Rainbow Connection Number and Radius

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Abstract

Rainbow connection number, $rc(G)$, of a connected graph G is the minimum number of colours needed to colour its edges, so that every pair of vertices is connected by at least one path in which no two edges are coloured the same. In this note we show that for every bridgeless graph G with radius r, $rc(G) \leq r(r+2)$. We demonstrate that this bound is the best possible for $rc(G)$ as a function of r, not just for bridgeless graphs, but also for graphs of any stronger connectivity. It may be noted that, for a general 1-connected graph G, $rc(G)$ can be arbitrarily larger than its radius ($K_{1,n}$ for instance). We further show that for every bridgeless graph G with radius r and chordality (size of a largest induced cycle) k, $rc(G) \leq rk$. Hitherto, the only reported upper bound on the rainbow connection number of bridgeless graphs is $4n/5 - 1$, where n is order of the graph [Caro et al., 2008]

Keywords: rainbow connectivity, rainbow colouring, radius, isometric cycle, chordality.

1. Introduction

Edge colouring of a graph is a function from its edge set to the set of natural numbers. A path in an edge coloured graph with no two edges sharing the same colour is called a *rainbow path*. An edge coloured graph is said to be *rainbow connected* if every pair of vertices is connected by at least one rainbow path. Such a colouring is called a *rainbow colouring* of the graph. The minimum number of colours required to rainbow colour a connected graph is called its *rainbow connection number*, denoted by $rc(G)$. For example, the rainbow connection number of a complete graph is 1, that of a path is its length, and that of a star is its number of leaves. For a basic introduction to the topic, see Chapter 11 in [1].

The concept of rainbow colouring was introduced by Chartrand, Johns, McKeon and Zhang in 2008 [2]. Chakraborty et al. showed that computing the rainbow connection number of a graph is NP-Hard [3]. To rainbow colour a graph, it is enough to ensure that every edge of some spanning tree in the graph gets a distinct colour. Hence order of the graph minus one is an upper bound for rainbow connection number. Many authors view rainbow connectivity as one 'quantifiable' way of strengthening the connectivity property of a graph [4, 3, 5]. Hence tighter upper bounds on rainbow connection number for graphs with higher connectivity have been a subject of investigation. The following are the results in this direction reported in literature: Let G be a graph of order n. If G is 2-edge-connected (bridgeless), then $rc(G) \leq 4n/5 - 1$ and if G is 2-vertex-connected, then $rc(G) \leq \min\{2n/3, n/2 + O(\sqrt{n})\}$ [4]. Krivelevich and Yuster showed that $rc(G) \leq 20n/\delta$ where δ is the minimum degree of G [5]. The result was recently improved by Chandran et al. where it was shown that

$rc(G) \leq 3n/(\delta+1)+3$ [6]. Hence it follows that $rc(G) \leq 3n/(\lambda+1)+3$ if G is λ-edge-connected.

All the above upper bounds grow with n. Diameter of a graph, and hence its radius, are obvious lower bounds for rainbow connection number. Hence it is interesting to see if there is an upper bound which is a function of the radius or diameter alone. Such upper bounds were shown for some special graph classes in [6]. But, for a general graph, the rainbow connection number cannot be upper bounded by a function of r alone. For instance, the star $K_{1,n}$ has a radius 1 but rainbow connection number n. Still, the question of whether such an upper bound exists for graphs with higher connectivity remains. Here we answer this question in the affirmative. In particular, we show that if G is bridgeless, then $rc(G) \leq r(r+2)$ where r is the radius of G (Corollary 5). Moreover, we also demonstrate that the bound cannot be improved even if we assume stronger connectivity (Example 6).

Since the above bound is quadratic in r, we tried to see what additional restriction would give an upper bound which is linear in r. To this end, we show that if the size of isometric cycles or induced cycles in a graph is bounded independently of r, then the rainbow connection number is linear in r. In particular, we show that if G is a bridgeless graph with radius r and the size of a largest isometric cycle ζ, then $rc(G) \leq r\zeta$ (Theorem 4). Since every isometric cycle is induced, it also follows that $rc(G) \leq rk$ where k is the chordality (size of a largest induced cycle) of G (Corollary 7).

1.1. Preliminaries

All graphs considered in this article are finite, simple and undirected. The *length* of a path is its number of edges. An edge in a connected graph is called a *bridge*, if its removal disconnects the graph. A connected graph with no bridges is called a *bridgeless* (or *2-edge-connected*) graph. If S is a subset of vertices of a graph G, the subgraph of G induced by the vertices in S is denoted by $G[S]$. The graph obtained by contracting the set S into a single vertex v_S is denoted by G/S. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$ respectively.

Definition 1. Let G be a connected graph. The *distance* between two vertices u and v in G, denoted by $d_G(u,v)$ is the length of a shortest path between them in G. The *eccentricity* of a vertex v is $ecc(v) := \max_{x \in V(G)} d_G(v,x)$. The *diameter* of G is $diam(G) := \max_{x \in V(G)} ecc(x)$. The *radius* of G is $rad(G) := \min_{x \in V(G)} ecc(x)$. Distance between a vertex v and a set $S \subseteq V(G)$ is $d_G(v,S) := \min_{x \in S} d_G(v,x)$. The neighbourhood of S is $N(S) := \{x \in V(G) | d_G(x,S) = 1\}$.

Definition 2. Given a graph G, a set $D \subseteq V(G)$ is called a *k-step dominating set* of G, if every vertex in G is at a distance at most k from D. Further if $G[D]$ is connected, then D is called a *connected k-step dominating set* of G.

Definition 3. A subgraph H of a graph G is called *isometric* if distance between any pair of vertices in H is the same as their distance in G. The size of a largest isometric cycle in G is denoted by $iso(G)$.

Definition 4. A graph is called *chordal* if it contains no induced cycles of length greater than 3. *Chordality* of a graph G is the length of a largest induced cycle in G.

Note that every isometric cycle is induced and hence $iso(G)$ is at most the chordality of G.

2. Our Results

The most important idea in this note is captured in Lemma 3 and all the upper bounds reported here will follow easily from it. Next important idea in this note, which is used in the construction of all the tight examples, is illustrated in Theorem 4. Before stating Lemma 3, we state and prove two small lemmas which are used in its proof.

Lemma 1. *For every edge e in a graph G, any shortest cycle in G containing e is isometric.*

Proof. Let C be a shortest cycle in G containing e. For contradiction, assume that there exists at least one pair $(x, y) \in V(C) \times V(C)$ such that $d_G(x, y) < d_C(x, y)$. Choose (x, y) to be one with minimum $d_G(x, y)$ among all such pairs. Let P be a shortest x–y path in G. First we show that $P \cap C = \{x, y\}$. If $P \cap C$ contains some vertex $z \notin \{x, y\}$, then $d_G(x, z) + d_G(z, y) = d_G(x, y) < d_C(x, y) \leq d_C(x, z) + d_C(z, y)$. First equality follows since P is a shortest x–y path, the strict inequality follows by assumption and the last is triangle inequality. Therefore, either $d_G(x, z) < d_C(x, z)$ or $d_G(y, z) < d_C(y, z)$. This contradicts the choice of (x, y). Now it is easy to see that P together with the segment of C between x and y containing e will form a cycle strictly smaller than C contradicting the minimality of C. Hence C is isometric. $\square$

Definition 5. Given a graph G and a set $D \subset V(G)$, a *D-ear* is a path $P = (x_0, x_1, \cdots, x_p)$ in G such that $P \cap D = \{x_0, x_p\}$. P may be a closed path, in which case $x_0 = x_p$. Further, P is called an *acceptable* D-ear if either P is a shortest D-ear containing (x_0, x_1) or P is a shortest D-ear containing (x_{p-1}, x_p).

Lemma 2. *If P is an acceptable D-ear in a graph G for some $D \subset V(G)$, then $d_G(x, D) = d_P(x, D)$ for every $x \in P$ where $d_P(x, D)$ is the length of a shortest x–D path along P.*

Proof. Without loss of generality, let $P = (x_0, x_1, \cdots, x_p)$ be a shortest D-ear containing $e = (x_0, x_1)$. It is easy to see that $P' = (v_D, x_1, x_2, \cdots, x_{p-1}, v_D)$ is a shortest cycle in $G' = G/D$ containing $e = (v_D, x_1)$. Hence by Lemma 1, P' is isometric in G'. Now the result follows since $d_G(x, D) = d_{G'}(x, v_D)$ and $d_P(x, D) = d_{P'}(x, v_D)$. $\square$

Lemma 3. *If G is a bridgeless graph, then for every connected k-step dominating set D^k of G, $k \geq 1$, there exists a connected $(k-1)$-step dominating set $D^{k-1} \supset D^k$ such that*

$$rc(G[D^{k-1}]) \leq rc(G[D^k]) + \min\{2k + 1, \zeta\},$$

where $\zeta = iso(G)$.

Proof. Given D^k, we rainbow colour $G[D^k]$ with $rc(G[D^k])$ colours. Let $m = \min\{2k + 1, \zeta\}$ and let $\mathcal{A} = \{a_1, a_2, \cdots\}$ and $\mathcal{B} = \{b_1, b_2, \cdots\}$ be two pools of colours, none of which are used to colour $G[D^k]$. A D^k-ear $P = (x_0, x_1, \cdots, x_p)$ will be called *evenly coloured* if its edges are coloured $a_1, a_2, \cdots, a_{\lceil \frac{p}{2} \rceil}, b_{\lfloor \frac{p}{2} \rfloor}, \cdots, b_2, b_1$ in that order. We prove the lemma by constructing a sequence of sets $D^k = D_0 \subset D_1 \subset \cdots \subset D_t = D^{k-1}$ and colouring the new edges in every induced graph $G[D_i]$ such that the following property is maintained for all $0 \leq i \leq t$.

Property 1. Every $x \in D_i \backslash D^k$ lies in an evenly coloured acceptable D^k-ear in $G[D_i]$.

Note that Property 1 is vacuously true for D_0. Given a D_i which satisfies Property 1, if $D_i \supset N(D^k)$, then D_i is a $(k-1)$-step dominating set and we stop the procedure by setting $t = i$. Otherwise we construct $D_{i+1} \supset D_i$ and colour the new edges of $G[D_{i+1}]$ as follows.

Pick any edge $e = (x_0, x_1) \in D^k \times (N(D^k) \backslash D_i)$ of G and let $Q = (x_0, x_1, \cdots, x_q)$ be a shortest D_k-ear containing e. Such an ear always exists since G is bridgeless. Let x_l be the first vertex of Q in D_i. If $x_l = x_q$, then evenly colour Q. Hence $P = Q$ is an evenly coloured acceptable D^k-ear. Otherwise, since D_i satisfies Property 1, x_l is on some evenly coloured acceptable D^k-ear P' in $G[D_i]$. Let R be the shorter segment of P' with respect to x_l. By Lemma 2, $|R| = d_{P'}(x_l, D^k) = d_G(x_l, D^k)$. Hence $L = (x_0, x_1, \cdots, x_l)$ together with R is also an acceptable D^k-ear, $P = L \cup R$, containing e. Colour the edges of L so that P is evenly coloured. This is possible because (i) R uses colours exclusively from one pool ($|R| \leq \lfloor |P'|/2 \rfloor$, since it is the shorter segment of P') and (ii) R forms the shorter segment of P ($|L| \geq d_G(x_l, D^k) = |R|$, by Lemma 2). Hence the colouring of R can be evenly extended to L. Set $D_{i+1} = D_i \cup P$. Remaining uncoloured edges of $G[D_{i+1}]$ can be assigned any used colour. Clearly D_{i+1} also satisfies Property 1.

Firstly, we claim that at most m new colours are used in the above procedure for constructing D^{k-1} from D^k, for which it is enough to show that $|P| \leq m$ in every iteration. Since D^k is a k-step dominating set and since the D^k-ear $P = (x_0, x_1, \cdots, x_p)$ added in each iteration is acceptable, it follows that $|P| \leq 2k+1$. Otherwise a middle vertex $x_{\lfloor \frac{p}{2} \rfloor}$ of P will be at a distance more than k from D^k (by Lemma 2). Let C be a shortest cycle containing $e = (x_0, x_1)$. C exists since G is bridgeless. By Lemma 1, C is isometric and hence $|C| \leq \zeta$. Further, $|P| \leq |C|$ since C includes a D^k-ear containing e. Thus $|P| \leq m = \min\{2k+1, \zeta\}$ in every iteration.

Next, we claim that the $G[D^{k-1}]$ constructed this way is rainbow connected. Any pair $(x, y) \in D^k \times D^k$, is rainbow connected in $G[D^k]$. For any pair $(x, y) \in (D^{k-1} \backslash D^k) \times D^k$, let $P = (x_0, x_1, \cdots, x_i = x, \cdots, x_p)$ be the evenly coloured (acceptable) D^k-ear containing x. Joining $(x = x_i, x_{i+1}, \cdots, x_p)$ with a x_p–y rainbow path in $G[D^k]$ gives a x–y rainbow path. For any pair $(x, y) \in (D^{k-1} \backslash D^k) \times (D^{k-1} \backslash D^k)$, let $P = (x_0, x_1, \cdots, x_i = x, \cdots, x_p)$ and $Q = (y_0, y_1, \cdots, y_j = y, \cdots, y_q)$ be evenly coloured (acceptable) D^k-ears containing x and y respectively. Without loss of generality, assume that the vertices of P and Q are ordered in such a way that their first halves get colours from Pool $\mathcal{A}$. We consider the following 4 cases. If $i \leq \lfloor \frac{p}{2} \rfloor$ and $j > \lfloor \frac{q}{2} \rfloor$, then joining $(y = y_j, y_{j+1} \cdots, y_q)$ (which is $\mathcal{B}$-coloured) to the y_q–x_0 rainbow path in $G[D^k]$ followed by $(x_0, x_1, \cdots, x_i = x)$ (which is $\mathcal{A}$-coloured) gives a x–y rainbow path. Case when $i > \lfloor \frac{p}{2} \rfloor$ and $j \leq \lfloor \frac{q}{2} \rfloor$ is similar. When $i \leq \lfloor \frac{p}{2} \rfloor$ and $j \leq \lfloor \frac{q}{2} \rfloor$ check if $i \leq j$. If yes, join $(y = y_j, y_{j+1}, \cdots, y_q)$ (which uses colours from $\{a_l \in \mathcal{A} : l \geq j+1\} \cup \mathcal{B}$) to the y_q–x_0 rainbow path in $G[D^k]$ followed by $(x_0, x_1, \cdots, x_i = x)$ (which uses colours from $\{a_l \in \mathcal{A} : l \leq i\}$) to get an x–y rainbow path. If $i > j$, then do the reverse. In the final case, when $i > \lfloor \frac{p}{2} \rfloor$ and $j > \lfloor \frac{q}{2} \rfloor$ check if $q - j \leq p - i$. If yes, join $(y = y_j, y_{j+1}, \cdots, y_q)$ (which uses colours from $\{b_l \in \mathcal{B} : l \leq q - j\}$ to the y_q–x_0 rainbow path in $G[D^k]$ followed by $(x_0, x_1, \cdots, x_i = x)$ (which uses colours from $\mathcal{A} \cup \{b_l \in \mathcal{B} : l \geq p - i + 1\}$) to get an x–y rainbow path. If $q - j > p - i$, then do the reverse. □

Theorem 4. *For every bridgeless graph G,*

$$rc(G) \leq \sum_{i=1}^{r} \min\{2i+1, \zeta\} \leq r\zeta,$$

where r is the radius of G and $\zeta = iso(G)$.

Moreover, for every two integers $r \geq 1$, and $3 \leq \zeta \leq 2r+1$, there exists a bridgeless graph G with radius r and $iso(G) = \zeta$ such that $rc(G) = \sum_{i=1}^{r} \min\{2i+1, \zeta\}$.

Proof. If u is a central vertex of G, i.e., $ecc(u) = r$, then $D^r = \{u\}$ is an r-step dominating set in G and $rc(G[D^r]) = 0$. The only 0-step dominating set in G is $V(G)$. Hence, repeated application of Lemma 3 gives the upper bound

To construct a tight example for a given $r \geq 1$ and $3 \leq \zeta \leq 2r+1$, consider the graph $H_{r,\zeta}$ in Figure 1. Note that (i) $H_{r,\zeta}$ is bridgeless, (ii) the size of largest isometric cycle in $H_{r,\zeta}$ is ζ, and (iii) $ecc(u) = r$ for any $\zeta \leq 2r+1$.

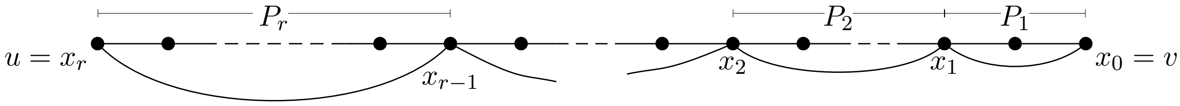

Figure 1: Graph $H_{r,\zeta}$. Every P_i is a x_i–x_{i-1} path of length $|P_i| = \min\{2i, \zeta-1\}$.

Let $m := \sum_{i=1}^{r} \min\{2i+1, \zeta\}$. Construct a graph G by taking $m^r + 1$ graphs $\{H^j\}_{j=0}^{m^r}$ where $V(H^j) = \{x^j : x \in V(H_{r,\zeta})\}$ and $E(H^j) = \{\{x^j, y^j\} : \{x, y\} \in E(H_{r,\zeta})\}$. Identify the vertex u^j as common in every copy ($u = u^j, 0 \leq j \leq m^r$). It can be easily verified that (i) G is bridgeless (ii) $rad(G) = r$ and (ii) size of the largest isometric cycle in G is ζ. Hence, by first part of this theorem, $k := rc(G) \leq m$. In any edge colouring $c : E(G) \to \{1, 2, \cdots, k\}$ of G, each r-length u–v^j path can be coloured in at most k^r different ways. By pigeonhole principle, there exist $p \neq q$, $0 \leq p, q \leq m^r$ such that $c(e_i^p) = c(e_i^q), 1 \leq i \leq r$ where $e_i^j = (x_{i-1}^j, x_i^j)$. Consider any rainbow path R between v^p and v^q. For every $1 \leq i \leq r$, $|R \cap \{e_i^p, e_i^q\}| \leq 1$ (since $c(e_i^p) = c(e_i^q)$) and hence $P_i^j \subset R$ for some $j \in \{p, q\}$. Thus $|R| \geq \sum_{i=1}^{r} (1 + |P_i|) = m$. Hence $k \geq m$ and G gives the required tight example. □

Corollary 5. *For every bridgeless graph G with radius r,*

$$rc(G) \leq r(r+2).$$

Moreover, for every integer $r \geq 1$, there exists a bridgeless graph with radius r and $rc(G) = r(r+2)$.

Proof. Noting that $\min\{2i+1, \zeta\} \leq 2i+1$, the upper bound follows from Theorem 4. The tight examples are obtained by setting $\zeta = 2r+1$ in the tight examples for Theorem 4 □

A natural question at this stage is whether the upper bound of $r(r+2)$ can be improved if we assume a stronger connectivity for G. But the following example shows that it is not the case.

Example 6 (*Construction of a κ-vertex-connected graph of radius r whose rainbow connection number is $r(r+2)$ for any two given integers $\kappa, r \geq 1$*). Let $s(0) := 0$, $s(i) := 2\sum_{j=r}^{r-i+1} j$ for $1 \leq i \leq r$ and $t := s(r) = r(r+1)$. Let $V = V_0 \uplus V_1 \uplus \cdots \uplus V_t$ where $V_i = \{x_{i,0}, x_{i,1}, \cdots, x_{i,\kappa-1}\}$ for $0 \leq i \leq t-1$ and $V_t = \{x_{t,0}\}$. Construct a graph $X_{r,\kappa}$ on V by adding the following edges. $E(X) = \{\{x_{i,j}, x_{i',j'}\} : |i - i'| \leq 1\} \cup \{\{x_{s(i),0}, x_{s(i+1),0}\} : 0 \leq i \leq r-1\}$. Figure 2 depicts $X_{3,2}$.

Let $m = r(r+2)$. Construct a new graph G by taking m^r+1 copies of $X_{r,k}$ and identifying the vertices in V_0 as common in every copy. It is easily seen that G is κ-connected and has a radius r with $x_{0,0}$ as the central vertex. By arguments similar to those in the tight examples for Theorem 4, we can see that $rc(G) = m$.

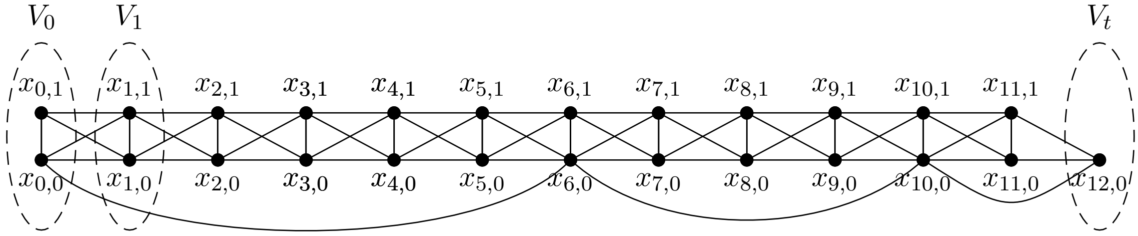

Figure 2: Graph $X_{3,2}$. Note: (i) $X_{3,2}$ is 2-connected and (ii) $ecc(x_{0,0}) = 3$.

Corollary 7. *For every bridgeless graph G with radius r and chordality k,*

$$rc(G) \leq \sum_{i=1}^{r} \min\{2i+1, k\} \leq rk.$$

Moreover, for every two integers $r \geq 1$ and $3 \leq k \leq 2r+1$, there exists a bridgeless graph G with radius r and chordality k such that $rc(G) = \sum_{i=1}^{r} \min\{2i+1, k\}$.

Proof. Since every isometric cycle is an induced cycle, the chordality of a graph is at least the size of its largest isometric cycle. i.e, $k \geq \zeta$. Hence the upper bound follows from that in Theorem 4. The tight example demonstrated in Theorem 4 suffices here too. □

This generalises a result from [6] that the rainbow connection number of any bridgeless chordal graph is at most three times its radius.

References

[1] G. Chartrand, P. Zhang, Chromatic graph theory, Chapman & Hall, 2008.

[2] G. Chartrand, G. L. Johns, K. A. McKeon, P. Zhang, Math. Bohem 133 (2008) 85–98.

[3] S. Chakraborty, E. Fischer, A. Matsliah, R. Yuster, Journal of Combinatorial Optimization (2009) 1–18.

[4] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, R. Yuster, the electronic journal of combinatorics 15 (2008) 1.

[5] M. Krivelevich, R. Yuster, Journal of Graph Theory 63 (2010) 185–191.

[6] L. S. Chandran, A. Das, D. Rajendraprasad, N. M. Varma, Arxiv preprint arXiv:1010.2296v1 [math.CO] (2010).