \cdots + \deg a_p$ while the degree of $l[a]m$ is $\deg l + \deg[a] + \deg m + p$ and the degree of $(l[a]m)[b]n$ is $\deg l + \deg[a] + \deg m + p + \deg[b] + \deg n + q$. The isomorphism α is given by

$$(l[a]m)[b]n) \;\mapsto\; (-1)^{(\deg l + \deg[a] + p)q}\, l[a](m[b]n)$$

Proof of the following lemma is a simple calculation of signs and degrees and is therefore omitted.

4.10. **Lemma.** *The bar construction $\boxtimes$ together with the associativity isomorphism α satisfy the associativity diagram (5) in* [13, VII.1].

4.11. **Definition.** Define the left unit map $el_M : A \boxtimes M \to M$ to be the composition $A \boxtimes M \xrightarrow{\sim} A \otimes_A M \xrightarrow{\cong} M$. Clearly this map is an equivalence. We define the right unit map $er_M : M \boxtimes A \to A$ similarly. It is easy to see that $el_A = er_A : A \boxtimes A \to A$. We will usually denote both units by e. We caution the reader that these unit maps do not satisfy diagram (7) from [13, VII.1].

The following properties are clear.

4.12. **Lemma.** *The bar construction preserves equivalences and short exact sequences in either variable. The bar construction is bilinear in the following sense: let $f, g : M \to N$ and $h : X \to Y$ be maps of A-bimodules then $(f + g) \boxtimes h = (f \boxtimes h) + (g \boxtimes h)$ and $h \boxtimes (f + g) = (h \boxtimes f) + (h \boxtimes g)$.*

4.13. **Signs.** For k-modules X and Y there are natural isomorphisms $\Sigma(X \otimes Y) \cong (\Sigma X) \otimes Y$ and $\Sigma(X \otimes Y) \cong X \otimes (\Sigma Y)$. The first isomorphism is simply $\Sigma(x \otimes y) \mapsto (\Sigma x) \otimes y$ while the second isomorphism is $\Sigma(x \otimes y) \mapsto (-1)^{|x|} x \otimes (\Sigma y)$. These standard isomorphisms induce in an obvious way isomorphisms of A^e-modules $\Sigma(X \boxtimes Y) \cong (\Sigma X) \boxtimes Y$ and $\Sigma(X \boxtimes Y) \cong X \boxtimes (\Sigma Y)$. From now on these are the isomorphisms we shall use (at times implicitly) whenever we consider a map of A^e-modules $X \otimes (\Sigma Y) \to M$ as an element of $\mathrm{Ext}^0_{A^e}(\Sigma(X \boxtimes Y), M) = \mathrm{Ext}^{-1}_{A^e}(X \boxtimes Y, M)$.

Similarly there are standard natural isomorphisms $(\mathbf{C}X) \otimes Y \cong \mathbf{C}(X \otimes Y) \cong X \otimes (\mathbf{C}Y)$ which give a commutative diagram

$$\begin{tikzcd}
\mathbf{C}(X) \otimes Y \arrow[r, leftrightarrow, "\cong"] \arrow[d] & \mathbf{C}(X \otimes Y) \arrow[r, leftrightarrow, "\cong"] \arrow[d] & X \otimes (\mathbf{C}Y) \arrow[d] \\
\Sigma(X) \otimes Y \arrow[r, leftrightarrow, "\cong"] & \Sigma(X \otimes Y) \arrow[r, leftrightarrow, "\cong"] & X \otimes (\Sigma Y)
\end{tikzcd}$$

Now consider the pushout $Q = \mathbf{C}X \coprod_X \mathbf{C}X$. Clearly Q is equivalent to ΣX. Indeed there are two natural equivalences $l : Q \to \mathbf{C}X \coprod_X 0 = \Sigma X$ and $r : Q \to 0 \coprod_X \mathbf{C}X = \Sigma X$ coming from maps of the appropriate pushout diagrams. It is also easy to see there is a natural map $\xi : \Sigma X \to Q$ such that $l\xi = -1$ while $r\xi = 1$.

Now let $P = (\mathbf{C}X) \otimes Y \coprod_{X\otimes Y} X \otimes (\mathbf{C}Y)$, then similarly there are two natural equivalences $l : P \to (\Sigma X) \otimes Y$ and $r : P \to X \otimes (\Sigma Y)$ coming from maps of the appropriate pushout diagrams. Combining the map ξ mentioned above with the various standard isomorphisms yields maps

$$\Sigma(X \otimes Y) \to \mathbf{C}(X \otimes Y) \coprod_{X\otimes Y} \mathbf{C}(X \otimes Y) \xrightarrow{\cong} (\mathbf{C}X) \otimes Y \coprod_{X\otimes Y} X \otimes (\mathbf{C}Y) = P$$

we denote this composition by ζ. Clearly, $r\zeta$ is the standard isomorphism while $l\zeta$ is -1 times the standard isomorphism. Moreover, there is the following natural short exact sequence

$$\begin{tikzcd}
\Sigma(X \otimes Y) \arrow[r] \arrow[d, "\zeta"] & \mathbf{C}\Sigma(X \otimes Y) \arrow[r] \arrow[d] & \Sigma^2(X \otimes Y) \arrow[d, "\cong"] \\
P \arrow[r] & \mathbf{C}X \otimes \mathbf{C}Y \arrow[r] & \Sigma X \otimes \Sigma Y
\end{tikzcd}$$

where the rightmost vertical isomorphism is the standard one.

These properties extend to the bar construction $\boxtimes$ in an obvious manner, and so we give the following lemma without proof.

4.14. **Lemma.** *Let X and Y be A^e-modules. Then there exists a natural morphism of short exact sequences of A^e-modules*

$$\begin{tikzcd}
\Sigma(X \boxtimes Y) \arrow[r] \arrow[d, "\zeta"] & \mathbf{C}\Sigma(X \boxtimes Y) \arrow[r] \arrow[d] & \Sigma^2(X \boxtimes Y) \arrow[d, "\cong"] \\
(\mathbf{C}X) \boxtimes Y \coprod_{X\boxtimes Y} X \boxtimes (\mathbf{C}Y) \arrow[r] & \mathbf{C}X \boxtimes \mathbf{C}Y \arrow[r] & \Sigma X \boxtimes \Sigma Y
\end{tikzcd}$$

where the rightmost vertical map is the standard isomorphism. In addition, the composition $\Sigma(X \boxtimes Y) \xrightarrow{\zeta} (\mathbf{C}X) \boxtimes Y \coprod_{X\boxtimes Y} X \boxtimes (\mathbf{C}Y) \to X \boxtimes (\Sigma Y)$ is the standard isomorphism, while $\Sigma(X \boxtimes Y) \xrightarrow{\zeta} (\mathbf{C}X) \boxtimes Y \coprod_{X\boxtimes Y} X \boxtimes (\mathbf{C}Y) \to (\Sigma X) \boxtimes Y$ is -1 times the standard isomorphism.

4.15. *Remark.* The signs here are analogous to the topological choice of orientation on the boundary of a manifold, compare [5, Remark 2.2].

5. Construction of the spectral sequence

5.1. **Filtering the dga.** Let $J(n)$ to be the sub-complex

$$0 \to \cdots \to 0 \to A^n \to A^{n+1} \to A^{n+1} \to \cdots$$

of A. Clearly $J(n)$ is an A-bimodule and we have a filtration of A by A-bimodules

$$(1) \qquad \cdots \to J(n) \xrightarrow{\iota} J(n-1) \xrightarrow{\iota} \cdots \to J(0) = A$$

There are also short exact sequences of A-bimodules

$$(2) \qquad 0 \to J(n+1) \xrightarrow{\iota} J(n) \xrightarrow{\theta} \Sigma^{-n} A^n \to 0$$

The tower (1) induces morphisms

$$(3) \qquad \cdots \to \mathrm{Ext}^*_{A^e}(A, J(n)) \xrightarrow{\iota} \mathrm{Ext}^*_{A^e}(A, J(n-1)) \to \cdots \to \mathrm{Ext}^*_{A^e}(A, A)$$

and the short exact sequences (2) induce long exact sequences sequences:

$$(4)\qquad \cdots \to \mathrm{Ext}^t_{A^e}(A, J(n+1)) \xrightarrow{\iota} \mathrm{Ext}^t_{A^e}(A, J(n)) \xrightarrow{\theta} \mathrm{Ext}^t_{A^e}(A, \Sigma^{-n}A^n) \xrightarrow{\kappa} \mathrm{Ext}^{t+1}_{A^e}(A, J(n+1)) \to \cdots$$

The indeterminacy of the connecting homomorphism can cause us trouble when trying to work out signs. Therefore we define $\kappa : \Sigma^{-n}A^n \to \Sigma J(n+1)$ to be the morphism in $\mathbf{D}(A^e)$ represented by the obvious maps $\Sigma^{-n}A^n \to \mathbf{C}\theta \xleftarrow{\sim} \Sigma J(n+1)$.

5.2. **The spectral sequence.** The spectral sequence we build shall be homologically graded, hence we set:

$$D^1_{p,q} = \mathrm{Ext}^{-p-q}_{A^e}(A, J(-p))$$
$$E^1_{p,q} = \mathrm{Ext}^{-p-q}_{A^e}(A, \Sigma^p A^{-p}) = \mathrm{Ext}^{-q}_{A^e}(A, A^{-p})$$

The morphisms κ, ι and θ now become

$$E^1_{p,q} \xrightarrow{\kappa} D^1_{p-1,q}$$
$$D^1_{p,q} \xrightarrow{\iota} D^1_{p+1,q-1}$$
$$D^1_{p,q} \xrightarrow{\theta} E^1_{p,q}$$

Together these yield an exact couple which gives rise to the desired spectral sequence (E^r, d^r).

5.3. **Lemma.** *Let A be a coconnected k-dga. The E^2-term of the spectral sequence we have just constructed is*

$$E^2_{p,q} \cong \mathit{Ext}^{-q}_{A^e}(A, H^{-p}(A))$$

This spectral sequence conditionally converges (see [2, Definition 5.10]*) to $\mathit{Ext}^{-p-q}_{A^e}(A, A)$. If A has bounded homology then the spectral sequence has strong convergence.*

Proof. To identify the E^2-term one need only note that there is an isomorphism of A^e-modules $A^n \cong \oplus_{\dim_k(A_n)} k$. To prove the other claims we need temporarily to extend our filtration of A by setting $J(-p) = A$ for all $p > 0$. Now we have a similar spectral sequence, only $D^1_{p,q} = \mathrm{Ext}^{-p-q}_{A^e}(A, A)$ for $p > 0$ (this leaves the E^1-term unchanged).

Fix an index q and consider the tower of graded abelian groups $M(p) = H_q(J(p))$. This tower satisfies the trivial Mittag-Leffler condition and therefore $\lim_p M(p) = \lim^1 M(p) = 0$. From this we conclude that the homotopy limit $\mathrm{holim}_p J(p)$ is equivalent to zero. Since

$$\mathrm{holim}_p \mathbf{R}\mathrm{Hom}_{A^e}(A, J(p)) \simeq \mathbf{R}\mathrm{Hom}_{A^e}(A, \mathrm{holim}_p J(p)) \simeq 0$$

then by a Milnor type short exact sequence (see for example [2, Theorem 4.9]) we see that $\lim_{p \to -\infty} D^1_{p,*} = \lim^1_{p \to -\infty} D^1_{p,*} = 0$. Thus our spectral sequence converges conditionally to $\mathrm{colim}_{p \to \infty} D^1_{p,*} = \mathrm{Ext}^{-p-q}_{A^e}(A, A)$.

Assuming that $H^*(A)$ is bounded implies that the $E^2_{p,q} = 0$ whenever $p > 0$ or $p < n$ for some fixed n. Hence the spectral sequence collapses at some finite stage r. By the remark following [2, Theorem 7.1] we see that the spectral sequence has strong convergence. $\square$

6. The multiplicative structure

6.1. **The pairing on the filtration.** The multiplication map $A \otimes_k A \to A$ yields an associative pairing

$$J(n) \otimes_k J(m) \xrightarrow{\phi_{n,m}} J(n+m)$$

Note that the maps $\phi_{n,m}$ are maps of A-bimodules, where the bimodule structure on $J(n) \otimes_k J(m)$ comes from the left A-module structure on $J(n)$ and the right A-module structure on $J(m)$. This is not sufficient, however, since we need to construct a pairing of A-bimodules:

$$J(n) \boxtimes J(m) \xrightarrow{\psi_{n,m}} J(n+m)$$

Let $J(n+m)_{(\bullet)}$ be the constant simplicial bimodule $J(n+m)$. Consider the map of simplicial of A-bimodules

$$\psi_{(\bullet),n,m} : B_{(\bullet)}(J(n), A, J(m)) \to J(n+m)_{(\bullet)}$$

which is given by the obvious multiplication map

$$\psi_{(t),n,m} : J(n) \otimes_k A^{\otimes t} \otimes_k J(m) \to J(n+m)$$

This results in a map of simplicial A-bimodules because the morphisms $\psi_{(t),m,n}$ commute with the face (and degeneracy) maps of $B_{(\bullet)}(J(n), A, J(m))$.

Upon taking realization of both simplicial bimodules we get a map $|\psi_{(\bullet),n,m}| : J(n) \boxtimes J(M) \to |J(n+m)_{(\bullet)}|$. Composing this map with the natural weak equivalence $|J(n+m)_{(\bullet)}| \xrightarrow{\simeq} J(n+m)$ yields

$$\psi_{n,m} : J(n) \boxtimes J(M) \to J(n+m)$$

which is the pairing we need. We will omit the subscripts n and m whenever they are clear from the context. Note that $\psi_{0,0}$ is the unit map e.

6.2. **Lemma.**

$$\psi(\iota \boxtimes 1) = \iota\psi = \psi(1 \boxtimes \iota)$$

Proof. We will only show $\psi(\iota \boxtimes 1) = \iota\psi$, the proof of the second equality being similar. Clearly, the diagram below commutes

$$\begin{tikzcd}
B_{(p)}\big(J(s), A, J(t)\big) \arrow[r, "\iota\otimes 1"] \arrow[d] & B_{(p)}\big(J(s-1), A, J(t)\big) \arrow[d] \\
J(s+t) \arrow[r, "\iota"] & J(s+t-1)
\end{tikzcd}$$

which implies the following diagram of A-modules commutes

$$\begin{tikzcd}
J(s) \boxtimes J(t) \arrow[r, "\iota\boxtimes 1"] \arrow[d, "\psi_{s,t}"] & J(s-1) \boxtimes J(t) \arrow[d, "\psi_{s-1,t}"] \\
|J(s+t)_{(\bullet)}| \arrow[r, "|\iota|"] \arrow[d, "\simeq"] & |J(s+t-1)_{(\bullet)}| \arrow[d, "\simeq"] \\
J(s+t) \arrow[r, "\iota"] & J(s+t-1)
\end{tikzcd}$$

□

6.3. **The pairing on $\mathbf{D^1}$.** We now construct a pairing on the D^1 term of the exact couple. Define maps

$$\bar{\psi}_{n,m} : \mathrm{Ext}^*_{A^e}(A, J(n)) \otimes \mathrm{Ext}^*_{A^e}(A, J(m)) \to \mathrm{Ext}^*_{A^e}(A, J(n+m))$$

in the following way: given $f \in \mathrm{Ext}^{-s}_{A^e}(A, J(n))$ and $g \in \mathrm{Ext}^{-t}_{A^e}(A, J(m))$ let $\bar{\psi}_{n,m}(f \otimes g)$ be the composition

$$\Sigma^{s+t} A \xrightarrow{\cong} \Sigma^s A \boxtimes \Sigma^t A \xrightarrow{f \boxtimes g} J(n) \boxtimes J(m) \xrightarrow{\psi_{n,m}} J(n+m)$$

in $\mathbf{D}(A^e)$, where the leftmost isomorphism is e^{-1}. Bilinearity of the bar construction $\boxtimes$ allows us to extend this to a pairing on $\mathrm{Ext}^*_{A^e}(A, J(n)) \otimes_k \mathrm{Ext}^*_{A^e}(A, J(m))$.

For what follows we shall need a concrete representation of this pairing on the category of A^e-modules. First, we must fix a cofibrant replacement of A as an A^e-module. Since $A \boxtimes A$ is cofibrant we define $e : A \boxtimes A \to A$ to be our cofibrant replacement. We will usually denote $A \boxtimes A$ by $\tilde{A}$. An element $f \in \mathrm{Ext}^{-s}_{A^e}(A, J(n))$ is now represented by the following maps of A^e-modules: $\Sigma^s A \xleftarrow{e} \Sigma^s \tilde{A} \xrightarrow{\tilde{f}} J(n)$, where $\tilde{f}$ represents fe^{-1}. We shall usually omit e from the description, simply saying $\tilde{f}$ represents f.

Given another element $g \in \mathrm{Ext}^{-t}_{A^e}(A, J(m))$ we choose a map of A^e-modules $\tilde{g} : \Sigma^t \tilde{A} \to J(m)$ representing g. Now we have the following maps of A^e-modules

$$\Sigma^{s+t} A \xleftarrow[\sim]{e} \Sigma^s A \boxtimes \Sigma^t A \xleftarrow[\sim]{e \boxtimes e} \Sigma^s \tilde{A} \boxtimes \Sigma^t \tilde{A} \xrightarrow{\tilde{f} \boxtimes \tilde{g}} J(n) \boxtimes J(m) \xrightarrow{\psi_{n,m}} J(n+m)$$

It is easy to see that $\bar{\psi}_{n,m}(f \otimes g)$ is equal to the composition $\psi_{n,m}(\tilde{f} \boxtimes \tilde{g})(e \boxtimes e)^{-1} e^{-1}$ in $\mathbf{D}(A^e)$.

From now on when we say that a map $f : \tilde{A} \boxtimes \tilde{A} \to X$ represents an element of $\mathrm{Ext}^*_{A^e}(A, X)$ we implicitly refer to the diagram $A \xleftarrow{e} \tilde{A} \xleftarrow{e \boxtimes e} \tilde{A} \boxtimes \tilde{A} \xrightarrow{f} X$. Thus we have just shown that $\psi_{n,m}(\tilde{f} \boxtimes \tilde{g})$ represents $\bar{\psi}_{n,m}(f \otimes g)$.

6.4. **Lemma.** *The pairing ψ is associative and so is the induced pairing $\bar{\psi}$.*

Proof. To show that ψ is associative we must show the following diagram commutes:

$$\begin{tikzcd}
J(l) \boxtimes \big(J(m) \boxtimes J(n)\big) \arrow[rr, "J(l)\boxtimes\psi_{m,n}"] \arrow[d, "\alpha"', "\cong"] & & J(l) \boxtimes J(m+n) \arrow[dd, "\psi_{l,m+n}"] \\
\big(J(l) \boxtimes J(m)\big) \boxtimes J(n) \arrow[d, "\psi_{l,m}\boxtimes J(n)"] & & \\
J(l+m) \boxtimes J(n) \arrow[rr, "\psi_{l+m,n}"] & & J(l+m+n)
\end{tikzcd}$$

Let $[a] = [a_1|\cdots|a_p] \in A^{\otimes p}$, $[b] = [b_1|\cdots|b_p] \in A^{\otimes q}$ and let $j_1[a](j_2[b]j_3)$ be a generic element in $J(l) \boxtimes (J(m) \boxtimes J(n))$. A simple calculation shows that

$$\begin{aligned}
\psi(1 \boxtimes \psi)(j_1[a](j_2[b]j_3)) &= \begin{cases} j_1 j_2 j_3 & p = q = 0 \\ 0 & \text{otherwise} \end{cases} \\
&= \psi(\psi \boxtimes 1)\alpha(j_1[a](j_2[b]j_3))
\end{aligned}$$

To show that $\bar{\psi}$ is also associative, we must show that our choice of isomorphism $A \cong A \boxtimes A$ in $\mathbf{D}(A^e)$ is coassociative. This reduces to showing that $A \boxtimes A \xrightarrow{e} A$ is associative. Since $e = \psi_{0,0}$, it is indeed associative by the first part of the proof. $\square$

6.5. **Lemma.** *The pairing $\bar{\psi}_{0,0}$ is the standard multiplication on $HH^*(A)$.*

Proof. This is well known and so we shall only sketch the proof, which uses the Eckmann-Hilton argument. The standard multiplication on $\mathrm{Ext}^*_{A^e}(A, A)$ is done by composition of arrows, denoted $f \circ g$. It is easy to show that the identity morphism 1_A is a unit also for $\bar{\psi}$. Thus, for any $f \in HH^*(A)$

$$f \circ 1 = 1 \circ f = f = \bar{\psi}(1 \otimes f) = \bar{\psi}(f \otimes 1)$$

It is also a simple exercise to show that for any $f_1, f_2, g_1, g_2 \in HH^*(A)$

$$\bar{\psi}((f_1 \circ f_2) \otimes (g_1 \circ g_2)) = \bar{\psi}(f_1 \otimes g_1) \circ \bar{\psi}(f_2 \otimes g_2)$$

Thus, by the Eckmann-Hilton argument both miltiplications are the same and are associative and graded-commutative. $\square$

6.6. **The multiplication on $\mathbf{E^1}$.** The multiplication on the E^1 term arises from the pairing ψ in a standard way which we will now follow (compare [5]). For convenience we shall denote the A-bimodule $\Sigma^{-n}A^n$ by $\mathcal{A}^n$.

6.7. **Lemma.** *There is a short exact sequence of A-bimodules*

$$0 \to \begin{pmatrix} J(n) \boxtimes J(m+1) + \\ J(n+1) \boxtimes J(m) \end{pmatrix} \to J(n) \boxtimes J(m) \to \mathcal{A}^n \boxtimes \mathcal{A}^m \to 0$$

where $J(n) \boxtimes J(m+1) + J(n+1) \boxtimes J(m)$ is the sum as subcomplexes of $J(n) \boxtimes J(m)$.

Proof. There are short exact sequences:

$$0 \to \begin{pmatrix} B_{(t)}(J(n), A, J(m+1)) + \\ B_{(t)}(J(n+1), A, J(m)) \end{pmatrix} \to B_{(t)}(J(n), A, J(m)) \to B_{(t)}(\mathcal{A}^n, A, \mathcal{A}^m) \to 0$$

which yield the desired short exact sequence after taking realization. $\square$

It is a simple observation that $J(n) \boxtimes J(m+1) + J(n+1) \boxtimes J(m)$ is in fact the pushout

$$J(n+1) \boxtimes J(m) \coprod_{J(n+1) \boxtimes J(m+1)} J(n) \boxtimes J(m+1)$$

The following lemma is an immediate consequence of this observation and the fact that ψ commutes with ι.

6.8. **Lemma.** *The following diagram of A-bimodules commutes*

$$\begin{array}{ccc} \begin{pmatrix} J(n) \boxtimes J(m+1) + \\ J(n+1) \boxtimes J(m) \end{pmatrix} & \longrightarrow & J(n) \boxtimes J(m) \\ \downarrow{\scriptstyle \psi'_{n,m}} & & \downarrow{\scriptstyle \psi_{n,m}} \\ J(n+m+1) & \xrightarrow{\iota} & J(n+m) \end{array}$$

Where ψ' is the obvious map of subcomplexes.

6.9. **Definition.** Define a pairing $\mu_{n,m} : \mathcal{A}^n \boxtimes \mathcal{A}^m \to \mathcal{A}^{n+m}$ of A-bimodules to be the pairing induced from the following diagram where both rows are short exact sequences of complexes:

$$\begin{array}{ccccc} \begin{pmatrix} J(n) \boxtimes J(m+1) + \\ J(n+1) \boxtimes J(m) \end{pmatrix} & \longrightarrow & J(n) \boxtimes J(m) & \longrightarrow & \mathcal{A}^n \boxtimes \mathcal{A}^m \\ \downarrow{\scriptstyle \psi'_{n,m}} & & \downarrow{\scriptstyle \psi_{n,m}} & & \downarrow{\scriptstyle \mu_{n,m}} \\ J(n+m+1) & \longrightarrow & J(n+m) & \longrightarrow & \mathcal{A}^{n+m} \end{array}$$

The multiplication on the E^1 term follows easily from the pairing above. Define

$$\bar{\mu}_{n,m} : \mathrm{Ext}^*_{A^e}(A, \mathcal{A}^n) \otimes_k \mathrm{Ext}^*_{A^e}(A, \mathcal{A}^m) \to \mathrm{Ext}^*_{A^e}(A, \mathcal{A}^{n+m})$$

as the composition $\mu_{n,m}(- \boxtimes -)e^{-1}$.

6.10. **Lemma.** *The pairing μ is associative and so is the multiplication $\bar{\mu}$.*

Proof. That μ is associative can be deduced from Lemma 6.4, but a direct calculation is much easier and makes the associativity obvious. As is the proof of Lemma 6.4, associativity of $\bar{\mu}$ is a consequence of the associativity of μ and of e. □

6.11. **Representing multiplication on $\mathbf{E^1}$.** We first show how to represent certain elements in E^1. The following lemma is well known in many settings, see for example [5, Lemma 3.3].

6.12. **Lemma.** *Let $x \in Ext^s_{A^e}(A, \mathcal{A}^p)$ and $a \in Ext^{s+1}_{A^e}(A, J(p+n+1))$ such that $\kappa x = \iota^n a$. Then there exists a commutative diagram of short exact sequences of A^e-modules*

$$\begin{tikzcd}
\Sigma^{-s-1}\tilde{A} \arrow[r] \arrow[d, "w"] & \mathbf{C}\Sigma^{-s-1}\tilde{A} \arrow[r] \arrow[d, "f"] & \Sigma^{-s}\tilde{A} \arrow[d, "\tilde{x}"] \\
J(p+1) \arrow[r, "\iota"] & J(p) \arrow[r, "\theta"] & \mathcal{A}^p
\end{tikzcd}$$

and a map of A^e-modules $\tilde{a} : \Sigma^{-s-1}\tilde{A} \to J(p+n+1)$ such that

(1) *$\tilde{a}$ represents a,*
(2) *$w = \iota^n \tilde{a}$ and so w represents κx,*
(3) *$\tilde{x}$ represents x.*

Proof. The case for $n = 0$ is well known and therefore omitted. As in the proof of [5, Lemma 3.3], the case for $n > 0$ is done using the homotopy extension property and we shall only sketch the argument. Recall that for an A^e-module M we denote by $M \wedge I$ the mapping cone of $M \xrightarrow{(1,-1)} M \oplus M$. Let the two obvious maps $M \to M \wedge I$ be denoted by i_o and i_1. Since $\tilde{A}$ and $\mathbf{C}\tilde{A}$ are cofibrant, $\tilde{A} \wedge I$ and $\mathbf{C}\tilde{A} \wedge I$ are very good cylinder objects for $\tilde{A}$ and $\mathbf{C}\tilde{A}$ respectively (see 3.3).

So suppose we have found $w : \Sigma^{-s-1}\tilde{A} \to J(p+1)$ and $f : \mathbf{C}\Sigma^{-s-1}\tilde{A} \to J(p)$ as for the case $n = 0$. Choose a map $\tilde{a} : \Sigma^{-s-1}\tilde{A} \to J(p+n+1)$ which represents a. Since $\iota\tilde{a}$ and w need not be equal, we need to replace f by an equivalent map f' which will make the following diagram commute

$$\begin{tikzcd}
\Sigma^{-s-1}\tilde{A} \arrow[r] \arrow[d, "\iota\tilde{a}"] & \mathbf{C}\Sigma^{-s-1}\tilde{A} \arrow[d, "f'"] \\
J(p+1) \arrow[r, "\iota"] & J(p)
\end{tikzcd}$$

Equivalence of $\iota\tilde{a}$ and w implies there is a map $h : \Sigma^{-s-1}\tilde{A} \wedge I \to J(p+1)$ which is a homotopy between these two maps, thus $hi_1 = \iota a$ and $hi_0 = w$. Let Y be the pushout of the diagram

$$\Sigma^{-s-1}\tilde{A} \wedge I \xleftarrow{i_0} \Sigma^{-s-1}\tilde{A} \to \mathbf{C}\Sigma^{-s-1}\tilde{A}$$

(one can liken Y to $\tilde{A} \times [0,1] \cup_{\tilde{A}\times\{0\}} \mathbf{C}\tilde{A} \times \{0\}$). Clearly there is a natural map $h' : Y \to J(p)$. Since $Y \to \mathbf{C}\Sigma^{-s-1}\tilde{A} \wedge I$ is a cofibration, we can extend h' to a homotopy $h'' : \mathbf{C}\Sigma^{-s-1}\tilde{A} \wedge I \to J(p)$. Now $h''i_1 : \mathbf{C}\Sigma^{-s-1}\tilde{A} \to J(p)$ is the map f' we need. □

6.13. **Lemma.** *Suppose* $x \in Ext_{A^e}^{-s-1}(A, \mathcal{A}^p)$ *and* $y \in Ext_{A^e}^{-t-1}(A, \mathcal{A}^u)$ *are represented by diagrams*

$$\begin{tikzcd} \Sigma^s \tilde{A} \arrow[r] \arrow[d, "w"] & \mathbf{C}\Sigma^s \tilde{A} \arrow[r] \arrow[d, "f"] & \Sigma^{s+1}\tilde{A} \arrow[d, "\tilde{x}"] \\ J(p+1) \arrow[r, "\iota"] & J(p) \arrow[r, "\theta"] & \mathcal{A}^p \end{tikzcd} \qquad \begin{tikzcd} \Sigma^t \tilde{A} \arrow[r] \arrow[d, "z"] & \mathbf{C}\Sigma^t \tilde{A} \arrow[r] \arrow[d, "g"] & \Sigma^{t+1}\tilde{A} \arrow[d, "\tilde{y}"] \\ J(u+1) \arrow[r, "\iota"] & J(u) \arrow[r, "\theta"] & \mathcal{A}^u \end{tikzcd}$$

Then $\kappa(xy)$ *is represented by*

$$\psi'_{p,u}(f \boxtimes z \coprod_{w \boxtimes z} w \boxtimes g)$$

Proof. The proof is simply given by the following commutative diagram

$$\begin{tikzcd} \mathbf{C}\Sigma^s \tilde{A} \boxtimes \Sigma^t \tilde{A} \coprod_{\Sigma^s \tilde{A} \boxtimes \Sigma^t \tilde{A}} \Sigma^s \tilde{A} \boxtimes \mathbf{C}\Sigma^t \tilde{A} \arrow[r] \arrow[d, "f \boxtimes z \coprod_{w\boxtimes z} w \boxtimes g"] & \mathbf{C}\Sigma^s \tilde{A} \boxtimes \mathbf{C}\Sigma^t \tilde{A} \arrow[r] \arrow[d, "f \boxtimes g"] & \Sigma^{s+1}\tilde{A} \boxtimes \Sigma^{t+1}\tilde{A} \arrow[d, "\tilde{x} \boxtimes \tilde{y}"] \\ J(p) \boxtimes J(u+1) \coprod_{J(p+1) \boxtimes J(u+1)} J(p+1) \boxtimes J(u) \arrow[r] \arrow[d, "\psi'"] & J(p) \boxtimes J(u) \arrow[r] \arrow[d, "\psi"] & \mathcal{A}^p \boxtimes \mathcal{A}^u \arrow[d, "\mu"] \\ J(p+u+1) \arrow[r] & J(p+u) \arrow[r] & \mathcal{A}^{p+u} \end{tikzcd}$$

Note that we are implicitly using here the natural isomorphisms described in Lemma 4.14. □

6.14. **The multiplicative property of the spectral sequence.** We shall employ a criterion of Massey from [14] to show that the spectral sequence is multiplicative. Note that translating the proof of [5, Proposition 5.1] to our setting would work equally well.

6.15. **Proposition.** *The pairings* μ *and* ψ *defined above make the spectral sequence* $E^r_{*,*}$ *into a multiplicative spectral sequence. Namely for every* r, $E^r_{*,*}$ *has an induced multiplication for which* d^r *is a derivation.*

Proof. Translated to our setting, Massey's criterion is

6.16. **Theorem** (Massey [14]). *Suppose the following conditions hold for the spectral sequence in Section 5.*

(1) E^1 *is a graded algebra.*
(2) *For every* $x, y \in E^1$ *and* $a, b \in D^1$ *such that* $\kappa(x) = \iota^n(a)$ *and* $\kappa(y) = \iota^n(b)$ *there exists* $c \in D^1$ *such that* $\kappa(xy) = \iota^n(c)$ *and* $\theta(c) = \theta(a)y + (-1)^{|x|}x\theta(b)$.

Then the spectral sequence is multiplicative.

In light of Lemma 6.10 the first condition reduces to verifying that our grading choice is correct, which is easily checked.

For the second condition we represent x and y by diagrams as in Lemma 6.12:

$$\begin{tikzcd} \Sigma^s \tilde{A} \arrow[r] \arrow[d, "w"] & \mathbf{C}\Sigma^s \tilde{A} \arrow[r] \arrow[d, "f"] & \Sigma^{s+1}\tilde{A} \arrow[d, "\tilde{x}"] \\ J(p+1) \arrow[r, "\iota"] & J(p) \arrow[r, "\theta"] & \mathcal{A}^p \end{tikzcd} \qquad \begin{tikzcd} \Sigma^t \tilde{A} \arrow[r] \arrow[d, "z"] & \mathbf{C}\Sigma^t \tilde{A} \arrow[r] \arrow[d, "g"] & \Sigma^{t+1}\tilde{A} \arrow[d, "\tilde{y}"] \\ J(u+1) \arrow[r, "\iota"] & J(u) \arrow[r, "\theta"] & \mathcal{A}^u \end{tikzcd}$$

where $\tilde{x}$ represents x and $\tilde{y}$ represents y. In addition, by the same lemma, there are maps $\tilde{a} : \Sigma^s \tilde{A} \to J(p+n+1)$ and $\tilde{b} : \Sigma^t \tilde{A} \to J(u+n+1)$ representing a and b such that $w = \iota^n \tilde{a}$ and $z = \iota^n b$.

Denote by P the pushout

$$J(p) \boxtimes J(u+n+1) \coprod_{J(p+n+1)\boxtimes J(u+n+1)} J(p+n+1) \boxtimes J(u)$$

and by B the pushout

$$\mathbf{C}\Sigma^s \tilde{A} \boxtimes \Sigma^t \tilde{A} \coprod_{\Sigma^s \tilde{A}\boxtimes \Sigma^t \tilde{A}} \Sigma^s \tilde{A} \boxtimes \mathbf{C}\Sigma^t \tilde{A}$$

Let ψ'' be the map

$$\psi_{p,u+n+1} \coprod_{\psi_{p+n+1,u+n+1}} \psi_{p+n+1,u} : P \to J(p+u+n+1)$$

We can now define a map $\tilde{c} : B \to J(p+u+n+1)$ by

$$\tilde{c} = \psi''(f \boxtimes \tilde{b} \coprod_{\tilde{a}\boxtimes\tilde{b}} \tilde{a} \boxtimes g)$$

We shall show that the morphism $c \in \mathrm{Ext}_{A^e}^{-s-t-1}(A, J(p+u+n+1))$ which is represented by $\tilde{c}$ is the desired element in D^1. As before, we are implicitly using the isomorphism $B \cong \Sigma^{s+t+1}(\tilde{A} \boxtimes \tilde{A})$ of Lemma 4.14.

It is easy to verify that the following diagram commutes

$$\begin{tikzcd}
B \arrow[d, "f\boxtimes\tilde{b}\coprod_{\tilde{a}\boxtimes\tilde{b}}\tilde{a}\boxtimes g"'] \arrow[dr, "(f\boxtimes z\coprod_{w\boxtimes z} w\boxtimes g)"] & \\
P \arrow[r, "\iota^n"] \arrow[d, "\psi''"'] & J(p)\boxtimes J(u+1) \coprod_{J(p+1)\boxtimes J(u+1)} J(p+1)\boxtimes J(u) \arrow[d, "\psi'"] \\
J(p+u+n+1) \arrow[r, "\iota^n"] & J(p+u+1)
\end{tikzcd}$$

Hence, by Lemma 6.13, we see that $\kappa(xy) = \iota^n c$.

It remains to show that $\theta(c) = \theta(a)y + (-1)^{|x|} x\theta(b)$. There is the following commutative diagram

$$\begin{tikzcd}
J(p)\boxtimes J(u+n+1) \arrow[d] & J(p+n+1)\boxtimes J(u+n+1) \arrow[l, "\iota^{n+1}\otimes 1"'] \arrow[r, "1\otimes\iota^{n+1}"] \arrow[d] & J(p+n+1)\boxtimes J(u) \arrow[d] \\
\mathcal{A}^p \boxtimes J(u+n+1) & 0 \arrow[l] \arrow[r] & J(p+n+1)\boxtimes \mathcal{A}^u
\end{tikzcd}$$

Taking the pushout of each row yields a map $\tau : P \to \mathcal{A}^p \boxtimes J(u+n+1) \oplus (p+n+1) \boxtimes \mathcal{A}^u$. We leave it to the reader to check that the following diagram commutes

$$\begin{tikzcd}
B \arrow[r, "f\boxtimes\tilde{b}\coprod_{\tilde{a}\boxtimes\tilde{b}}\tilde{a}\boxtimes g"] \arrow[ddr, "\tilde{c}"] & P \arrow[r, "\tau"] \arrow[dd, "\psi''"] & \begin{pmatrix} \mathcal{A}^p \boxtimes J(u+n+1) \oplus \\ J(p+n+1) \boxtimes \mathcal{A}^u \end{pmatrix} \arrow[d, "1\boxtimes\theta\oplus\theta\boxtimes 1"] \\
 & & \begin{pmatrix} \mathcal{A}^p \boxtimes \mathcal{A}^{u+n+1} \oplus \\ \mathcal{A}^{p+n+1} \boxtimes \mathcal{A}^u \end{pmatrix} \arrow[d, "\mu+\mu"] \\
 & J(s+t+n+1) \arrow[r, "\theta"] & A^{s+t+n+1}
\end{tikzcd}$$

From this diagram one sees that $\theta\tilde{c}$ is equal to $\mu(\theta\tilde{a} \boxtimes \tilde{y}) + \mu(\tilde{x} \boxtimes \theta\tilde{b})$. Using the sign convention from Lemma 4.14 we see that $\theta(c) = \theta(a)y + (-1)^{|x|}x\theta(b)$. □

7. The Hochschild cohomology shearing map

In this section we give three equivalent definitions for the Hochschild cohomology shearing mapping

$$\chi : HH^*(A) \to \mathrm{Ext}^*_{A^e}(A, k)$$

This morphism is well known and there is no claim to originality here. However, this is only one possible variant of the shearing map; a more thorough discussion can be found in [1].

7.1. **Two descriptions of the shearing map.** Let A_l denote the dga $A \otimes k$ and let $\lambda : A^e \to A_l$ be the obvious map of dgas. Clearly A_l is isomorphic to A. The map λ induces an adjunction $F : A^e\text{–mod} \rightleftarrows A_l\text{–mod} : G$ where the left adjoint is $F(M) = M \otimes_A k$ and the right adjoint is $G(N) = N \cong N \otimes k$ with the left action of A on N and the right action of A on k.

Since G preserves fibrations and weak equivalences, F and G constitute a Quillen pair and thereby yield a pair of adjoint functors on the derived categories

$$\mathcal{F} : \mathbf{D}(A^e) \rightleftarrows \mathbf{D}(A_l) : \mathcal{G}$$

It is easy to see that $\mathcal{F}(A)$ is isomorphic in $\mathbf{D}(A_l)$ to k.

7.2. **Definition.** Choose an isomorphism $\mathcal{F}(A) \xrightarrow{\varphi} k$ in $\mathbf{D}(A_l)$. Define the *shearing map for* A to be the morphism $\chi : \mathrm{Ext}^*_{A^e}(A, A) \to \mathrm{Ext}^*_{A_l}(k, k)$ given by the composition

$$\mathrm{Ext}^*_{A^e}(A, A) \xrightarrow{\mathcal{F}} \mathrm{Ext}^*_{A^l}(\mathcal{F}(A), \mathcal{F}(A)) \xrightarrow{p} \mathrm{Ext}^*_{A_l}(k, k)$$

where $p(f) = \varphi f \varphi^{-1}$. We note two immediate properties of the shearing map. First, it is clear that the shearing map is a map of graded rings. Second, this definition is in fact independent of the choice of φ. This is because the automorphism group of k in $\mathbf{D}(A_l)$ is the commutative group k^*, which is also central in $\mathrm{Ext}^*_{A_l}(k, k)$.

Let $\eta_A : A \to \mathcal{G}\mathcal{F}(A)$ be the unit map of this adjunction. We can now define another map $\alpha : \mathrm{Ext}^*_{A^e}(A, A) \to \mathrm{Ext}^*_{A_l}(k, k)$, as the composition

$$\mathrm{Ext}^*_{A^e}(A, A) \xrightarrow{\eta_A} \mathrm{Ext}^*_{A^e}(A, \mathcal{G}\mathcal{F}(A)) \cong \mathrm{Ext}^*_{A^l}(\mathcal{F}(A), \mathcal{F}(A)) \cong \mathrm{Ext}^*_{A_l}(k, k)$$

where the left isomorphism comes from the adjunction. As above, the map α does not depend on our choice of isomorphism $\mathcal{F}(A) \cong k$. From classical category theory we learn that the map $\mathrm{Ext}^*_{A^e}(A,A) \xrightarrow{\mathcal{F}} \mathrm{Ext}^*_{A^l}(\mathcal{F}(A),\mathcal{F}(A))$ is equal to the composition $\mathrm{Ext}^*_{A^e}(A,A) \xrightarrow{\eta_A} \mathrm{Ext}^*_{A^e}(A,\mathcal{GF}(A)) \xrightarrow{\cong} \mathrm{Ext}^*_{A^l}(\mathcal{F}(A),\mathcal{F}(A))$. Hence we conclude that α and χ are equal.

7.3. **The unit map.** For what follows we need to identify the unit map η_A. First we define a map $\mu : k \boxtimes k \to k$ to be the composition of the natural map $k \boxtimes k \to k \otimes_A k$ from Lemma 4.6 with the isomorphism $k \otimes_A k \cong k$. One can think of μ as extending the pairing $\mu_{p,q}$ of Definition 6.9.

Next we choose a model for the derived functor of F, set

$$\mathbf{L}F(M) = F(M \boxtimes A) \cong M \boxtimes k$$

we leave it to the reader to ascertain this indeed yields the derived functor of F. Now $\mathbf{L}F(A) = F(A \boxtimes A)$ and $GF(A \boxtimes A) = A \boxtimes k$ and the unit map $A \boxtimes A \to GF(A \boxtimes A)$ is clearly $1 \boxtimes \mathfrak{a}$. Commutation of the following diagram

$$\begin{tikzcd} A \boxtimes A \arrow[r, "1\boxtimes\mathfrak{a}"] \arrow[d, "\psi"] & A \boxtimes k \arrow[d, "\mu(\mathfrak{a}\boxtimes 1)"] \\ A \arrow[r, "\mathfrak{a}"] & k \end{tikzcd}$$

shows that χ is equal to the composition

$$\mathrm{Ext}^*_{A^e}(A,A) \xrightarrow{\mathfrak{a}} \mathrm{Ext}^*_{A^e}(A,k) \cong \mathrm{Ext}^*_{A_l}(k,k)$$

Since the map $I : \mathrm{Ext}^*_{A^e}(A,A) \to \mathrm{Ext}^*_{A_l}(k,k)$ defined by [11] is induced by $\mathrm{Ext}^*_{A^e}(A,\mathfrak{a})$ we have proven

7.4. **Lemma.** *The shearing map $\chi : Ext^*_{A^e}(A,A) \to Ext^*_{A_l}(k,k)$ is equal to the morphism I defined in* [11].

In addition we can now identify the morphism adjoint to a given element $g \in \mathrm{Ext}^*_{A_l}(k,k)$. The map $\mu(\mathfrak{a} \boxtimes 1) : A \boxtimes k \to k$, is clearly a cofibrant replacement of k. We represent g by the map $\tilde{g} : A \boxtimes k \to \Sigma^n k$. By classical category theory the adjoint map to $\tilde{g}$ is $g(1 \boxtimes \mathfrak{a}) : A \boxtimes A \to \Sigma^n k$.

7.5. **The adjunction.** To specify the adjunction we need to identify the counit map $\epsilon_k : \mathcal{FG}(k) \to k$ as well.

7.6. **Lemma.** *The counit $\epsilon_k : \mathcal{FG}(k) \to k$ map is represented by $\mu : k \boxtimes k \to k$.*

Proof. The counit map is the composition $\mathbf{L}F(G(k)) \xrightarrow{l_{G(k)}} FG(k) \xrightarrow{\varepsilon_k} k$ where l is the natural map $\mathbf{L}F \to F$ and ε is the counit for the adjunction of F and G. Now ε_k is simply the isomorphism $k \otimes k \cong k$. From the definition it is easy to see that $l_{G(k)}$ is μ. □

7.7. **Corollary.** (1) *Let $f \in Ext^*_{A^e}(A,k)$ be represented by $\tilde{f} : \tilde{A} \to \Sigma^n k$, then the adjoint is represented by $\mu(\widetilde{f \boxtimes k})$.*
(2) *Let $g \in Ext^*_{A_l}(k,k)$ be represented by $\tilde{g} : A \boxtimes k \to \Sigma^n k$, then the adjoint is represented by $g(1 \boxtimes \mathfrak{a})$.*

7.8. **The induced pairing.** Since $\mathrm{Ext}^*_{A^e}(A,k)$ is isomorphic to $\mathrm{Ext}^*_{A_l}(k,k)$, there is a graded algebra structure on $\mathrm{Ext}^*_{A^e}(A,k)$. We detail this structure.

7.9. **Definition.** Define a pairing $m : \mathrm{Ext}^i_{A^e}(A,k) \otimes \mathrm{Ext}^j_{A^e}(A,k) \to \mathrm{Ext}^{i+j}_{A^e}(A,k)$ in the following way. Let $\tilde{f} : \tilde{A} \to \Sigma^i k$ represent an element $f \in \mathrm{Ext}^i_{A^e}(A,k)$ and let $\tilde{g} : \tilde{A} \to \Sigma^j k$ represent an element $g \in \mathrm{Ext}^j_{A^e}(A,k)$. Let $m(f \otimes g)$ be the morphism represented by the composition

$$\tilde{A} \xrightarrow{\sim} \tilde{A} \boxtimes \tilde{A} \xrightarrow{\tilde{f} \boxtimes \tilde{g}} \Sigma^i k \boxtimes \Sigma^j k \xrightarrow{\mu} \Sigma^{i+j} k$$

7.10. **Lemma.** *Under the isomorphism $Ext^*_{A^e}(A,k) \cong Ext^*_{A_l}(k,k)$ given by the adjunction, the pairing m corresponds to the multiplication on $Ext^*_{A_l}(k,k)$.*

Proof. By Corollary 7.7 we need first to represent the composition of $\mu(f \boxtimes k)$ with $\mu(g \boxtimes k)$. One easily sees that this composition is represented by

$$(*) \quad \tilde{A} \boxtimes \tilde{A} \boxtimes k \xrightarrow{1 \boxtimes \tilde{g} \boxtimes 1} \tilde{A} \boxtimes \Sigma^j k \boxtimes k \xrightarrow{1 \boxtimes \mu} \tilde{A} \boxtimes \Sigma^j k \xrightarrow{\tilde{f} \boxtimes 1} \Sigma^i k \boxtimes \Sigma^j k \xrightarrow{\mu} \Sigma^{i+j} k$$

This composition is equal to $\mu(1 \boxtimes \mu)(\tilde{f} \boxtimes \tilde{g} \boxtimes \tilde{1}_k)$. It is easy to see that μ satisfies the associativity relation $\mu(1 \boxtimes \mu) = \mu(\mu \boxtimes 1)$. This, together with the associativity of $\boxtimes$, shows that the composition $(*)$ is equal to $\mu(\mu(\tilde{f} \boxtimes \tilde{g}) \boxtimes 1_k)$, whose adjoint is $\mu(\tilde{f} \boxtimes \tilde{g})$ by Corollary 7.7. $\square$

7.11. **The third description of the shearing map.** Every element $x \in HH^n(A)$ induces a natural transformation of functors $\zeta(x) : 1_{\mathbf{D}(A)} \to \Sigma^n 1_{\mathbf{D}(A)}$ which we now describe. For an A-module M the morphism $\zeta(x)_M$ is represented by

$$M \xleftarrow{u} \tilde{A} \boxtimes M \xrightarrow{\tilde{x} \boxtimes M} \Sigma^n \tilde{A} \boxtimes M \xrightarrow{u} M$$

where $\tilde{x} : \tilde{A} \to \Sigma^n \tilde{A}$ represents x and $u = e(e \boxtimes 1_M)$. Addition and multiplication of elements in $HH^*(A)$ become addition and composition of such natural transformations. In this way we get a map ζ of graded commutative rings from $HH^*(A)$ to the graded commutative ring of natural transformations $\{1_{\mathbf{D}(A)} \to \Sigma^n 1_{\mathbf{D}(A)}\}_n$. This is a well known map whose target is called the *centre of* $\mathbf{D}(A)$.

We can now define a third map $HH^*(A) \to \mathrm{Ext}^*_A(k,k)$ by $x \mapsto \zeta(x)_k$. It is easy to see this map is the shearing map as defined in Definition 7.2. From this description it is also clear that the image of the shearing map lies in the graded commutative centre of $\mathrm{Ext}^*_A(k,k)$.

8. Identifying the E^2-term

8.1. **Lemma.** *Suppose A is coconnected of finite type, then there is an isomorphism of graded algebras*

$$E^2_{p,q} \cong H^{-p}(A) \otimes_k Ext^{-q}_A(k,k)$$

Proof. Let B be the differential bigraded algebra given by

$$B_{p,q} = A^{-p} \otimes_k \mathrm{Ext}^{-q}_A(k,k), \quad d^B(a \otimes f) = d^A(a) \otimes f$$

where d^A is the given differential on A. We will show there exists a morphism of differential bigraded algebras $\phi : B \to E^1$ which induces an isomorphism $H_*(\phi) : H_*(B) \to H_*(E^1) = E^2$.

Composition of morphisms yields natural morphisms

$$\mathrm{Ext}^0_{A^e}(k, A^n) \otimes_k \mathrm{Ext}^i_{A^e}(A,k) \to \mathrm{Ext}^i_{A^e}(A, A^n)$$

Note that there is an obvious monomorphism $A^n \cong \mathrm{Ext}^0_k(k, A^n) \to \mathrm{Ext}^0_{A^e}(k, A^n)$. In this way we get morphisms

$$\lambda_{p,q} : A^{-p} \otimes_k \mathrm{Ext}^{-q}_{A^e}(A, k) \to \mathrm{Ext}^{-q}_{A^e}(A, A^{-p})$$

Recall that there is an isomorphism $\mathrm{Ext}^i_A(k, k) \cong \mathrm{Ext}^i_{A^e}(A, k)$. Thus we have a morphism of graded vector spaces:

$$\phi_{p,q} : A^{-p} \otimes_k \mathrm{Ext}^{-q}_A(k, k) \to E^1_{p,q} = \mathrm{Ext}^{-q}_{A^e}(A, A^{-p})$$

Since the differential on E^1 is simply $\mathrm{Ext}^*_{A^e}(A, d^A)$ we see that ϕ yields a morphism of differential bigraded vector spaces $\phi : B \to E^1$.

Next we show that ϕ is a quasi-isomorphism. The observation we need here is that A^n is a k-module whose A^e-module structure is induced by the augmentation $A^e \to k$ (one can use [7, Proposition 3.9] to see this, but in our setting it is obvious). Hence A^n is isomorphic, as an A^e-module, to $d^A(A^{n-1}) \oplus H^n(A) \oplus (A^n/\ker(d^A))$. Since $H^n(A)$ is finite dimensional, then the map $H^n(A) \otimes \mathrm{Ext}^m_A(k, k) \to \mathrm{Ext}^m_{A^e}(A, H^n(A))$ induced by ϕ is an isomorphism. It is now clear that ϕ is a quasi-isomorphism.

It remains to show that ϕ is a morphism of graded algebras. Let $\tilde{k}$ be a cofibrant replacement of k over A. Suppose given elements $x \in A^{-t}$, $u \in A^{-s}$ and maps $y : \tilde{k} \to \Sigma^n k$ and $v : \tilde{k} \to \Sigma^m k$. We shall now compute the product $\phi(x \otimes y) \cdot \phi(u \otimes v)$.

Choose maps $x : k \to A^{-t}$ and $u : k \to A^{-s}$ representing the elements x and y. Recall that $\tilde{A}$ is a cofibrant replacement for A over A^e and let $a : \tilde{A} \to \tilde{k}$ be a map equivalent to the augmentation map $\mathfrak{a} : A \to k$. From the definition ϕ and Corollary 7.7 one sees that $\phi(x \otimes y)$ is represented by the composition xya. The product $\phi(x \otimes y) \cdot \phi(u \otimes v)$ is then represented by the composition

$$(*) \quad \tilde{A} \xrightarrow{\sim} \tilde{A} \boxtimes \tilde{A} \xrightarrow{ay \boxtimes av} \Sigma^n k \boxtimes \Sigma^m k \xrightarrow{x \boxtimes u} \Sigma^n A^{-t} \boxtimes \Sigma^m A^{-s} \xrightarrow{\mu_{s,t}} \Sigma^{n+m} A^{-s-t}$$

Let $w : k \to A^{-s-t}$ be a map representing the element $xu \in A^{-s-t}$, i.e. $w(1) = xu$. We leave it to the reader to ascertain that the composition $(*)$ above is equal to the composition

$$\tilde{A} \xrightarrow{\sim} \tilde{A} \boxtimes \tilde{A} \xrightarrow{ay \boxtimes av} \Sigma^n k \boxtimes \Sigma^m k \xrightarrow{\mu} \Sigma^{n+m} k \xrightarrow{w} \Sigma^{n+m} A^{-s-t}$$

From Lemma 7.10 we see that, under the isomorphism $\mathrm{Ext}^*_{A^e}(A, k) \cong \mathrm{Ext}^*_A(k, k)$, the composition

$$\tilde{A} \xrightarrow{\sim} \tilde{A} \boxtimes \tilde{A} \xrightarrow{ay \boxtimes av} \Sigma^n k \boxtimes \Sigma^m k \xrightarrow{\mu} \Sigma^{n+m} k$$

corresponds to the composition $y \circ v \in \mathrm{Ext}^*_A(k, k)$. We conclude that

$$\phi^{-1}(\phi(x \otimes y) \cdot \phi(u \otimes v)) = xu \otimes (y \circ v)$$

□

8.2. **Lemma.** *Under the isomorphism $E^2_{0,*} \cong Ext^*_A(k, k)$, the infinite cycles in $E^2_{0,*}$ can be identified with the image of the shearing map $\chi : HH^*(A) \to Ext^*_A(k, k)$.*

Proof. Consider the trivial filtration on k given by $I(0) = k$ and $I(p) = 0$ for $p > 0$. As the filtration of A gave a spectral sequence for $\mathrm{Ext}_{A^e}(A, A)$ in Section 5, so does this filtration of k yield a spectral sequence $E'^r_{p,q}$ for $\mathrm{Ext}^*_{A^e}(A, k)$, one which collapses on the E'^1 stage. Indeed $E'^1_{0,*} = \mathrm{Ext}^*_{A^e}(A, k)$ and is zero otherwise.

The augmentation map $\mathfrak{a} : A \to k$ is then a map of filtered A-bimodules and so induces a map of the aforementioned spectral sequences. This map is easily described on the E^2-term: it is the identity map $E^2_{0,*} \to E'^2_{0,*}$ and zero elsewhere. It follows that the infinite cycles in $E^2_{0,*}$ are the image of the map $\mathrm{Ext}^*_{A^e}(A, A) \to \mathrm{Ext}^*_{A^e}(A, k)$ induced by the augmentation $\mathfrak{a} : A \to k$. As we saw in 7.1 this indeed gives the image of the shearing map after composing with the adjunction isomorphism $\mathrm{Ext}^*_{A^e}(A, k) \cong \mathrm{Ext}^*_A(k, k)$. $\square$

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