Orbit types of the compact Lie group E_7 in the complex Freudenthal vector space $\mathfrak{P}^C$

Takashi Miyasaka and Ichiro Yokota

1 Introduction

Let $\mathfrak{J}$ be the exceptional Jordan algebra over $\boldsymbol{R}$ and $\mathfrak{J}^C$ its complexification. Then the simply connected compact exceptional Lie group F_4 acts on $\mathfrak{J}$ and F_4 has three orbit types which are

$$F_4/F_4, \quad F_4/Spin(9), \quad F_4/Spin(8).$$

Similarly the simply connected compact exceptional Lie group E_6 acts on $\mathfrak{J}^C$ and E_6 has five orbit types which are

$$E_6/E_6, \quad E_6/F_4, \quad E_6/Spin(10), \quad E_6/Spin(9), \quad E_6/Spin(8)$$

([6]). In this paper, we determine the orbit types of the simply connected compact exceptional Lie group E_7 in the complex Freudenthal vector space $\mathfrak{P}^C$. As results, E_7 has seven orbit types which are

$$E_7/E_7, \quad E_7/E_6, \quad E_7/F_4, \quad E_7/Spin(11), \quad E_7/Spin(10), \quad E_7/Spin(9), \quad E_7/Spin(8).$$

2 Preliminaries

Let $\mathfrak{C}$ be the division Cayley algebra and $\mathfrak{J} = \{X \in M(3,\mathfrak{C}) \,|\, X^* = X\}$ be the exceptional Jordan algebra with the Jordan multiplication $X \circ Y$, the inner product (X, Y) and the Freudenthal multiplication $X \times Y$ and let $\mathfrak{J}^C$ be the complexification of $\mathfrak{J}$ with the Hermitian inner product $\langle X, Y\rangle$. (The definitions of $X \circ Y$, (X, Y), $X \times Y$ and $\langle X, Y\rangle$ are found in [2]). Moreover, let $\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C$ be the Freudenthal C-vector space with the Hermitian inner product $\langle P, Q\rangle$. For $P, Q \in \mathfrak{P}^C$, we can define a C-linear mapping $P \times Q$ of $\mathfrak{P}^C$. (The definitions of $\langle P, Q\rangle$ and $P \times Q$ are found in [2]). The complex conjugation in the complexified spaces $\mathfrak{C}^C, \mathfrak{J}^C$ or $\mathfrak{P}^C$ is denoted by τ. Now, the simply connected compact exceptional Lie groups F_4, E_6 and E_7 are defined by

$$F_4 \quad = \quad \{\alpha \in \mathrm{Iso}_R(\mathfrak{J}) \,|\, \alpha(X \circ Y) = \alpha X \circ \alpha Y\},$$

$$\begin{aligned}
E_6 &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C) \,|\, \tau\alpha\tau(X \times Y) = \alpha X \times \alpha Y, \langle \alpha X, \alpha Y\rangle = \langle X, Y\rangle\},\\
E_7 &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \,|\, \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q\rangle = \langle P, Q\rangle\}\\
&= \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C) \,|\, \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \alpha(\tau\lambda) = (\tau\lambda)\alpha\}
\end{aligned}$$

(where λ is the C-linear transformation of $\mathfrak{P}^C$ defined by $\lambda(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi)$), respectively. Then we have the natural inclusion $F_4 \subset E_6 \subset E_7$, that is,

$$\begin{aligned}
E_6 &= \{\alpha \in E_7 \,|\, \alpha(0, 0, 1, 0) = (0, 0, 1, 0)\} \subset E_7,\\
F_4 &= \{\alpha \in E_6 \,|\, \alpha E = E\} \subset E_6 \subset E_7,
\end{aligned}$$

where E is the 3×3 unit matrix. The groups F_4, E_6 and E_7 have subgroups

$$\begin{aligned}
Spin(8) &= \{\alpha \in F_4 \,|\, \alpha E_k = E_k, k = 1, 2, 3\} \subset F_4 \subset E_6 \subset E_7,\\
Spin(9) &= \{\alpha \in F_4 \,|\, \alpha E_1 = E_1\} \subset F_4 \subset E_6 \subset E_7,\\
Spin(10) &= \{\alpha \in E_6 \,|\, \alpha E_1 = E_1\} \subset E_6 \subset E_7,\\
Spin(11) &= \{\alpha \in E_7 \,|\, \alpha(E_1, 0, 1, 0) = (E_1, 0, 1, 0)\} \subset E_7,
\end{aligned}$$

(the last fact $(E_7)_{(E_1,0,1,0)} = Spin(11)$ will be proved in Theorem 6.(4)), where E_k is the usual notation in $\mathfrak{J}^C$, e.g. $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ([2]).

3 Orbit types of F_4 in $\mathfrak{J}$ and E_6 in $\mathfrak{J}^C$

We shall review the results of orbit types of F_4 in $\mathfrak{J}$ and E_6 in $\mathfrak{J}^C$.

Lemma 1 ([1]). *Any element $X \in \mathfrak{J}$ can be transformed to a diagonal form by some $\alpha \in F_4$:*

$$\alpha X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, \quad \xi_k \in \boldsymbol{R}, \quad \text{(which is briefly written by } (\xi_1, \xi_2, \xi_3)).$$

The order of ξ_1, ξ_2, ξ_3 can be arbitrarily exchanged under the action of F_4.

Theorem 2 ([6]). *The orbit types of the group F_4 in $\mathfrak{J}$ are as follows.*

(1) *The orbit through (ξ, ξ, ξ) is F_4/F_4.*

(2) *The orbit through (ξ_1, ξ, ξ) (where $\xi_1 \neq \xi$) is $F_4/Spin(9)$.*

(3) *The orbit through (ξ_1, ξ_2, ξ_3) (where ξ_1, ξ_2, ξ_3 are distinct) is $F_4/Spin(8)$.*

Lemma 3 ([2],[4]). *Any element $X \in \mathfrak{J}^C$ can be transformed to the following diagonal form by some $\alpha \in E_6$:*

$$\alpha X = \begin{pmatrix} r_1 a & 0 & 0 \\ 0 & r_2 a & 0 \\ 0 & 0 & r_3 a \end{pmatrix}, \quad r_k \in \boldsymbol{R},\, a \in C, |a| = 1$$

(which is briefly written by (r_1a, r_2a, r_3a)). *The order of* r_1, r_2, r_3 *can be arbitrarily exchanged under the action of* E_6.

Theorem 4 ([6]). *The group* E_6 *has the following five orbit type in* $\mathfrak{J}^C$:

$$E_6/E_6, \quad E_6/F_4, \quad E_6/Spin(10), \quad E_6/Spin(9), \quad E_6/Spin(8)$$

More details, ((r_1a, r_2a, r_3a) *is considered up to a constant*),

(1) *The orbit through* $(0, 0, 0)$ *is* E_6/E_6.

(2) *The orbit through* $(1, 1, 1)$ *is* E_6/F_4.

(3) *The orbit through* $(1, 0, 0)$ *is* $E_6/Spin(10)$.

(4) *The orbit through* $(1, r, r)$ (*where* $0 < r, r \neq 1$) *is* $E_6/Spin(9)$.

(5) *The orbit through* (r, s, t) (*where* $0 \leq r, 0 \leq s, 0 \leq t$ *and* r, s, t *are distinct*) *is* $E_6/Spin(8)$.

4 Orbit types of E_7 in $\mathfrak{P}^C$

Lemma 5 ([2]). *Any element* $P \in \mathfrak{P}^C$ *can be transformed to the following diagonal form by some* $\alpha \in E_7$:

$$\alpha P = (\begin{pmatrix} ar_1 & 0 & 0 \\ 0 & ar_2 & 0 \\ 0 & 0 & ar_3 \end{pmatrix}, \begin{pmatrix} br_1 & 0 & 0 \\ 0 & br_2 & 0 \\ 0 & 0 & br_3 \end{pmatrix}, ar, br), \qquad \begin{matrix} r_k, r \in \boldsymbol{R}, 0 \leq r_k, 0 \leq r, \\ a, b \in C, |a|^2 + |b|^2 = 1. \end{matrix}$$

Moreover, any element $P \in \mathfrak{P}^C$ *can be transformed to the following diagonal form by some* $\varphi(A)\alpha \in \varphi(SU(2))E_7$:

$$\varphi(A)\alpha P = (\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, r, 0), \quad r_k, r \in \boldsymbol{R}, 0 \leq r_k, 0 \leq r,$$

(which is briefly written by $(r_1, r_2, r_3; r)$), *where* $\varphi(A) \in \varphi(SU(2)) \subset E_8$ *and commutes with any element* $\alpha \in E_7$. *The order of* r_1, r_2, r_3, r *can be arbitrarily exchanged under the action of* E_7. (As for the definitions of the groups E_8 and $\varphi(SU(2))$, see [2]). The action of $\varphi(A), A \in SU(2)$, on $\mathfrak{P}^C$ is given by

$$\varphi(A)P = \varphi(\begin{pmatrix} a & -\tau b \\ b & \tau a \end{pmatrix})(X, Y, \xi, \eta) = (aX + \tau(bY), aY - \tau(bX), a\xi + \tau(b\eta), a\eta - \tau(b\xi)).$$

Theorem 6. *The group* E_7 *has the following seven orbit types in* $\mathfrak{P}^C$:

$$E_7/E_7, \quad E_7/E_6, \quad E_7/F_4, \quad E_7/Spin(11), \quad E_7/Spin(10), \quad E_7/Spin(9), \quad E_7/Spin(8).$$

More details,

(1) *The orbit through* $(0, 0, 0; 0)$ *is* E_7/E_7.

(2) *The orbit through* $(0, 0, 0; 1)$ *or* $(1, 1, 1; 1)$ *is* E_7/E_6.

(3) *The orbit through* $(1,\ 1,\ 1;\ 0)$ *or* $(1,1,1;r)$ (*where* $0 < r, 1 \neq r$) *is* E_7/F_4.

(4) *The orbit through* $(1,\ 0,\ 0;\ 1)$ *or* $(1,r,r;1)$ (*where* $0 < r, 1 \neq r$) *is* $E_7/Spin(11)$.

(5) *The orbit through* $(1,0,0;r)$ (*where* $0 < r, 1 \neq r$) *is* $E_7/Spin(10)$.

(6) *The orbit through* $(1,1,r;0)$ *or* $(1,1,r;s)$ (*where* $0 < r, 0 < s$ *and* $1,r,s$ *are distinct*) *is* $E_7/Spin(9)$.

(7) *The orbit through* $(1,r,s;0)$ *or* $(1,r,s;t)$ (*where* r,s,t *are positive and* $1,r,s,t$ *are distinct*) *is* $E_7/Spin(8)$.

Proof. From Lemma 5, the representatives of orbit types (up to a constant) can be given by the following.

$$\begin{array}{llll} (0,0,0;0), & (0,0,0;1), & (0,0,1;1), & (0,0,1;r), \\ (0,1,1;1), & (0,1,1;r), & (0,1,r;s), & (1,1,1;1), \\ (1,1,1;r), & (1,1,r;r), & (1,1,r;s), & (1,r,s;t) \end{array}$$

where r,s,t are positive, $0,1,r,s,t$ are distinct and the order of $0,1,r,s,t$ can be arbitrarily exchanged.

(1) The isotropy subgroup $(E_7)_{(0,0,0;0)}$ is obviously E_7. Therefore the orbit through $(0,\ 0,\ 0;\ 0)$ is E_7/E_7.

(2) The isotropy subgroup $(E_7)_{(0,0,0;1)}$ is E_6. Therefore the orbit through $(0,0,0;1)$ is E_7/E_6.

(2′) The isotropy subgroup $(E_7)_{(1,1,1;1)}$ is conjugate to E_6 in E_7. In fact, we know that the following realization of the homogeneous space E_7/E_6 : $E_7/E_6 = \{P \in \mathfrak{P}^C \,|\, P \times P = 0, \langle P, P\rangle = 1\} = \mathfrak{M}$ ([4]). Since $\frac{1}{2\sqrt{2}}(E,E,1,1)$ and $(0,0,1,0) \in \mathfrak{M}$, there exists $\delta \in E_7$ such that

$$\delta\big(\frac{1}{2\sqrt{2}}(E,E,1,1)\big) = (0,0,1,0).$$

Hence the isotropy subgroup $(E_7)_{(E,E,1,1)}$ is conjugate to the isotropy subgroup $(E_7)_{(0,0,1,0)}$ in E_7 : $(E_7)_{(E,E,1,1)} \sim (E_7)_{(0,0,1,0)}$. On the other hand, since

$$\varphi(\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix})(E,0,1,0) = \frac{1}{\sqrt{2}}(E,E,1,1),$$

we have $(E_7)_{(E,0,1,0)} = (E_7)_{(E,E,1,1)} \sim (E_7)_{(0,0,1,0)} = E_6$. Therefore the orbit through $(1,1,1;1)$ is E_7/E_6.

(3) The isotropy subgroup $(E_7)_{(1,1,1;0)}$ is F_4. In fact, for $\alpha \in E_7$ and $P \in \mathfrak{P}^C$, we have $\alpha(\tau\lambda((P \times P)P)) = \tau\lambda(\alpha((P \times P)P)) = \tau\lambda(\alpha(P \times P)\alpha^{-1}\alpha P) = \tau\lambda((\alpha P \times \alpha P)\alpha P)$. Now, let $P = (1,1,1;0)$. Since $\tau\lambda((P \times P)P) = \frac{3}{2}(0,0,0;1)$, if $\alpha \in E_7$ satisfies $\alpha P = P$, then α also satisfies $\alpha(0,0,0;1) = (0,0,0;1)$. Hence $\alpha \in E_6$, so together with $\alpha E = E$, we have $\alpha \in F_4$. Therefore the orbit through $(1,1,1;0)$ is E_7/F_4.

(3′) The isotropy subgroup $(E_7)_{(1,1,1;r)}$ is F_4. In fact, let $P = (1,1,1;r)$. Since $\tau\lambda((P \times P)P) = \frac{3}{2}(r,r,r;1)$, if $\alpha \in E_7$ satisfies $\alpha P = P \cdots$ (i), then α also satisfies $\alpha(r,r,r;1) = (r,r,r;1) \cdots$ (ii). Take (i) − (ii), then we have $\alpha(1-r, 1-r, 1-r; r-1) = (1-r, 1-r, 1-r; r-1)$. Since $1 - r \neq 0$, we have $\alpha(1,1,1;-1) = (1,1,1;-1)$. Together with $\alpha P = P$, we have $\alpha(0,0,0;1) = (0,0,0;1)$ and $\alpha(1,1,1;0) = (1,1,1;0)$. Hence $\alpha \in E_6$ and hence $\alpha \in F_4$. Therefore the orbit through $(1,1,1;r)$ is E_7/F_4.

(4) The isotropy subgroup $(E_7)_{(1,0,0;1)}$ is $Spin(11)$. In fact, we know that $(E_7)_{(E_1,E_1,1,1)} = Spin(11)$ ([3]). On the other hand, since

$$\varphi(\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix})(E_1, 0, 1, 0) = \frac{1}{\sqrt{2}}(E_1, E_1, 1, 1),$$

we have $(E_7)_{(E_1,0,1,0)} = (E_7)_{(E_1,E_1,1,1)} = Spin(11)$. therefore the orbit through $(1,0,0;1)$ is $E_7/Spin(11)$. Therefore the orbit through $(1,0,0;1)$ is $E_7/Spin(11)$.

(4′) The isotropy subgroup $(E_7)_{(1,r,r;1)}$ is $Spin(11)$. In fact, let $P = (1,r,r;1)$. Since $\tau\lambda((P \times P)P) = \frac{3}{2}(r^2, r, r; r^2)$, if $\alpha \in E_7$ satisfies $\alpha P = P \cdots$ (i), then α also satisfies $\alpha(r^2, r, r; r^2) = (r^2, r, r; r^2) \cdots$ (ii). Take (i) − (ii), then we have $\alpha(1 - r^2, 0, 0; 1 - r^2) = (1 - r^2, 0, 0; 1 - r^2)$. Since $1 - r^2 \neq 0$, we have $\alpha(1,0,0,1) = (1,0,0;1)$. Hence $\alpha \in Spin(11)$. Therefore the orbit through $(1,r,r;1)$ is $E_7/Spin(11)$.

(5) The isotropy subgroup $(E_7)_{(1,0,0;r)}$ is $Spin(10)$. In fact, for $\alpha \in E_7$ and $P \in \mathfrak{P}^C$, we have $\alpha((P \times P)\tau\lambda P) = (\alpha(P \times P)\alpha^{-1})\alpha(\tau\lambda P) = (\alpha P \times \alpha P)\tau\lambda(\alpha P)$. Now, let $P = (1,0,0;r)$. Since $(P \times P)\tau\lambda P = -\frac{1}{2}(r^2, 0, 0; r)$, if $\alpha \in E_7$ satisfies $\alpha P = P \cdots$ (i), then α also satisfies $\alpha(r^2, 0, 0; r) = (r^2, 0, 0; r) \cdots$ (ii). Take (i) − (ii), then we have $\alpha(1 - r^2, 0, 0; 0) = (1 - r^2, 0, 0; 0)$. Since $1 - r^2 \neq 0$, we have $\alpha(1,0,0;0) = (1,0,0;0) \cdots$ (iii). Take (i) − (iii), then $\alpha(0,0,0;r) = (0,0,0;r)$, that is, $\alpha(0,0,0;1) = (0,0,0;1)$. Hence $\alpha \in E_6$ and $\alpha E_1 = E_1$. Thus $\alpha \in Spin(10)$. Therefore the orbit through $(1,0,0;r)$ is $E_7/Spin(10)$.

(6) The isotropy subgroup $(E_7)_{(1,1,r;0)}$ is $Spin(9)$. In fact, let $P = (1,1,r;0)$. Since $\tau\lambda((P \times P)P) = \frac{3}{2}(0,0,0;r)$, if $\alpha \in E_7$ satisfies $\alpha P = P$, then α also satisfies $\alpha(0,0,0;1) = (0,0,0;1)$. Hence $\alpha \in E_6$, so together with $\alpha P = P$, we have $\alpha \in Spin(9)$ (Theorem 4.(4)). Therefore the orbit through $(1,1,r;0)$ is $E_7/Spin(9)$.

(6′) The isotropy subgroup $(E_7)_{(1,1,r;s)}$ is $Spin(9)$. In fact, let $P = (1,1,r;s)$. Since $\tau\lambda((P \times P)P) = \frac{3}{2}(rs, rs, s; r)$, if $\alpha \in E_7$ satisfies $\alpha P = P \cdots$ (i), then α also satisfies $\alpha(rs, rs, s; r) = (rs, rs, s; r) \ldots$ (ii). Take (i) $\times\, r$ − (ii) $\times\, s$, then we have $\alpha(r(1-s^2), r(1-s^2), r^2-s^2; 0) = (r(1-s^2)s, r(1-s^2), r^2-s^2; 0)$. Since $r(1-s^2), r^2 - s^2$ are non-zero and $r(1-s^2) \neq r^2 - s^2$, from (6) we have $\alpha \in Spin(9)$. Therefore the orbit through $(1,1,r;s)$ is $E_7/Spin(9)$.

(7) The isotropy subgroup $(E_7)_{(1,r,s;0)}$ is $Spin(8)$. In fact, let $P = (1,r,s;0)$. Since $\tau\lambda((P \times P)P) = \frac{3}{2}(0,0,0;rs)$, if $\alpha \in E_7$ satisfies $\alpha P = P$, then α also satisfies $\alpha(0,0,0;1) = (0,0,0;1)$. Hence $\alpha \in E_6$, so together with $\alpha P = P$, we

have $\alpha \in Spin(8)$ (Theorem 4.(5)). Therefore the orbit through $(1, r, s; 0)$ is $E_7/Spin(8)$.

(7′) The isotropy subgroup $(E_7)_{(1,r,s;t)}$ is $Spin(8)$. In fact, let $P = (1, r, s; t)$. Since $\tau\lambda((P \times P)P) = \frac{3}{2}(rst, st, rt; rs)$, if $\alpha \in E_7$ satisfies $\alpha P = P \cdots$ (i), then α also satisfies $\alpha(rst, st, rt; rs) = (rst, st, rt; rs) \cdots$ (ii). Take (i) $\times\ rs$ $-$ (ii) $\times\ t$, then we have $\alpha(rs(1-t^2), s(r^2-t^2), r(s^2-t^2); 0) = (rs(1-t^2), s(r^2-t^2), r(s^2-t^2); 0)$. Since $rs(1-t^2), s(r^2-t^2)$ and $r(s^2-t^2)$ are non-zero and distinct, from (7) we have $\alpha \in Spin(8)$. Therefore the orbit through $(1, r, s; t)$ is $E_7/Spin(8)$.

References

[1] H. Freudenthal, *Oktaven, Ausnahmegruppen und Oktavengeometrie,* Math. Inst. Rijksuniv. te Utrecht, 1951.

[2] T. Miyasaka, O. Yasukura and I. Yokota, *Diagonalization of an element P of $\mathfrak{P}^C$ by the compact Lie group E_7*, Tsukuba J. Math., to appear.

[3] O. Yasukura and I. Yokota, *Subgroup $(SU(2) \times Spin(12))/\boldsymbol{Z}_2$ of compact simple Lie group E_7 and non-compact simple Lie group $E_{7,\sigma}$ of type $E_{7(-5)}$*, Hiroshima Math. J. 12(1982), 59-76.

[4] I. Yokota, *Simply connected compact simple Lie group $E_{6(-78)}$ of type E_6 and its involutive automorphisms*, J. Math., Kyoto Univ., 20(1980), 447-473.

[5] I. Yokota, *Realization of involutive automorphisms σ of exceptional Lie groups G,* part II, $G = E_7$, Tsukuba J. Math., 14(1990), 379 - 404.

[6] I. Yokota, *Orbit types of the compact Lie group E_6 in the complex exceptional Jordan algebra $\mathfrak{J}^C$*, Inter. Symp. on nonassociative algebras and related topics, Hiroshima, Japan, World Scientific,1990, 353-359.

[7] I. Yokota, *Exceptional simple Lie groups* (in Japanese), Gendai-Sugakusha, (1992).